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Uniform Boundedness of Kantorovich Operators in Variable Exponent Lebesgue Spaces

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Abstract. In this paper, the Kantorovich operators K_n, $n \in \mathbb{N}$ are shown to be uniformly bounded in variable exponent Lebesgue spaces on the closed interval $[0,1]$. Also an upper estimate is obtained for the difference $K_n(f) - f$ for functions f of regularity of order 1 and 2 measured in variable exponent Lebesgue spaces, which is of interest on its own and can be applied to other problems related to the Kantorovich operators.

1. Introduction

The variable exponent Lebesgue space $L^{p(\cdot)}$ is a special case of Orlicz-Musielak spaces treated by Musielak [8]. Many results for variable exponent spaces were obtained, we can refer [8],[13] and the references therein. Since the spaces $L^{p(\cdot)}$ are not invariant to translations, they do not have some undesired properties. For instance, the translation operator is in general not continuous on $L^{p(\cdot)}$. Especially, for every $L^{p(\cdot)}$ with p non-constant there exist $f \in L^{p(\cdot)}$ and a translation τ_h, such that $\tau_h f \notin L^{p(\cdot)}$ (see e.g. [7]). The convolution is in general not continuous, particularly, Young's inequality for convolutions does not hold in the spaces $L^{p(\cdot)}$ (see e.g. [8]). The Hardy-Littlewood maximal operator is in general not bounded on the $L^{p(\cdot)}$. For these reasons the exponent $p(.)$ is a strange manifestation of the technique of proof which does not correspond to anything in the constant exponent case.

We denote by $\Im\,[0,1]$ the class of all measurable functions $p : [0,1] \to [1,\infty)$ satisfying the condition

$$1 \le p^- := \inf_{0\le x\le 1} p(x) \le p(x) \le p^+ := \sup_{0\le x\le 1} p(x) < +\infty.$$

For any $p \in \Im\,[0,1]$, we define the variable exponent Lebesgue space by

$$L^{p(.)}([0,1]) = \left\{ f \mid f : [0,1] \to \mathbb{R} \text{ is measurable}, \int_0^1 \left|f(x)\right|^{p(x)} dx < \infty \right\},$$

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then $L^{p(\cdot)}([0,1])$ endowed with the norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \int_0^1 \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

The modular of $L^{p(\cdot)}([0,1])$ which is the mapping $\rho_{p(.)} : L^{p(\cdot)}([0,1]) \to \mathbb{R}$ is defined by

$$\rho_{p(.)}(f) = \int_0^1 |f(x)|^{p(x)} dx < \infty.$$

We have the following relations:

Proposition 1.1. *(see [8], [13]). If $f \in L^{p(\cdot)}([0,1])$, $f \neq 0$, then*

(i) $\|f\|_{p(.)} < 1\ (= 1; > 1) \Leftrightarrow \rho_{p(.)}(f) < 1\ (= 1; > 1)$;

(ii) $\min\left\{\|f\|_{p(.)}^{p^-}, \|f\|_{p(.)}^{p^+}\right\} \leq \rho_{p(.)}(f) \leq \max\left\{\|f\|_{p(.)}^{p^-}, \|f\|_{p(.)}^{p^+}\right\}$;

(iii) $\rho_{p(.)}(f) \leq C \Leftrightarrow \|f\|_{p(.)} \leq \overline{C}$;

(iv) $\rho_{p(.)}\left(\frac{f}{\|f\|_{p(.)}}\right) = 1.$

Definition 1.2. *Let $p : [0,1] \to \mathbb{R}$. We say that $p(\cdot)$ is log-Hölder continuous if there is a $C > 0$ such that*

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \tag{1}$$

for all $x, y \in [0,1]$ with $|x-y| \leq \frac{1}{2}$.

Fortunately, under some conditions one has proved the continuity of the Hardy–Littlewood maximal function (see [5], [7]). In this articles the authors consider the Hardy-Littlewood maximal operator,

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B\cap\Omega} |f(y)|\, dy$$

where the supremum is taken over all balls B which contain x and for which $|B \cap \Omega| > 0$. It has been known that the condition (1) plays a crucial role for the action of integral operators on $L^{p(\cdot)}(\Omega)$. In particular, [5], [7] showed that if p is log-Hölder continuous $M : L^{p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$ is bounded (continous). In other words, there is a constant $C = C(p^-, p^+, \Omega)) > 0$ such that

$$\|M(f)\|_{p(.)} \leq C(p^-, p^+, \Omega)\, \|f\|_{p(.)}, \tag{2}$$

for $f \in L^{p(\cdot)}(\Omega)$, $1 < p^- \leq p^+ < \infty$.

Bernstein polynomials are used for a constructive proof of the Weierstrass theorem, which dates back to 1911 (see [1], [14]).

Let f be a continuous function on $[0,1]$. Denote by $g_{n,k}(x)$ the Bernoulli polynomial:

$$g_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Bernstein proposed to use polynomials of the form

$$B_n(f)(x) := \sum_{k=0}^{n} f(k/n) g_{n,k}(x),\ f \in C([0,1]),\ x \in [0,1],\ n \in \mathbb{N}.$$

He showed that these polynomials converge uniformly over $[0,1]$ to the original function $f \in C([0,1])$:

$$\lim_{n\to\infty} B_n(f) = f$$

uniformly on $[0,1]$. Some other authors have dealt with the degree of L^p-approximation of the Bernstein operators (see [9] and in case $p=1$ see [2], [11]).

The Kantorovich polynomials introduced by Kantorovich [12]

$$K_n(f)(x) := (n+1)\left(\sum_{k=0}^{n} g_{n,k}(x)\right)\int_{I_{k,n+1}} f(y)dy$$

for every $n \geq 1$ and $0 \leq k \leq n$, $f \in L^1([0,1])$, $0 \leq x \leq 1$. Each K_n is a polynomial of degree not greater than n and every K_n is a positive linear operator from $L^p([0,1])$ into $L^p([0,1])$ (and, in particular, from $C([0,1])$ into $C([0,1])$). For additional information on these operators, see to Chap.10 in [6] and [14].

In [4], Theorem 2.1.2 in [14], [16] and [18] proved that for $f \in L^p([0,1])$, $p \geq 1$

$$\lim_{n\to\infty} \left\|K_n(f) - f\right\|_p = 0.$$

In case $p=1$ the degree of this approximation process for a very special subclass of Lebesgue integrable functions was given by [10].

In [15] the author proved that for $f \in L^p([0,1])$, $p>1$, then

$$\left\|K_n(f) - f\right\|_p = O(n^{-1}),\ n \in \mathbb{N}.$$

It should be noted that, in [3] Kantorovich operators K_n, $n \in \mathbb{N}$ are shown to be uniformly bounded in Morrey spaces on the closed interval $[0,1]$. Also an upper estimate is obtained for the difference $K_n(f) - f$ for functions f of regularity of order 1 measured in Morrey spaces by authors.

In [17], the author obtained the convergence of a sequence of operators of Bernstein – Kantorovich $\{K_n(f)\}_{n=1}^{\infty}$ to the function f in Lebesgue spaces with variable exponent $L^{p(.)}([0,1])$. In this article, $p(x)$ variable exponent satisfied log-Hölder continous and

$$p(x) = \begin{cases} q_1, & 0 \leq x \leq \delta, \\ q_2, & 1-\delta \leq x \leq 1, \end{cases}$$

where $0 < \delta \ll 1$ and $q_1, q_2 \geq 1$ are constants. The conditions on the variable exponent at which this sequence is uniformly bounded in these spaces are obtained and, as a corollary, it is shown that if $n \to \infty$ then $K_n(f)$ converges to function f in the metric of space $L^{p(.)}([0,1])$ defined by the norm.

In the paper [19], the author consider the approximation on an open set Ω by positive linear operators on a variable space $L^{p(\cdot)}(\Omega)$ associated with a general exponent function $p : \Omega \to [1,\infty)$. Under an assumption of log-Hölder continuity of the exponent function p, the author provide quantitative estimates for the approximation when the approximated function lies in a variable Sobolev space. The uniform boundedness of the Kantorovich operators and the Durrmeyer operators on the variable spaces is proved when the exponent function p is Lipschitz α with $0 \leq \alpha \leq 1$, which yields rates of approximation. The technical difficulty arising from the uniform boundedness is overcome by the Lipschiz continuity of the exponent function and localization of Bernstein type positive linear operators.

In this paper we deal with a Kantorovich operators and characterize the convergence in the variable exponent Lebesgue space $L^{p(.)}([0,1])$. Also we an upper estimate is obtained for the difference $K_n(f) - f$ for functions f of regularity of order 1 and 2 measured in $L^{p(.)}([0,1])$ spaces. We note that, the conditions we put on $p(.)$ variable exponent in our article are simplified according to conditions of $p(.)$ variable exponent which using in [17], [19].

Let $p \in \Im[0,1]$ and defined Kantorovich operators $K_n : L^{p(.)}([0,1]) \to L^{p(.)}([0,1])$ by

$$K_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (n+1) \int_{I_{k,n+1}} f(y)dy, \tag{3}$$

where

$$I_{k,n+1} = [k/(n+1), (k+1)/(n+1)].$$

Let $x \in [0,1]$. Define

$$\sum_{k=0}^{n} g_{n,k}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

We need to recall some properties of convex functions.

Consider a real interval I of $\mathbb{R}$. A function $\varphi : I \to \mathbb{R}$ is said to be convex if

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y)$$

for every $x, y \in I$ and $0 \leq \alpha \leq 1$.

If I is open and φ is convex, then for every finite family $(x_k)_{0\leq k\leq n}$ in I and $(\alpha_k)_{0\leq k\leq n}$ in $[0,1]$ such that $\sum_{k=0}^{n} \alpha_k = 1$,

$$\varphi\left(\sum_{k=0}^{n} \alpha_k x_k\right) \leq \sum_{k=0}^{n} \alpha_k \varphi(x_k)$$

(Jensen's inequality).

The function $|t|^p$ ($t \in \mathbb{R}$), $1 \leq p < \infty$, is convex. Given a probability space (Ω, F, μ), an open interval I of $\mathbb{R}$ and a μ-integrable function $f : \Omega \to I$, then

$$\int_\Omega f d\mu \in I.$$

Furthermore, if $\varphi : I \to \mathbb{R}$ is convex and $\varphi \circ f : \Omega \to R$ is μ-integrable, then

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega (\varphi \circ f)\, d\mu$$

(Integral Jensen inequality). In particular, we have

$$\left|\int_\Omega f d\mu\right|^p \leq \int_\Omega \left|f\right|^p d\mu. \tag{4}$$

We denote

$$1 < p_n^- = \inf_{x\in[k/(n+1),(k+1)/(n+1)]} p(x),\ 0 \leq k \leq n \in \mathbb{N}.$$

Lemma 1.3. *Let $p \in \mathfrak{I}[0,1]$ be log-Hölder continuous, if there is a constant $C_0 > 0$ such that*

$$\left|p(x) - p_n^-\right| \leq \frac{C_0}{\ln \frac{n+1}{x}},\ x \in [0,1],\ n \in \mathbb{N}. \tag{5}$$

Then there is a positive constant $\widehat{C}$ such that, for every $x \in [0,1]$

$$(n+1)^{p(x)-p_n^-} \leq \widehat{C}. \tag{6}$$

Proof. By using (5) *we obtain*

$$(n+1)^{p(x)-p_n^-} \leq (n+1)^{\frac{C_0}{\ln \frac{n+1}{x}}} \leq (n+1)^{\frac{C_0}{\ln(n+1)}} = e^{C_0}.$$

Lemma1.1 is proved. □

By $\mathfrak{R}^{\log}[0,1]$ the class of all exponents $p(.)$, satisfying the condition (5).

2. Main results

We can now state our main result.

Theorem 2.1. *If $p \in \mathfrak{R}^{\log}[0,1] \cap \mathfrak{I}[0,1]$ and $f \in L^{p(.)}([0,1])$, then operators of Kantorovich $\{K_n(f)\}_{n=1}^{\infty}$ uniformly bounded in $L^{p(.)}([0,1])$. In other words, there is a constant $C = C(p^-, p^+) > 0$ such that*

$$\left\|K_n(f)\right\|_{p(.)} \leq C \left\|f\right\|_{p(.)},\ n \in \mathbb{N}. \tag{7}$$

Proof. By the Proposition 1.1 (i) we assume that

$$\left\|f\right\|_{p(.)} = 1 \Longleftrightarrow \int_0^1 \left|f(x)\right|^{p(x)} dx = 1. \tag{8}$$

Then, we need to show that there exists a constant C independent of function f such that

$$\left\|K_n(f)\right\|_{p(.)} \leq C.$$

Consider the operators (3), then we have

$$\begin{aligned}\int_0^1 \left|K_n(f)(x)\right|^{p(x)} dx &\leq \int_0^1 \sum_{k=0}^{n} \left|g_{n,k}(x)(n+1)\int_{I_{k,n+1}} f(y)dy\right|^{p(x)-p_n^-} \\ &\quad \times \sum_{k=0}^{n} \left|g_{n,k}(x)(n+1)\int_{I_{k,n+1}} f(y)dy\right|^{p_n^-} dx \\ &:= \int_0^1 \left|J_{1,n}(x)\right| . \left|J_{2,n}(x)\right| dx. \end{aligned} \tag{9}$$

Note that since

$$\sum_{k=0}^{n} g_{n,k}(x) = 1, \tag{10}$$

and

$$x^k (1-x)^{n-k} \geq 0$$

for every $x \in [0,1]$. Then by (6), (8), (9) and (10) we have

$$\begin{aligned}\left|J_{1,n}(x)\right| &\leq (n+1)^{p(x)-p_n^-} \sum_{k=0}^{n} \left(g_{n,k}(x)\int_{I_{k,n+1}} \left|f(y)\right| dy\right)^{p(x)-p_n^-} \\ &\leq \widehat{C}\left(\sum_{k=0}^{n} g_{n,k}(x)\int_0^1 \left|f(y)\right| dy\right)^{p(x)-p_n^-} \\ &\leq \widehat{C}\left(\int_0^1 \left|f(y)\right| dy\right)^{p(x)-p_n^-} \\ &\leq \widehat{C}\left(\int_0^1 \left(\left|f(y)\right|_{\chi\{y:f>1)\}} + \left|f(y)\right|_{\chi\{y:f\leq 1)\}}\right) dy\right)^{p(x)-p_n^-} \\ &\leq \widehat{C}\left(\int_0^1 \left(\left|f(y)\right|^{p(y)} + 1\right) dy\right)^{p(x)-p^-} \\ &\leq \widehat{C}2^{p^+-p^-} \\ &= C_1. \end{aligned}$$

On the other hand, by considering the beta function

$$B(u,v) := \int_0^1 t^{u-1}(1-t)^{v-1}\,dt,\ \ (u,v>0)$$

it is not difficult to show that, for $0 \le k \le n$,

$$\int_0^1 x^k (1-x)^{n-k}\,dx = B(k+1, n-k+1) = \frac{1}{(n+1)\binom{n}{k}}. \tag{11}$$

Hence by the convexity of the function $t \to |t|^{p_n^-}$ ($p_n^- > 1$, $n \in \mathbb{N}$), Jensen's inequality applied to the measure $(n+1)dy$ for every $f \in L^{p_n^-}([0,1])$ and by relations (4), (8), (10), (11) we get

$$\begin{aligned}
\int_0^1 \left|K_n(f)(x)\right|^{p(x)} dx &\le C_1 \int_0^1 \left|J_{2,n}\right| dx \\
&\le C_1 \int_0^1 \left(\sum_{k=0}^n g_{n,k}(x)(n+1)\int_{I_{k,n+1}} \left|f(y)\right| dy\right)^{p_n^-} dx \\
&\le C_1 (n+1)\sum_{k=0}^n \binom{n}{k}\left(\int_0^1 x^k(1-x)^{n-k}\,dx\right)\left(\int_{I_{k,n+1}} \left|f(y)\right|^{p_n^-} dy\right) \\
&\le C_1 \sum_{k=0}^n \int_{I_{k,n+1}} \left|f(y)\right|^{p_n^-} dy \le C_2 \int_0^1 \left(\left|f(y)\right|^{p(y)} + 1\right) dy \\
&= 2C_1 \\
&= C_2.
\end{aligned}$$

Therefore,

$$\int_0^1 \left|K_n(f)(x)\right|^{p(x)} dx \le C_2. \tag{12}$$

By the Proposition 1.1(iii), (8) and (12) we get

$$\left\|K_n(f)\right\|_{p(.)} \le C\left\|f\right\|_{p(.)},$$

then for every $f \in L^{p(\cdot)}([0,1])$, i.e., $\|K_n\|_{p(.)} \le C$.

Now we show that Kantorovich operators converge in $L^{p(\cdot)}([0,1])$. We suppose that f is continuous on $[0,1]$, i. e, $f \in C([0,1])$. In this case, for any $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon)$ such that for $\left|x-y\right| < \delta_0$, $x, y \in [0,1]$ the inequality $\left|f(x) - f(y)\right| < \varepsilon$ holds. Hence, for a given $\varepsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$

$$\begin{aligned}
\left\|K_n(f) - f\right\|_{p(.)} &\le \left\|\sum_{k=0}^n g_{n,k}(x)(n+1)\int_{I_{k,n+1}} \left|f(y) - f(x)\right| dy\right\|_{p(.)} \\
&\le \varepsilon \left\|\sum_{k=0}^n g_{n,k}(x)\right\|_{p(.)} \\
&\le \varepsilon.
\end{aligned} \tag{13}$$

Thus $\{K_n(f)\}_{n=1}^{\infty}$ uniformly bounded in $f \in C([0,1])$. Next, it is well-known (see, for example Corollary 3.4.10 in [8]) that the continuous functions h on $[0,1]$ are dense in $L^{p(\cdot)}([0,1])$. As a result, we can easily see that the functions $h(.)$ which are continuous on $[0,1]$ are dense in $L^{p(\cdot)}([0,1])$. We have

$$\left\|K_n(f) - f\right\|_{p(.)} \le \left\|f - h\right\|_{p(.)} + \|K_n(h) - h\|_{p(.)} + \left\|K_n(f) - K_n(h)\right\|_{p(.)}. \tag{14}$$

Given $\varepsilon > 0$, we choose a function h such that

$$\left\|f - h\right\|_{p(.)} < \varepsilon. \tag{15}$$

By (13), for the continuous function h a given $\varepsilon > 0$ and for $\left|x - y\right| < \delta_0$, $x, y \in [0, 1]$ such that

$$\|K_n(h) - h\|_{p(.)} < \varepsilon. \tag{16}$$

Finally, by (7) and (15) implies that

$$\left\|K_n(f) - K_n(h)\right\|_{p(.)} = \left\|K_n(f - h)\right\|_{p(.)} \leq C(p^-, p^+)\left\|f - h\right\|_{p(.)} \leq C\varepsilon, \tag{17}$$

and now, by (14)-(17) we obtain

$$\left\|K_n(f) - f\right\|_{p(.)} \leq C\varepsilon,\ n \in \mathbb{N}.$$

Since $C([0, 1])$ is dense in $L^{p(\cdot)}([0, 1])$ we have operators $\{K_n(f)\}_{n=1}^{\infty}$ uniformly bounded in $L^{p(\cdot)}([0, 1])$ for arbitrary functions $f \in L^{p(\cdot)}([0, 1])$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *If $f \in L^1([0, 1])$, then*

$$\left\|K_n(f)\right\|_1 \leq (n + 1)\left\|f\right\|_1,\ n \in \mathbb{N}.$$

Proof. Applying twice the Jensen inequality and (10) we can write

$$\left\|K_n(f)\right\|_1 \leq (n + 1)\left(\int_0^1 \sum_{k=0}^{n} g_{n,k}(x)\,dx\right)\int_{I_{k,n+1}} \left|f(y)\right| dy = (n + 1)\left\|f\right\|_1,\ n \in \mathbb{N}$$

namely $K_n(f) \in L^1([0, 1])$ whenever $f \in L^1([0, 1])$. This completes the proof of Theorem 2.2. □

Let $L^{1,p(\cdot)}([0, 1]) = \{f : f \text{ absolutely continuous on } [0, 1],\ f' \in L^{p(\cdot)}([0, 1])\}$.

Theorem 2.3. *If $p \in \mathfrak{R}^{\log}[0, 1] \cap \mathfrak{I}[0, 1]$ and $f \in L^{1,p(\cdot)}([0, 1])$, then there exists $C = C(p^-, p^+) > 0$ such that*

$$\left\|K_n(f) - f\right\|_{p(.)} \leq \frac{C}{n}\left\|f'\right\|_{p(.)},\ n \in \mathbb{N}. \tag{18}$$

Proof. Let $f \in L^{p(\cdot)}([0, 1])$ for every $x \in [0, 1]$, we have

$$K_n(f)(x) - f(x) = K_n(f(.) - f(x))(x) = (n + 1)\left(\sum_{k=0}^{n} g_{n,k}(x)\right)\int_{I_{k,n+1}} (f(t) - f(x))\,dt.$$

By the Proposition 1.1(i) let's assume that

$$\left\|f'\right\|_{p(.)} = 1 \Longleftrightarrow \int_0^1 \left|f'(x)\right|^{p(x)} dx = 1. \tag{19}$$

In order to prove (18) we have to prove

$$n\left\|K_n(f) - f\right\|_{p(.)} \leq C,\ n \in \mathbb{N}.$$

So, one can write

$$\begin{aligned}
& \int_0^1 \left|n\left(K_n\left(f\right)(x)-f(x)\right)\right|^{p(x)} dx \\
= & \int_0^1 \left(n\left(n+1\right)\sum_{k=0}^{n} g_{n,k}\left(x\right)\int_{I_{k,n+1}} \left|f(y)-f(x)\right| dy\right)^{p(x)} dx \\
= & \int_0^1 \left(n\left(n+1\right)\sum_{k=0}^{n} g_{n,k}\left(x\right)\int_{I_{k,n+1}} \int_x^y \left|f'(t)\right| dtdy\right)^{p(x)} dx \\
\leq & \int_0^1 \left(n\left(n+1\right)\sum_{k=0}^{n} g_{n,k}\left(x\right)\int_{I_{k,n+1}} dy \int_{I_{k,n+1}} \left|f'(t)\right| dt\right)^{p(x)} dx \\
\leq & \int_0^1 \left(n\sum_{k=0}^{n} g_{n,k}\left(x\right)\left(\int_{I_{k,n+1}} \left|f'(t)\right| dt\right)\right)^{p(x)} dx \\
\leq & \int_0^1 \left[\left(n\sum_{k=0}^{n} g_{n,k}\left(x\right)\int_{I_{k,n+1}} \left|f'(t)\right| dt\right)^{p(x)-p_n^-} \left(n\sum_{k=0}^{n} g_{n,k}\left(x\right)\int_{I_{k,n+1}} \left|f'(t)\right| dt\right)^{p_n^-}\right] dx \\
:= & \int_0^1 \left|G_{1,n}\left(x\right)\right| . \left|G_{2,n}\left(x\right)\right| dx.
\end{aligned} \tag{20}$$

and by (19)

$$\begin{aligned}
\left|G_{1,n}(x)\right| & \leq (n+1)^{p(x)-p_n^-}\left(\sum_{k=0}^{n}\int_{I_{k,n+1}} g_{n,k}\left(x\right)\left|f'(t)\right| dt\right)^{p(x)-p_n^-} \\
& \leq \widehat{C}\left(\sum_{k=0}^{n} g_{n,k}\left(x\right)\int_0^1 \left|f'(t)\right| dt\right)^{p(x)-p_n^-} \\
& \leq C_1\left(\int_0^1 \left|f'(t)\right| dt\right)^{p(x)-p_n^-} \\
& \leq \widehat{C}\left(\int_0^1 \left(\left|f'(t)\right|_{\chi\{y:f>1)\}} + \left|f'(t)\right|_{\chi\{y:f\leq 1)\}}\right) dt\right)^{p(x)-p_n^-} \\
& \leq \widehat{C}\left(\int_0^1 \left(\left|f'(t)\right|^{p(t)}+1\right) dt\right)^{p(x)-p^-} \\
& \leq \widehat{C}2^{p^+-p^-} \\
& = C_3.
\end{aligned} \tag{21}$$

Hence by using (20) we obtain

$$\begin{aligned}
\int_0^1 \left|n\left(K_n\left(f\right)(x)-f(x)\right)\right|^{p(x)} dx & \leq C_3\int_0^1 \left|G_{2,n}(x)\right| dx \\
& \leq C_3\sum_{k=0}^{n}\binom{n}{k}\left(\int_0^1 x^k\left(1-x\right)^{n-k} dx\right)\left(\left(n+1\right)\int_{I_{k,n+1}} \left|f(t)\right|^{p_n^-} dt\right)
\end{aligned}$$

$$\begin{aligned} &\leq C_3 \sum_{k=0}^{n} \int_{I_{k,n+1}} \left|f(t)\right|^{p_n^-} dt \leq C \int_0^1 \left(\left|f(t)\right|^{p(t)} + 1\right) dt \\ &= 2C_3. \end{aligned} \tag{22}$$

Then from (21) and (22) we get

$$\int_0^1 \left|n\left(K_n\left(f\right)(x) - f(x)\right)\right|^{p(x)} dx \leq C(p^-, p^+),$$

or, what is the same,

$$\int_0^1 \left|\frac{n\left(K_n\left(f\right)(x) - f(x)\right)}{C(p^-, p^+)^{\frac{1}{p(x)}}}\right|^{p(x)} dx \leq 1.$$

Hence,

$$\int_0^1 \left|\frac{K_n\left(f\right)(x) - f(x)}{\left((1 + C(p^-, p^+))^{\frac{1}{p^-}}\right)/n}\right|^{p(x)} dx \leq 1,$$

and so

$$\left\|K_n(f) - f\right\|_{p(.)} \leq \frac{(1 + C(p^-, p^+))^{\frac{1}{p^-}}}{n} = \frac{C}{n}\left\|f'\right\|_{p(.)},\ n \in \mathbb{N}.$$

Theorem 2.3 is proved. □

Let $L^{2,p(.)}([0,1]) = \{f : f'$ absolutely continuous on $[0,1]$, $f'' \in L^{p(.)}([0,1])\}$.

Theorem 2.4. *If $p \in \mathfrak{I}[0,1]$ satisfies (1) and $f \in L^{2,p(.)}([0,1])$, then there are constants $C_0 = C_0(p^-, p^+) > 0$ and $C_1 = C_1(p^-, p^+) > 0$ such that*

$$\left\|K_n\left(f\right) - f\right\|_{p(.)} \leq \frac{C_0}{2\sqrt{n}}\left\|f'\right\|_{p(.)} + \frac{C_1}{4n}\left\|f''\right\|_{p(.)},\ n \in \mathbb{N}.$$

Proof. Let $f \in L^{p(.)}([0,1])$. For some $z \in (x,t)$

$$f(t) = f(x) + (t - x)\, f'(z).$$

Thus we have

$$\begin{aligned} f(t) - f(x) &= (t-x)\, f'(x) + (t-x)\left(f'(z) - f'(x)\right) \\ &\leq (t-x)\, f'(x) + (t-x)\int_x^z f''(u)du. \end{aligned}$$

Since all $g_{n,k}(x) \geq 0$, we have

$$\begin{aligned} \left|K_n\left(f\right)(x) - f(x)\right| &\leq K_n\left(\left|f(.) - f(x)\right|\right)(x) \\ &\leq \left|f'(x)\right| K_n\left(|. - x|\right)(x) + \left|\int_x^z f''(u)du\right| K_n\left(|. - x|\right)(x) \\ &\leq \left|f'(x)\right| K_n\left(|. - x|\right)(x) + K_n\left((. - x)^2\right)(x) \sup_{\substack{0 \leq t \leq 1 \\ t \neq x}} \frac{1}{t-x}\int_x^t \left|f''(u)\right| du \\ &= \left|f'(x)\right| K_n\left(|. - x|\right)(x) + \left|M\left(f''\right)(x)\right| K_n\left((. - x)^2\right)(x), \end{aligned}$$

where

$$K_n\left(|.-x|\right)(x)=\sum_{k=0}^{n} g_{n,k}(x)(n+1)\int_{I_{k,n+1}} |t-x|\,dt.$$

We obtain by direct computation

$$K_n(1)(x)=1,$$

$$K_n(t)(x)=\frac{n}{n+1}x+\frac{1}{2(n+1)},$$

and

$$K_n\left(t^2\right)(x)=\frac{n(n-1)}{(n+1)^2}x^2+\frac{2n}{(n+1)^2}x+\frac{1}{3(n+1)^2}.$$

Thus by (4) and since $x(1-x)\le\frac{1}{4}$ we obtain

$$\begin{aligned} K_n^2\left(|.-x|\right)(x) &= \sum_{k=0}^{n} g_{n,k}(x)(n+1)\int_{I_{k,n+1}} (t-x)^2\,dt \\ &\le \frac{(n-1)}{(n+1)^2}x(x-1)+\frac{1}{3(n+1)^2} \\ &\le \frac{n-1}{4(n+1)^2}+\frac{1}{3(n+1)^2} \\ &\le \frac{3n+1}{12(n+1)^2} \\ &\le \frac{1}{4n}. \end{aligned}$$

This yields

$$\left|K_n(f)(x)-f(x)\right|\le\frac{1}{2\sqrt{n}}\left|f'(x)\right|+\frac{1}{4n}\left|M(f'')(x)\right|.$$

We can immediately obtain

$$\left|K_n(f)(x)-f(x)\right|^{p(x)}\le C\left(\frac{1}{2\sqrt{n}}\left|f'(x)\right|\right)^{p(x)}+C\left(\frac{1}{4n}\left|M(f'')(x)\right|\right)^{p(x)}.$$

Let

$$\mu_n=C_0\left(\frac{1}{2\sqrt{n}}\left\|f'\right\|_{p(.)}+\frac{1}{4n}\left\|M(f'')\right\|_{p(.)}\right)\ne 0,\ C_0>0$$

will be taken in the following. Then

$$\begin{aligned} \int_0^1\left|\frac{K_n(f)(x)-f(x)}{\mu_n}\right|^{p(x)}dx &\le C\left(\int_0^1\left|\frac{\frac{1}{2\sqrt{n}}f'(x)}{\mu_n}\right|^{p(x)}dx+\int_0^1\left|\frac{\frac{1}{4n}M(f'')(x)}{\mu_n}\right|^{p(x)}dx\right) \\ &\le C\left(\int_0^1\left|\frac{f'(x)}{C_0\left\|f'\right\|_{p(.)}}\right|^{p(x)}dx+\int_0^1\left|\frac{M(f'')(x)}{C_0\left\|M(f'')\right\|_{p(.)}}\right|^{p(x)}dx\right). \end{aligned}$$

If $2C \leq 1$, let $C_0 = 1$, then (by Proposition 1.1(iv))

$$\int_0^1 \left| \frac{K_n(f)(x) - f(x)}{\mu_n} \right|^{p(x)} dx \leq 2C \leq 1.$$

If $2C > 1$, let $C_0 = 2C$ then

$$\int_0^1 \left| \frac{K_n(f)(x) - f(x)}{\mu_n} \right|^{p(x)} dx \leq \frac{2C}{C_0^{p^-}} \leq 1.$$

Hence we can find a constant C_0 that only depends on p, but is independent of f, such that (by Proposition 1.1(iv))

$$\left\| K_n(f) - f \right\|_{p(.)} \leq \frac{C_0}{2\sqrt{n}} \left\| f' \right\|_{p(.)} + \frac{C_0}{4n} \left\| M(f'') \right\|_{p(.)}.$$

Since $p(.)$ satisfy (1) and by (2) we have

$$\left\| K_n(f) - f \right\|_{p(.)} \leq \frac{C_0}{2\sqrt{n}} \left\| f' \right\|_{p(.)} + \frac{C_1}{4n} \left\| f'' \right\|_{p(.)}, \; n \in \mathbb{N}.$$

Theorem 2.4 is proved. □

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References

[1] S.N. Bernstein, Demostration du theoreme de Weierstrass fond´ee sur le calcul de probabilit´es, Comm. Soc. Math. Kharkow 13(2) (1912-1913), 1–2.

[2] R. Bojanic and O. Shisha, Degree of L_1 approximation to integrable functions by modified Bernstein polynomials, J. Approximation Theory, 13(1975), 66–72.

[3] V. Burenkov, A. Ghorbanalizadeh and Y. Sawano, Uniform boundedness of Kantorovich operators in Morrey spaces, Positivity, 22(4) (2018), 1097–1107.

[4] M. Campiti and G. Metafune, L^p-convergence of Bernstein-Kantorovich-type operators, Ann. Polon. Math., 63(3) (1996), 273–280.

[5] D. Cruz-Uribe, SFO, A. Fiorenza, C.J. Neugebauer, The maximal function on variable L^p spaces, Ann. Acad. Sci. Fenn. Math. 28 (2003) 223–238; 29 (2004), 247–249.

[6] R.A. De Vore and G. G. Lorentz, Constructive Approximation, Grundlehren der mathematischen Wissenschaften, 303, Springer-Verlag, Berlin, Heidelberg, 1993.

[7] L. Diening, Maximal function on generalized $L^{p(.)}$ spaces, Math. Inequal. Appl. 7 (2004), 245–253.

[8] L. Diening, P. Harjuletho, P. Hästö, and M. Růžička, Lebesgue and Sobolev spaces with variable exponents. Springer-Verlag, 509, 2011, Berlin.

[9] Z. Ditzian and C.P. May, L_p saturation and inverse theorems for modified Bernstein polynomials, Indiana Univ. Math. J., 25 (1976), 733–751.

[10] A. Grundmann, Güiteabschüzungen für den Kantorovich-Operator in der L_1-Norm, Math. Z.,150 (1976), 45–47.

[11] W. Hoeffding, The L_1-norm of the approximation error for Bernstein-type polynomials, J. Approximation Theory, 4 (1971), 347–356.

[12] L.V. Kantorovich, Sur certains developlaements suivant les polynomes de la forme de S. Bernstein. Iet II, C. R. Acad. Sci. URSS (1930), 563–568 et 595–600.

[13] O. Kováčik, J. Rǎkosnik, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41(116) (1991), 592–628.

[14] G.G. Lorentz, Bernstein polynomials, 2nd edn. Chelsea Publishing Co., New York,1986.

[15] V. Maier, L_p-approximation by Kantorovich operators, Analysis Mathematica, 4 (1978), 289–295.

[16] M.W. Müller, Die Gtite der L_p-Approximation durch Kantorovich-Polynome, Math. Z.,152 (1976), 243–247.

[17] T.N. Shakh-Emirov, On Convergence of Bernstein-Kantorovich Operators sequence in Variable Exponent Lebesgue Spaces, Izvestiya of Saratov University. New Series. Series Mathematics. Mechanics. Informatics, 16(3) (2016), 322-330.

[18] V. Totik, L^p ($p > 1$)-Approximation by Kantorovich polinomials, Analysis, 3 (1983), 79-100.

[19] D.X. Zhou, Approximation by Positive Linear Operators on Variable $L^{p(.)}$ Spaces, Journal of Applied Functional Analysis, 9(1/2) (2014), 379-391.