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Weyl Type Theorems for Cesàro-Hypercyclic Operators

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Abstract. In this paper we study the relations between Cesàro-hypercyclic operators and the operators for which Weyl type theorem holds.

1. Introduction

Throughout this note let $B(\mathcal{H})$ denote the algebra of bounded linear operators acting on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$. If $T \in B(\mathcal{H})$, write $N(T)$ and $R(T)$ for the null space and the range of T; $\sigma(T)$ for the spectrum of T; $\pi_{00}(T) = \pi_0(T) \cap \text{iso}\sigma(T)$, where $\pi_0(T) = \{\lambda \in \mathbb{C} : 0 < \dim N(T-\lambda I) < \infty\}$ are the eigenvalues of finite multiplicity. Let $p_{00}(T)$ denote the set of Riesz points of T (i.e., the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is Fredholm of finite ascent and descent [1]). An operator $T \in B(\mathcal{H})$ is called upper semi-Fredholm if it has closed range with finite dimensional null space and if $R(T)$ has finite co-dimension, $T \in B(\mathcal{H})$ is called a lower semi-Fredholm operator. We call $T \in B(\mathcal{H})$ Fredholm if it has closed range with finite dimensional null space and its range is of finite co-dimension. The index of a Fredholm operator $T \in B(\mathcal{H})$ if given by

$$\text{ind}(T) = \dim N(T) - \dim R(T)^{\perp}(= \dim N(T) - \dim N(T^*)).$$

An operator $T \in B(\mathcal{H})$ is called Weyl if it is Fredholm of index zero. And $T \in B(\mathcal{H})$ is called Browder if it is Fredholm of finite ascent and descent: equivalently [9] if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, the Browder spectrum $\sigma_b(T)$, the upper semi-Fredholm spectrum and the lower semi-Fredholm spectrum of $T \in B(\mathcal{H})$ are defined by

$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}$,
$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$,
$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}$,
$\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm}\}$,
$\sigma_{SF_-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Fredholm}\}$.

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In keeping with current usage [1, 11], we say that an operator $T \in B(\mathcal{H})$ satisfies Browder's theorem (respectively Weyl's theorem) if $\sigma(T)\backslash\sigma_w(T) = p_{00}(T)$, equivalently $\sigma_w(T) = \sigma_b(T)$ (respectively $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$). The following implications hold [11]: Weyl's theorem for $T \Rightarrow$ Browder's theorem for $T \Leftrightarrow$ Browder's theorem for T^*. Let $\pi^a_{00}(T)$ denote the set of $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma_a(T), \lambda \in \mathrm{iso}\sigma_a(T)$, and $0 < \dim N(T-\lambda I) < \infty$, where $\sigma_a(T)$ denotes the approximate point spectrum of the operator T. Then $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi^a_{00}(T)$. T is said to satisfy a-Weyl's theorem if $\sigma_a(T)\backslash\sigma_{ea}(T) = \pi^a_{00}(T)$, where we write $\sigma_{ea}(T)$ for the essential approximate point spectrum of T (i.e., $\sigma_{ea}(T) = \bigcap\{\sigma_a(T+K) : K \in K(H)\}$: a-Weyl's theorem for $T \Rightarrow$ Weyl's theorem for T, but the converse is generally false [15]. It is well known that $\sigma_{ea}(T)$ coincides with $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T-\lambda I \notin SF^-_+\}$, where $SF^-_+(\mathcal{H}) = \{T \in B(\mathcal{H}) : T$ is upper semi-Fredholm of $\mathrm{ind}(T) \leq 0\}$. We say that T satisfies a-Browder's if $\sigma_{ea}(T) = \sigma_{ab}(T)$, (equivalently, $\sigma_a(T)\backslash\sigma_{ea}(T) = p^a_{00}(T)$, where $p^a_{00}(T) = \{\lambda \in \sigma_a(T) : \lambda \notin \sigma_{ab}(T)\}$ [14] and $\sigma_{ab}(T)$ the Browder essential approximate point spectrum (i.e., $\lambda \notin \sigma_{ab}(T)$ if and only if $T-\lambda I$ is upper semi-Fredholm and $T-\lambda I$ has finite ascent). Evidently, a-Browder's theorem implies Browder's theorem (but the converse is generally false).

We turn to a variant of the essential approximate point spectrum. $T \in B(\mathcal{H})$ is called a generalized upper semi-Fredholm operator if there exists T-invariant subspaces M and N such that $\mathcal{H} = M \oplus N$ and $T_{|M} \in SF^-_+(M), T_{|N}$ is quasinilpotent. Clearly, if T is generalized upper semi-Fredholm, there exists $\epsilon > 0$ such that $T-\lambda I \in SF^-_+(\mathcal{H})$ and $N(T-\lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0 < |\lambda| < \epsilon$. Clearly, if $\lambda \in \mathrm{iso}\sigma(T), T-\lambda I$ is generalized upper semi-Fredholm. The new spectrum set is defined as follows. Let

$$\rho_1(T) = \{\lambda \in \mathbb{C} : \text{there exists } \epsilon > 0 \text{ such that } T-\mu I \text{ is generalized upper semi-Fredholm if } 0 < |\mu-\lambda| < \epsilon\}$$

and let $\sigma_1(T) = \mathbb{C}\backslash\rho_1(T)$. Then

$$\sigma_1(T) \subseteq \sigma_{ea}(T) \subseteq \sigma_{ab}(T) \subseteq \sigma_a(T).$$

T is called approximate isoloid (a-isoloid) (or isoloid) if $\lambda \in \mathrm{iso}\sigma_a(T)(\mathrm{iso}\sigma(T)) \Rightarrow N(T-\lambda I) \neq \{0\}$ and T is called finite approximate isoloid (f-a-isoloid) (or finite isoloid, f-isoloid) operator if the isolated points of approximate point spectrum (of the spectrum) are all eigenvalues of finite multiplicity. Clearly, f-a-isoloid implies a-isoloid and finite isoloid, but the converse is not true.

Recall that an operator $T \in B(\mathcal{H})$ has the single-valued extension property at a point $\lambda_0 \in \mathbb{C}$, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to H$ satisfying $(T-\lambda I)f(\lambda) = 0$ is the function $f \equiv 0$. T has SVEP if it has SVEP at every point of $\mathbb{C}$ (= the complex plane). It is known [5, Lemma 2.18] that a Banach space operator T with SVEP satisfies a-Browder's theorem.

A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is called hypercyclic if there is some vector $x \in \mathcal{H}$ such that $Orb(T,x) = \{T^n x : n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, where such a vector x is said hypercyclic for T.

The first example of hypercyclic operator was given by Rolewicz in [16]. He proved that if B is a backward shift on the Banach space l^p, then λB is hypercyclic if and only if $|\lambda| > 1$.

Let $\{e_n\}_{n\geq 0}$ be the canonical basis of $l^2(\mathbb{N})$. If $\{w_n\}_{n\in\geq 1}$ is a bounded sequence in $\mathbb{C}\backslash\{0\}$, then the unilateral backward weighted shift $T : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ is defined by $Te_n = w_n e_{n-1}, \ \ n \geq 1, Te_0 = 0$, and let $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis of $l^2(\mathbb{Z})$. If $\{w_n\}_{n\in\mathbb{Z}}$ is a bounded sequence in $\mathbb{C}\backslash\{0\}$, then the bilateral weighted shift $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is defined by $Te_n = w_n e_{n-1}$.

The definition and the properties of supercyclicity operators were introduced by Hilden and Wallen [12]. They proved that all unilateral backward weighted shifts on a Hilbert space are supercyclic.

A bounded linear operator $T \in B(\mathcal{H})$ is called supercyclic if there is some vector $x \in \mathcal{H}$ such that the projective orbit $\mathbb{C}.Orb(T,x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X. Such a vector x is said supercyclic for T. Refer to [2][8][6][19] for more informations about hypercyclicity and supercyclicity.

In [17] and [18], Salas characterized the bilateral weighted shifts that are hypercyclic and those that are supercyclic in terms of their weight sequence. In [7], N. Feldman gave a characterization of the invertible bilateral weighted shifts that are hypercyclic or supercyclic.

For the following theorem, see [7, Theorem 4.1].

Theorem 1.1. *Suppose that* $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ *is a bilateral weighted shift with weight sequence* $(w_n)_{n\in\mathbb{Z}}$ *and either* $w_n \geq m > 0$ *for all* $n < 0$ *or* $w_n \leq m$ *for all* $n > 0$. *Then:*

1. *T is hypercyclic if and only if there exists a sequence of integers* $n_k \to \infty$ *such that* $\lim_{k\to\infty} \prod_{j=1}^{n_k} w_j = 0$ *and* $\lim_{k\to\infty} \prod_{j=1}^{n_k} \frac{1}{w_{-j}} = 0$.
2. *T is supercyclic if and only if there exists a sequence of integers* $n_k \to \infty$ *such that* $\lim_{k\to\infty}(\prod_{j=1}^{n_k} w_j)(\prod_{j=1}^{n_k} \frac{1}{w_{-j}}) = 0$.

Let $\mathcal{M}_n(T)$ denote the arithmetic mean of the powers of $T \in B(\mathcal{H})$, that is

$$\mathcal{M}_n(T) = \frac{1 + T + T^2 + ... + T^{n-1}}{n}, n \in \mathbb{N}^*.$$

If the arithmetic means of the orbit of x are dense in $\mathcal{H}$ then the operator T is said to be Cesàro-hypercyclic. In [13], Fernando León-Saavedra proved that an operator is Cesàro-hypercyclic if and only if there exists a vector $x \in \mathcal{H}$ such that the orbit $\{n^{-1}T^n x\}_{n\geq 1}$ is dense in $\mathcal{H}$ and characterized the bilateral weighted shifts that are Cesàro-hypercyclic.
For the following proposition, see [13, Proposition 3.4].

Proposition 1.2. *Let* $T : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ *be a bilateral weighted shift with weight sequence* $(w_n)_{n\in\mathbb{Z}}$. *Then T is Cesàro-hypercyclic if and only if there exists an increasing sequence* n_k *of positive integers such that for any integer q,*

$$\lim_{k\to\infty} \prod_{i=1}^{n_k} \frac{w_{i+q}}{n_k} = \infty \text{ and } \lim_{k\to\infty} \prod_{i=0}^{n_k-1} \frac{w_{q-i}}{n_k} = 0.$$

Hypercyclic and supercyclic (Hilbert space) operators satisfying a Browder-Weyl type theorem have recently been considered by Cao [3]. In [4] B.P. Duggal gave the necessary and sufficient conditions for hypercyclic and supercyclic operators to satisfy a-Weyl's theorem.
In this paper we will give an example of a hypercyclic and supercyclic operator which is not Cesàro-hypercyclic and vice versa. Furthermore, we study the relations between Cesàro-hypercyclic operators and the operators for which Weyl type theorem holds.

2. Main results

Definition 2.1. *An operator* $T \in B(\mathcal{H})$ *is Cesàro-hypercyclic if and only if there exists a vector* $x \in \mathcal{H}$ *such that the orbit* $\{n^{-1}T^n x\}_{n\geq 1}$ *is dense in* $\mathcal{H}$

The following example gives an operator which is Cesàro-hypercyclic but not hypercyclic.

Example 2.2. *[13] Let T the bilateral backward shift with the weight sequence*

$$w_n = \begin{cases} 1 & \text{if } n \leq 0, \\ 2 & \text{if } n \geq 1. \end{cases}$$

Then T is not hypercyclic, but it is Cesàro-hypercyclic.

Now, we will give an example of a hypercyclic and supercyclic operator which is not Cesàro-hypercyclic.

Example 2.3. *Let T the bilateral backward shift with the weight sequence*

$$w_n = \begin{cases} 2 & \text{if } n < 0, \\ \frac{1}{2} & \text{if } n \geq 0. \end{cases}$$

Then T is not Cesàro-hypercyclic, but it is hypercyclic and supercyclic.

Proof. By applying Theorem 1.1 and taking $n_k = n$, we have

$$\lim_{n\to\infty} \prod_{j=1}^{n} w_j = \lim_{n\to\infty} \frac{1}{2^n} = 0;$$

and

$$\lim_{n\to\infty}\prod_{j=1}^{n}\frac{1}{w_{-j}}=\lim_{n\to\infty}\frac{1}{2^n}=0.$$

Furthermore, we have

$$\lim_{n\to\infty}(\prod_{j=1}^{n}w_j)(\prod_{j=1}^{n}\frac{1}{w_{-j}})=\lim_{n\to\infty}(\frac{1}{2^n})(\frac{1}{2^n})=0.$$

Therefore by Theorem 1.1 the operator T is hypercyclic and supercyclic. However, for all increasing sequence $n_k = n$ of positive integers and taking $q = 0$, we have

$$\lim_{n\to\infty}\prod_{i=1}^{n}\frac{w_{i+q}}{n}=\lim_{n\to\infty}\frac{1}{n2^n}=0,$$

from Proposition 1.2, T is not Cesàro-hypercyclic. □

The following example gives us an operator which is Cesàro-hypercyclic but not hypercyclic and supercyclic.

Example 2.4. *Let T the bilateral backward shift with the weight sequence*

$$w_n=\begin{cases}\frac{1}{2} & \text{if } n<0,\\ n+1 & \text{if } n\geq 0.\end{cases}$$

Then T is Cesàro-hypercyclic, but it is not hypercyclic and supercyclic.

Proof. By applying Proposition 1.2 and taking $n_k = n$ and $q = 0$, we have

$$\lim_{n\to\infty}\prod_{i=1}^{n}\frac{w_{i+q}}{n}=\lim_{n\to\infty}\frac{(n+1)!}{n}=\infty,$$

and

$$\lim_{n\to\infty}\prod_{i=0}^{n}\frac{w_{q-i}}{n}=\lim_{n\to\infty}\frac{1}{n2^n}=0.$$

Therefore by Proposition 1.2 the operator T is Cesàro-hypercyclic. On the other hand, we have

$$\lim_{n\to\infty}\prod_{j=1}^{n}w_j=\lim_{n\to\infty}((n+1)!)=\infty;$$

and

$$\lim_{n\to\infty}(\prod_{j=1}^{n}w_j)(\prod_{j=1}^{n}\frac{1}{w_{-j}})=\lim_{n\to\infty}((n+1)!)(2^n)=\infty.$$

Therefore by Theorem 1.1 the operator T is not hypercyclic and supercyclic. □

We denote by $CH(\mathcal{H})$ the class of all cesàro-hypercyclic operators in $B(\mathcal{H})$ and $\overline{CH(\mathcal{H})}$ the norm-closure of the class $CH(\mathcal{H})$. The following lemma [13, Theorem 5.1] give the essential facts for hypercyclic operators and supercyclic operators that we will need to prove the main theorem.

Lemma 2.5. *$\overline{CH(\mathcal{H})}$ is the class of all those operators $T \in B(\mathcal{H})$ satisfying the conditions:*

1. *$\sigma_w(T) \cup \partial D$ is connected, where ∂D the boundary of the open unit disk;*
2. *$\sigma(T)\backslash\sigma_b(T) = \emptyset$;*
3. *$ind(T - \lambda I) \geq 0$ for every $\lambda \in \rho_{SF}(T)$, where $\rho_{SF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is semi-Fredholm $\}$.*

Lemma 2.6. *Let $T \in \overline{CH(\mathcal{H})}$. If $T \in B(\mathcal{H})$ is f-isoloid and the Weyl's theorem holds for T, then $\lambda \notin \sigma_1(T)$ implies that $\lambda \notin \sigma(T)$ or $\lambda \in iso\sigma(T)$.*

Proof. Suppose $T \in \overline{CH(\mathcal{H})}$. Let $\lambda_0 \notin \sigma_1(T)$. Then there exists $\epsilon > 0$ such that $T - \lambda I$ is generalized upper semi-Fredholm. For every λ, there exists ϵ' such that $T - \lambda' I \in SF_+^-(\mathcal{H})$ and $N(T - \lambda' I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda' I)^n]$ if $0 < |\lambda' - \lambda| < \epsilon'$. Since $T \in \overline{CH(\mathcal{H})}$, it induces that $\text{ind}(T - \lambda I) \geq 0$ by Lemma 2.5(3). Then $T - \lambda' I$ is Weyl if $0 < |\lambda' - \lambda| < \epsilon$. Since the Weyl's theorem holds for T, then $T - \lambda' I$ is Browder and hence $T - \lambda' I$ is invertible if $0 < |\lambda' - \lambda| < \epsilon$. It implies $\lambda \in \text{iso}\sigma(T) \cup \rho(T)$, where $\rho(T) = \mathbb{C} \backslash \sigma(T)$. We claim that $\lambda \notin \text{iso}\sigma(T)$. If not, since T is finite isoloid and theWeyl's theorem holds for T, it follows that $\lambda \in \pi_{00} = \sigma(T) \backslash \sigma_w(T)$. Then $T - \lambda I$ is Browder. It is in contradiction to the fact that $T \in \overline{CH(\mathcal{H})}$ by Lemma 2.5(2). Thus $\lambda \notin \sigma(T)$. It induces that $\lambda_0 \in \text{iso}\sigma(T) \cup \rho(T)$. Using the same way, we prove that $T - \lambda_0 I$ is invertible, which means that $\lambda \notin \sigma(T)$. □

Let $H(T)$ be the class of complex-valued functions which are analytic in a neighborhood of $\sigma(T)$ and are not constant on any neighborhood of any component of $\sigma(T)$. Our results are:

Theorem 2.7. *If $T \in B(\mathcal{H})$ is f-isoloid and the Weyl's theorem holds for T (or T is f-a-isoloid and the a-Weyl's theorem holds for T), then $T \in \overline{CH(\mathcal{H})} \Leftrightarrow \sigma(T) = \sigma_1(T)$ and $\sigma(T) \cup \partial D$ is connected*

Proof. For the forward implication, since T satisfies Weyl's theorem and T is isoloid imply $\sigma_b(T) = \sigma_w(T) = \sigma(T)$, $\pi_{00}(T) = \emptyset$, hence $\sigma(T) \cup \partial D$ is connected and it induces that $\sigma(T) = \sigma_1(T)$ by Lemma 2.6.

Conversely, $\sigma_1(T) \subseteq \sigma_{ae}(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$, (1) and (2) in Lemma 2.5 follow and (3) is evident. □

Corollary 2.8. *Suppose $T \in \overline{CH(\mathcal{H})}$ and the a-Weyl's theorem holds for T. Then a-Weyl's theorem holds for $f(T)$ for any $f \in H(T)$.*

Proof. Since $T \in \overline{CH(\mathcal{H})}$, it induces that for each pair $\lambda, \mu \in \mathbb{C} \backslash \sigma_{SF_+}(T)$, $\text{ind}(T - \lambda I)\text{ind}(T - \mu I) \geq 0$. Theorem 2.2 in [10] tells us that the a-Weyl's theorem holds for $f(T)$ for any $f \in H(T)$. □

Theorem 2.9. *If $T \in CH(\mathcal{H})$, then T and T^* satisfy a-Browder's theorem.*

Proof. Since $\sigma_p(T^*) = \emptyset$ for cesàro-hypercyclic T, T^* has SVEP, hence T^* satisfies a-Browder's theorem. Evidently, $\sigma_{ea}(T) \subseteq \sigma_{ab}(T)$. Thus to prove that T satisfies a-Browder's theorem it would suffice to prove that $\sigma_{ab}(T) \subseteq \sigma_{ea}(T)$ [5, Lemma 2.18]. Let $\lambda \notin \sigma_{ea}(T)$. Then $T - \lambda I$ is upper semi-Fredholm and $\text{ind}(T - \lambda I) \leq 0$. Since T^* has SVEP, $\text{dsc}(T - \lambda I) < \infty$ [1, Theorem 3.17] $\Rightarrow \text{ind}(T - \lambda I) \geq 0$. Thus $\text{ind}(T - \lambda I) = 0$ and $T - \lambda I$ is Fredholm. But then, since $\text{dsc}(T - \lambda I) < \infty$, $\text{asc}(T - \lambda I) = \text{dsc}(T - \lambda I) < \infty$ [1, Theorem 3.4], which implies that $\lambda \notin \sigma_{ab}(T)$. □

The following example gives us an operator which satisfies a-Browder's theorem but not Cesàro-hypercyclic.

Example 2.10. *Let T be defined by*

$$T(\tfrac{x_0}{2}, \tfrac{x_1}{3}, \tfrac{x_2}{4}, ...) \text{ for all } (x_n) \in l^2(\mathbb{N}).$$

Then T is quasi-nilpotent, so has SVEP and consequently satisfies a-Browder's theorem. On the other hand, by Proposition 1.2 the operator T is not Cesàro-hypercyclic.

Theorem 2.11. *If $T \in CH(\mathcal{H})$, then T^* satisfies Weyl's theorem. If also $\pi_{00}(T) \subseteq \pi_{00}(T^*)$, then T satisfies a-Weyl's theorem.*

Proof. Evidently, if $T \in CH(\mathcal{H})$, then $p_{00}(T) = p_{00}(T^*) = \pi_{00}(T^*) = \emptyset$. Since $\sigma_p(T^*) = \emptyset$ for cesàro-hypercyclic T, T^* has SVEP, hence T^* satisfies satisfies Browder's theorem, it follows that T^* satisfies Weyl's theorem. Since $p_{00}(T) \subseteq \pi_{00}(T)$ for every operator T, and since operators T, satisfy Browder's theorem, we have that $\sigma(T)\backslash\sigma_w(T) = p_{00}(T) \subseteq \pi_{00}(T)$. Hence, if $\pi_{00}(T) \subseteq \pi_{00}(T^*)$, then $\sigma(T)\backslash\sigma_w(T) = p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}(T^*) = p_{00}(T^*) = p_{00}(T)$, i.e., T satisfies Weyl's theorem. To complete the proof, we prove now that T satisfies a-Weyl's theorem.
Since $\sigma_p(T^*) = \emptyset$ for cesàro-hypercyclic T, T^* has SVEP, hence $\sigma(T) = \sigma_a(T)$ and $\pi_{00}(T) = \pi^a_{00}(T)$. Let $\lambda \notin \sigma_{ea}(T)$. then $T - \lambda I$ is upper semi-Fredholm and $\text{ind}(T - \lambda I) \leq 0$. Arguing as in the proof of Theorem 2.9, it is seen that $T - \lambda I$ is Fredholm and $\text{ind}(T - \lambda I) = 0$, i.e., $\lambda \notin \sigma_w(T)$. Since $\sigma_w(T) \supseteq \sigma_{ea}(T)$ for every operator T, we conclude that $\sigma_w(T) = \sigma_{ea}(T)$. But then, since T satisfies Weyl's theorem, $\sigma_a(T)\backslash\sigma_{ea}(T) = \sigma(T)\backslash\sigma_w(T) = \pi_{00}(T) = \pi^a_{00}(T)$. □

Corollary 2.12. *$T \in CH(\mathcal{H})$ satisfies a-Weyl's theorem if and only if $\pi_{00}(T) = \emptyset$.*

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