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Study of New Class of $q-$Fractional Integral Operator

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Abstract. In this paper, we study on the new class of $q-$fractional integral operator. In the aid of iterated Cauchy integral approach to fractional integral operator, we applied $t^p f(t)$ in these integrals and a new class of q-fractional integral operator with parameter p, is introduced. Recently, the q-analogue of fractional differential integral operator is studied and all of the operators defined in these studies are $q-$analogue of Riemann fractional differential operator. We show that our new class of operator generalize all the operators in use, and additionally, it can cover the q-analogue of Hadamard fractional differential operator, as well. Some properties of this operator are investigated.

Keyword: q-fractional differential integral operator, Hadamard fractional differential operator

1. Introduction

Fractional calculus has a long history and has recently gone through a period of rapid development. q-differential operators were defined by Jackson (1908)[1], after that , $q-$calculus became a bridge between mathematics and physics. It has lots of applications in different areas of mathematics such as combinatorics, number theory, basic hypergeometric functions and other sciences: quantum theory, mechanics, theory of relativity, capacitor theory, electrical circuits, particle physics, viscoelastic, electro analytical chemistry, neurology, diffusion systems, control theory and statistics. The $q-$Riemann-Liouville fractional integral operator was first introduced by Al-Salam [2], and then some studies on q-analogues of Riemann operator were done [3], [4], [5], [6], [7], [8].

On the other hand, recent studies on fractional differential equations indicate that a variety of interesting and important results concerning existence and uniqueness of solutions, stability properties of solutions, and analytic and numerical methods of solutions for these equations have been obtained, and the surge for investigating more and more results is underway. Several real world problems were modeled. Nowadays, fractional-order differential equations can be traced in a variety of applications such as diffusion processes, biomathematics, thermo-elasticity [9], etc. However, most of the work on this topic is based on Riemann-Liouville, and Caputo-type fractional differential equations . q-analogue of these operators are defined [2] and application of them is investigated [5], [6], [10]. Another type of fractional derivative that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard operator, introduced in 1892 [11], which contains logarithmic function of arbitrary exponent in the kernel of the integral appearing in its definition. In the paper [12], Leonhard Euler (1707–1783)

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introduced the series which can be considered as $q-$analogue of logarithm function. Since that time, a lot of mathematician have tried to define $q-$logarithm function. Because of the difficulty of working with this function, there is no $q-$analogue of Hadamard fractional differential integral operator. Hadamard-type integrals arise in the formulation of many problems in mechanics such as fracture analysis. For more details and applications of Hadamard fractional derivative and integral, we refer the reader to a new book that gathered all of these applications [13] and to articles [14–16].

In the following paper, new q-integral operator is introduced, then, Some properties and relations will be investigated. In fact, a parameter is used to generalize the Riemann operator to define new class of $q-$fractional difference operator. In the first section, let us introduce some familiar concepts of q-calculus. Most of these definitions and concepts are available in [18] and [19]. We use $[n]_q$ as a $q-$analogue of any complex number. Naturally, we can define $[n]_q!$ as

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \neq 1)\,; \quad [0]_q! = 1; \quad [n]_q! = [n]_q\,[n-1]_q \quad n \in \mathbb{N},\ a \in \mathbb{C}\,.$$

The q-shifted factorial and q-polynomial coefficient are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n \in \mathbb{N},$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j a\right), \quad \left|q\right| < 1,\ a \in \mathbb{C}.$$

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = \frac{(q;q)_n}{(q;q)_{n-k}\,(q;q)_k},$$

Let the function $|f(x)x^\alpha|$ be bounded on the interval $(0, A]$ for some $0 \leq \alpha < 1$, then Jackson integral is defined as [18]

$$\int f(x)d_qx = (1-q)x\sum_{i=0}^{\infty} q^i f(q^i x)$$

and it converges to a function $F(x)$ on $(0, A]$, which is a $q-$antiderivative of $f(x)$. Suppose $0 < a < b$, the definite $q-$integral is defined as

$$\int_0^b f(x)d_qx = (1-q)b\sum_{i=0}^{\infty} q^i f(q^i b),$$

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

$q-$analogue of integral by part can be written as

$$\int_a^b g(x)D_qf(x)d_qx = (g(b)f(b) - g(a)f(a)) - \int_a^b f(qx)D_qg(x)d_qx.$$

In addition, we can interchange the order of double q-integral by

$$\int_0^x \int_0^v f(s)d_qs d_qv = \int_0^x \int_{qs}^x f(s)d_qv d_qs.$$

It is noticeable that, the limit of integration is changed here. q-shifted factorial may extend to the following definition

$$(x-a)^{(\alpha)} = x^{\alpha} \prod_{k=0}^{\infty} \frac{\left(1-\frac{x}{a}q^{k}\right)}{\left(1-\frac{x}{a}q^{k+\alpha}\right)} = \frac{x^{\alpha}\left(\frac{x}{a};q\right)_{\infty}}{\left(q^{\alpha}\frac{x}{a};q\right)_{\infty}}. \tag{1}$$

We can write q-Gamma function by using this definition as [4]

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}.$$

Let us generalize the relation (1) to the following form

$$(x-y)^{(\alpha)}_{q^{p+1}} = x^{\alpha} \prod_{k=0}^{\infty} \frac{\left(x-y\left(q^{p+1}\right)^{k}\right)}{\left(x-y\left(q^{p+1}\right)^{k+\alpha}\right)} = \frac{x^{\alpha}\left(\frac{y}{x};q^{p+1}\right)_{\infty}}{\left(q^{\alpha(p+1)}\frac{y}{x};q^{p+1}\right)_{\infty}}. \tag{2}$$

2. Iterated q-integral to approach new class of operators

There are several approaches to fractional differential operators. One of these discussions is obtained by the iterated Cauchy integrals. The Riemann–Liouville fractional integral is a generalization of the following iterated Cauchy integral:

$$\int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} f(t_n)dt_n = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt.$$

In the aid of this formula, for any positive real value $0 < \alpha$, we have

$${}_aI^{\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt.$$

By putting $\frac{1}{t_i}$ in the chain of integration, we can reach to Hadamard operator. The related iterated integral is written as

$$\int_a^x \frac{1}{t_1}dt_1 \int_a^{t_1} \frac{1}{t_2}dt_2 ... \int_a^{t_{n-1}} \frac{1}{t_n} f(t_n)dt_n = \frac{1}{\Gamma(n)} \int_a^x \left(Log\left(\frac{x}{t}\right)\right)^{n-1} f(t)\frac{dt}{t}.$$

In 1892, Hadamard began the publication of series of articles under the common title [11]. Third section of this article gave an underlying idea for creating different form of fractional integral operators. In this section, Hadamard investigated the relation between coefficients of series with unit radius of convergent and singularity of function. This operator is defined by

$${}_aJ^{\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(Log\left(\frac{x}{t}\right)\right)^{\alpha-1} f(t)\frac{dt}{t}.$$

The author in [21] assumed t_i^p in the chain of integration and reached to the general formula for fractional integral operator.There are four different models of q-analogues of Riemann-Liouville fractional integral operators. There are some trying to investigated the Hadamard type but there is no q-analogue of this operator. First let us rewrite the definition of $q-$fractional integral operator in all introduced forms. In fact, for $\alpha \geq 0$ and $f : [a, b] \to \mathbb{R}$, the α order fractional q-integral of a function $f(x)$ is defined by

$$I_{q,a}^{\alpha}(f(x)) = \frac{1}{\Gamma_q(\alpha)} \int_a^x K_q(t, x) f(t) d_q t$$

The kernel of this integral is defined as $K_q(t, x) = (x-qt)^{(\alpha-1)}$ [3], $K_q(t, x) = (x-qt)_q^{\alpha-1}$ [4], $K_q(t, x) = x^{\alpha-1}(\frac{qt}{x}; q)_{\alpha-1}$ [6], and $K_q(t, x) = (x - qt)_{\alpha-1}$ [5]. In fact, some alternative definitions were introduced, [22] but difficulty of defining q-analogue of logarithm still remains problem.

Lemma 2.1. *On behalf of the Hadamard integral operator for $k \in \mathbb{N}$, the new q-integral operator is found as below*

$$J_{p,q}^{k}(f(a)) = \frac{1}{\prod_{n=1}^{k-1} [n(p+1)]_q} \int_0^a w^p f(w) \prod_{n=0}^{k-1} \left(a^{p+1} - (wq)^{p+1} q^{n(p+1)}\right) d_q w$$

Proof. Clearly, the base case holds. For $n = 2$, we have:

$$J_{p,q}^{2}(f(a)) = \int_0^a \int_0^x x^p y^p f(y) d_q y d_q x = \int_0^a \int_{qy}^a x^p y^p f(y) d_q x d_q y = \frac{1}{[p+1]_q} \int_0^a y^p f(y) \left[a^{p+1} - q^{p+1} y^{p+1}\right] d_q y.$$

Assume that for $n = k - 1$ the relation is true, then we have

$$\begin{aligned} J_{p,q}^{k}(f(x)) &= J_{p,q}^{k-1}\left(J_{p,q} f(x)\right) \\ &= \frac{1}{[p+1]_q \prod_{n=1}^{k-1} [n(p+1)]} \int_0^x \int_0^y y^p w^p f(w) \left[x^{p+1} - q^{p+1} y^{p+1}\right] \prod_{n=0}^{k-1} \left(a^{p+1} - (wq)^{p+1} q^{n(p+1)}\right) d_q w d_q y \\ &= \frac{1}{\prod_{n=1}^{k} [n(p+1)]} \int_0^x w^p f(w) \times \\ &\left[\frac{[k(p+1)]}{[p+1]} \int_{qy}^x y^p \left[x^{p+1} - q^{p+1} y^{p+1}\right] \prod_{n=0}^{k-1} \left(a^{p+1} - (wq)^{p+1} q^{n(p+1)}\right) d_q y \right] d_q w. \end{aligned}$$

Inner part of integral can be computed as special case of Lemma 3.3 which is mention later. It is completed the proof by induction. □

This mentioned relation motivates us to define q-analogue of integral operator as follows;

Definition 2.2. *Let $\alpha > 0$ and $x > 0$, if Jackson integral of $f(x)$exists, then we define q-fractional integral as*

$$\begin{aligned} J_{p,q}^{\alpha}(f(a)) = &= \frac{(1-q)^{\alpha-1}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha-1)}} \int_0^a w^p f(w) (a^{p+1} - (wq)^{p+1})_{q^{p+1}}^{(\alpha-1)} d_q w \\ &= \frac{\left([p+1]_q\right)^{1-\alpha}}{\Gamma_{q^{p+1}}(\alpha)} \int_0^a w^p f(w) (a^{p+1} - (wq)^{p+1})_{q^{p+1}}^{(\alpha-1)} d_q w. \end{aligned}$$

Remark 2.3. *Alternatively, we may define*

$$[p+1]^{(\alpha)} = \prod_{k=1}^{\infty} \frac{[p+1]_{q^k}}{[p+1]_{q^{k+\alpha}}} = \frac{\left(1-q^{p+1}\right)\left(1-q^{2p+2}\right)...}{(1-q^{(\alpha+1)(p+1)})\,(1-q^{(\alpha+2)(p+1)})\,...}\,\frac{(1-q^{\alpha+1})(1-q^{\alpha+2})...}{(1-q)(1-q^2)...}$$
$$= \frac{\left(1-q^{p+1}\right)_{q^{p+1}}^{(\alpha)}}{(1-q)^{(\alpha)}} = \frac{\left(1-q^{p+1}\right)_{q^{p+1}}^{(\alpha)}}{(1-q)^{\alpha}\,\Gamma_q(\alpha+1)} = \frac{\Gamma_{q^{p+1}}(\alpha+1)}{\Gamma_q(\alpha+1)}\left([p+1]_q\right)^{\alpha}, \tag{3}$$

which is another version of $q-$analogue of exponent. Then last definition can be written as

$$J_{p,q}^{\alpha}\left(f(a)\right) = = \frac{1}{[p+1]^{(\alpha-1)}\,\Gamma_q(\alpha)} \int_0^a w^p f(w)(a^{p+1}-(wq)^{p+1})_{q^{p+1}}^{(\alpha-1)} d_q w.$$

For any natural number $k \in \mathbb{N}$ we have:

$$[k\,(p+1)]_q = \frac{1-q^{k(p+1)}}{1-q} = \frac{1-q^{k(p+1)}}{1-q^k}\,\frac{1-q^k}{1-q} = [p+1]_{q^k}\,[k]_q\,.$$

Moreover, this definition is the unification of q-analogue of Reimann and Hadamard integral operator. To show this fact, let $q \to 1^-$ then we have

$$\lim_{q\to 1^-} J_{p,q}^{\alpha}\left(f(a)\right) = \frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^a w^p f(w)\left(a^{p+1}-w^{p+1}\right)^{\alpha-1} dw.$$

This is exactly the operator introduced in [21]. On the other hand, if we let $p \to -1^+$ and use L'Hopital, we get

$$\lim_{p\to -1^+}\lim_{q\to 1^-} J_{p,q}^{\alpha}\left(f(a)\right) = \frac{1}{\Gamma(\alpha)} \int_0^a \lim_{p\to -1^+}\left(\frac{a^{p+1}-w^{p+1}}{p+1}\right)^{\alpha-1} w^p f(w)dw = \frac{1}{\Gamma(\alpha)} \int_0^a \left(Log\left(\frac{a}{w}\right)\right)^{\alpha-1} f(w)\frac{dw}{w}.$$

When $p=0$, we get the well-known $q-$fractional Reimann integral [3].

3. Some properties of the new q-fractional integral operator

In this section, we study some familiar properties of fractional integral operator as semi-group properties of it. This property is essentially useful to solve the related $q-$difference equation. In addition, we will define inverse operator as q-fractional derivative and at the end, properties of these operators will be studied. In this procedure, we prove some useful identities and relations as well. $q-$fractional Reimann integral operators were extensively investigated in several resources [3]. In the aid of Hine's transform for q-hypergeometric functions, useful identities were introduced and a lot of identities were studied [3] [6]. Let us start by the following lemma that is proved in [3]. This relation acts an important role to show the semi-group property of our operator.

Lemma 3.1. *For $\alpha, \beta, \mu \in \mathbb{R}^+$, the following identity is valid*

$$\sum_{t=0}^{\infty} \frac{(1-\mu q^{1-t})^{(\alpha-1)}(1-q^{1+t})^{(\beta-1)}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}}\left(q^t\right)^{\alpha} = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}}.$$

Remark 3.2. *To find an appropriate condition for our aim, we have to modify the Lemma 3.1 by using relation (3) and substituting q by q^{p+1}. Then, we will get;*

$$\sum_{t=0}^{\infty}(1-\mu\left(q^{p+1}\right)^{1-t})_{q^{p+1}}^{(\alpha-1)}(1-\left(q^{p+1}\right)^{1+t})_{q^{p+1}}^{(\beta-1)}\left(q^{1+p}\right)^{t\alpha}=\frac{(1-q^{p+1})_{q^{p+1}}^{(\alpha-1)}(1-q^{p+1})_{q^{p+1}}^{(\beta-1)}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha+\beta-1)}}(1-\mu q^{p+1})_{q^{p+1}}^{(\alpha+\beta-1)}.$$

In this step, let us calculate the following q-integral by using the above remark.

Lemma 3.3. *The following Jackson integral for real positive α and $\lambda>-1$ holds.*

$$\int_a^x t^p(x^{p+1}-(qt)^{p+1})_{q^{p+1}}^{(\alpha-1)}(t^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}d_qt=\frac{1}{[p+1]_q}\left(\frac{\Gamma_{q^{p+1}}(\alpha)\Gamma_{q^{p+1}}(\lambda-1)}{\Gamma_{q^{p+1}}(\alpha+\lambda-1)}\right)\left[(x^{p+1}-a^{p+1})_{q^{p+1}}^{(\alpha+\lambda)}\right].$$

Proof. In the aid of definition of Jackson integral, left hand side of this inequality can be written as

$$\int_a^x t^p(x^{p+1}-(qt)^{p+1})_{q^{p+1}}^{(\alpha-1)}(t^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}d_qt$$
$$=\int_0^x t^p(x^{p+1}-(qt)^{p+1})_{q^{p+1}}^{(\alpha-1)}(t^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}d_qt-\int_0^a t^p(x^{p+1}-(qt)^{p+1})_{q^{p+1}}^{(\alpha-1)}(t^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}d_qt.$$

We know that for some $i\in\mathbb{N}$ the factor $(\left(aq^i\right)^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}=0$ so we can expand the integral as

$$\int_0^a t^p(x^{p+1}-(qt)^{p+1})_{q^{p+1}}^{(\alpha-1)}(t^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}d_qt=$$
$$a(1-q)\sum_{i=0}^{\infty}q^i\left(aq^i\right)^p(x^{p+1}-\left(aq^{i+1}\right)^{p+1})_{q^{p+1}}^{(\alpha-1)}(\left(aq^i\right)^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}=0.$$

Now using the Lemma 3.1 we have

$$\int_0^x t^p(x^{p+1}-(qt)^{p+1})_{q^{p+1}}^{(\alpha-1)}(t^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}d_qt$$
$$=\left(x^{p+1}\right)^{\alpha+\lambda}(1-q)\sum_{i=0}^{\infty}\left(q^i\right)^{(p+1)(\lambda+1)}(1-\left(q^{i+1}\right)^{p+1})_{q^{p+1}}^{(\alpha-1)}\left(1-\left(\frac{a}{xq}\right)^{p+1}\left(q^{1-i}\right)^{p+1}\right)_{q^{p+1}}^{(\lambda)}$$
$$=\left(x^{p+1}\right)^{\alpha+\lambda}(1-q)\frac{(1-q^{p+1})_{q^{p+1}}^{(\alpha-1)}(1-q^{p+1})_{q^{p+1}}^{(\lambda)}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha+\lambda)}}\left(1-\left(\frac{a}{xq}\right)^{p+1}q^{p+1}\right)_{q^{p+1}}^{(\lambda+\alpha)}$$
$$=(1-q)\left(\frac{(1-q^{p+1})_{q^{p+1}}^{(\alpha-1)}(1-q^{p+1})_{q^{p+1}}^{(\lambda)}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha+\lambda)}}\right)\left(x^{p+1}-a^{p+1}\right)_{q^{p+1}}^{(\lambda+\alpha)}.$$

□

Corollary 3.4. *We interpret logarithm function by limit of expression in Remark 2.3. Hadamard integral operator has the following property [23];*

$$J_{a^+}^{\alpha}\left(\left(\log\left(\frac{t}{a}\right)\right)^{\lambda}\right)(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)}\left(\log\left(\frac{t}{a}\right)\right)^{\lambda+\alpha}.$$

On the other hand, we have

$$\lim_{p\to -1^+}\lim_{q\to 1^-}\left(\frac{(t^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}}{[p+1]^{(\lambda)}}\right) = \left(\log\left(\frac{t}{a}\right)\right)^{\lambda}.$$

Now, by considering the Lemma 3.3, we get

$$J_{a^+,p,q}^{\alpha}\left(\frac{(t^{p+1}-a^{p+1})_{q^{p+1}}^{(\lambda)}}{[p+1]^{(\lambda)}}\right) = \frac{(1-q)_{q^{p+1}}^{(\alpha-1)}\Gamma_q(\lambda+1)}{(1-q)^{\alpha-1}\Gamma_q(\lambda+\alpha+1)\,[p+1]^{(\alpha+\lambda)}}\left(x^{p+1}-a^{p+1}\right)_{q^{p+1}}^{(\lambda+\alpha)}.$$

Proposition 3.5. *The given q-fractional integral operator has semi-group property. That means*

$$J_{p,q}^{\alpha}\left(J_{p,q}^{\beta}f(x)\right) = J_{p,q}^{\alpha+\beta}f(x)$$

Proof. Let us start by left hand side of this equality

$$\begin{aligned}
J_{p,q}^{\alpha}\left(J_{p,q}^{\beta}f(x)\right) &= \frac{(1-q)^{\alpha-1}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha-1)}}\int_0^x w^p(x^{p+1}-(wq)^{p+1})_{q^{p+1}}^{(\alpha-1)}\left(J_{p,q}^{\beta}f(w)\right)d_qw\\
&= \frac{(1-q)^{\alpha+\beta-2}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha-1)}(1-q^{p+1})_{q^{p+1}}^{(\beta-1)}}\int_0^x s^p f(s)\\
&\times\left(\int_{qs}^x w^p(x^{p+1}-(wq)^{p+1})_{q^{p+1}}^{(\alpha-1)}(w^{p+1}-(sq)^{p+1})_{q^{p+1}}^{(\beta-1)}d_qw\right)d_qs.
\end{aligned}$$

Now apply the Lemma 3.3 to have

$$\begin{aligned}
J_{p,q}^{\alpha}\left(J_{p,q}^{\beta}f(x)\right) &= \frac{(1-q)^{\alpha+\beta-2}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha-1)}(1-q^{p+1})_{q^{p+1}}^{(\beta-1)}}\int_0^x s^p f(s)\\
&\times\left((1-q)\left(\frac{(1-q^{p+1})_{q^{p+1}}^{(\alpha-1)}(1-q^{p+1})_{q^{p+1}}^{(\beta-1)}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha+\beta-1)}}\right)\left[(x^{p+1}-(sq)^{p+1})_{q^{p+1}}^{(\alpha+\beta-1)}\right]\right)d_qs\\
&=\frac{(1-q)^{\alpha+\beta-1}}{(1-q^{p+1})_{q^{p+1}}^{(\alpha+\beta-1)}}\int_0^x s^p f(s)(x^{p+1}-(sq)^{p+1})_{q^{p+1}}^{(\alpha+\beta-1)}d_qs = J_{p,q}^{\alpha+\beta}f(x).
\end{aligned}$$

□

Definition 3.6. *Let $\alpha \geq 0$ and $n = \lfloor\alpha\rfloor+1$ such that $n \geq \alpha$, and let $p > 0$. The corresponding generalized $q-$fractional derivatives is defined by*

$$(D_{p,q}^{0}f)(x) = f(x)$$

$$(D_{p,q}^{\alpha}f)(x) = (x^{-p}D_q)^n\left(J_{p,q}^{n-\alpha}\right)f(x) = \frac{\left([p+1]_q\right)^{\alpha-n+1}}{\Gamma_{q^{p+1}}(n-\alpha)}(x^{-p}D_q)^n\int_0^x w^p f(w)(x^{p+1}-(wq)^{p+1})_{q^{p+1}}^{(n-\alpha-1)}d_qw$$

if the integral does exist.

Now, the relation between the defined q-derivative and q-integral operator is as follows;

$$(D^{\alpha}_{p,q}J^{\alpha}_{p,q}f)(x) = (x^{-p}D_q)^n \left(J^{n-\alpha}_{p,q}\right)(J^{\alpha}_{p,q}f)(x) = (x^{-p}D_q)^n(J^{n}_{p,q}f)(x).$$

It is easy to see that $(x^{-p}D_q)^n(J^{n}_{p,q}f)(x) = f(x)$. The case $n = 1$ can be derived by using fundamental theorem of q-calculus, i.e

$$(x^{-p}D_q)(J_{p,q}f)(x) = (x^{-p}D_q)\left(\int_0^x w^p f(w)d_qw\right) = x^{-p}\left(x^p f(x)\right).$$

General case for arbitrary natural number n can be proved by induction easily. For instance, let us consider the case for $0 < \alpha < 1$ in next proposition.

Proposition 3.7. *Assume that* $0 < \alpha < 1$, $p > 0$ *and integral does exist, then the following identity holds;*

$$(D^{\alpha}_{p,q}J^{\alpha}_{p,q}f)(x) = f(x).$$

Proof. Direct calculation of the identity in the aid of Lemma 3.3 shows that

$$\begin{aligned}(D^{\alpha}_{p,q}J^{\alpha}_{p,q}f)(x) &= \frac{\left([p+1]_q\right)}{\Gamma_{q^{p+1}}(\alpha)\Gamma_{q^{p+1}}(1-\alpha)}(x^{-p}D_q)\int_0^x\int_0^w w^p s^p f(s)(w^{p+1}-(sq)^{p+1})^{(\alpha-1)}_{q^{p+1}} \\ &\times (x^{p+1}-(wq)^{p+1})^{(-\alpha)}_{q^{p+1}}d_qsd_qw \\ &= \frac{\left([p+1]_q\right)}{\Gamma_{q^{p+1}}(\alpha)\Gamma_{q^{p+1}}(1-\alpha)}(x^{-p}D_q)\int_0^x s^p f(s)\left(\frac{\Gamma_{q^{p+1}}(\alpha)\Gamma_{q^{p+1}}(1-\alpha)}{[p+1]_q}\right)d_qs = f(x)\end{aligned}$$

□

4. Conclusion

In this paper, we defined the class of generalized q-fractional difference integral operator and the inverse operator also is defined. A few properties of these operators were investigated, but still there are a lot of identities and formulae related to this operator which can be studied as a future work. q-calculus is the world of mathematics without limit and the introduced operator can be necessary and important as a part of these objects.

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