Filomat 33:17 (2019), 5561–5588
https://doi.org/10.2298/FIL1917561S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence, Blow-Up and Exponential Decay Estimates for the Nonlinear Kirchhoff-Carrier Wave Equation in an Annular with Nonhomogeneous Dirichlet Conditions

Le Huu Ky Son[a,c], Le Thi Phuong Ngoc[b], Nguyen Thanh Long[a]

[a] *Department of Mathematics and Computer Science, VNUHCM - University of Science, 227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam.*
[b] *University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam*
[c] *Department of Fundamental sciences, Ho Chi Minh City University of Food Industry, 140 Le Trong Tan Str., Tan Phu Dist., Ho Chi Minh City, Vietnam.*

Abstract. This paper is devoted to the study of a nonlinear Kirchhoff-Carrier wave equation in an annular associated with nonhomogeneous Dirichlet conditions. At first, by applying the Faedo-Galerkin, we prove existence and uniqueness of the solution of the problem considered. Next, by constructing Lyapunov functional, we prove a blow-up result for solutions with a negative initial energy and establish a sufficient condition to obtain the exponential decay of weak solutions.

1. Introduction

In this paper, we are concerned with the following nonlinear Kirchhoff-Carrier wave equation in the annular

$$u_{tt} - \mu\left(t, \|u(t)\|_0^2, \|u_x(t)\|_0^2\right)\left(u_{xx} + \frac{1}{x}u_x\right) = f\left(x, t, u, u_x, u_t, \|u(t)\|_0^2, \|u_x(t)\|_0^2\right),\ \rho < x < 1,\ 0 < t < T, \tag{1.1}$$

associated with nonhomogeneous Dirichlet conditions

$$u(\rho, t) = g_0(t),\ u(1, t) = g_1(t), \tag{1.2}$$

and initial conditions

$$u(x, 0) = \tilde{u}_0(x),\ u_t(x, 0) = \tilde{u}_1(x), \tag{1.3}$$

where $\mu, f, g_0, g_1, \tilde{u}_0(x), \tilde{u}_1(x)$ are given functions; $\rho \in (0, 1)$ is given constant. In Eq. (1.1), the nonlinear terms $\mu\left(t, \|u(t)\|_0^2, \|u_x(t)\|_0^2\right)$, $f\left(x, t, u, u_x, u_t, \|u(t)\|_0^2, \|u_x(t)\|_0^2\right)$, depend on the integrals $\|u(t)\|_0^2 = \int_\rho^1 xu^2(x, t)dx$ and $\|u_x(t)\|_0^2 = \int_\rho^1 xu_x^2(x, t)dx$.

2010 *Mathematics Subject Classification*. 35L20, 35L70, 35Q74, 37B25.
Keywords. Nonlinear Kirchhoff-Carrier wave equation; Blow-up; Exponential decay.
Received: 10 December 2018; Revised: 14 February 2019; Accepted: 09 March 2019
Communicated by Dragan S. Djordjević
Email addresses: kyson85@gmail.com (Le Huu Ky Son), ngoc1966@gmail.com (Le Thi Phuong Ngoc), longnt2@gmail.com (Nguyen Thanh Long)

Eq. (1.1) herein is the bidimensional nonlinear wave equation describing nonlinear vibrations of the annular membrane $\Omega = \{(x, y) : \rho^2 < x^2 + y^2 < 1\}$. In the vibration processing, the area of the annular membrane and the tension at various points change in time. The conditions $u(\rho, t) = g_0(t)$ and $u(1, t) = g_1(t)$ on the boundaries $\Gamma_1 = \{(x, y) : x^2 + y^2 = \rho^2\}$ and $\Gamma_2 = \{(x, y) : x^2 + y^2 = 1\}$ describe vibrations of a membrane which depends on g_0 and g_1 at the both boundaries of the annular membrane.

It is known that Kirchhoff [6] first investigated the following nonlinear vibration of an elastic string

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right) u_{xx}, \tag{1.4}$$

where $u = u(x, t)$ is the lateral displacement at the space coordinate x and the time t, ρ is the mass density, h is the cross-section area, L is the length, E is the Young modulus, P_0 is the initial axial tension.

In [3], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2(y,t)dy\right) u_{xx} = 0, \tag{1.5}$$

where $u(x, t)$ is the $x-$derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross - section of a string, L is the length of a string and ρ is the density of a material. Clearly, if properties of a material depends on x and t, there is a hyperbolic equation of the type (Larkin [7])

$$u_{tt} - B\left(x, t, \int_0^1 u^2(y,t)dy\right) u_{xx} = 0. \tag{1.6}$$

The Kirchhoff - Carrier equations of the form Eq. (1.1) received much attention. We refer the reader to, e.g., Cavalcanti et al. [1], [2], Ebihara, Medeiros and Miranda [4], Miranda et al. [15], Lasiecka and Ong [8], Hosoya, Yamada [5], Larkin [7], Long et al. [10]-[12], Medeiros [14], Menzala [16], Messaoudi [17], Ngoc et al. [18]-[22], Park et al. [23], [24], Rabello et al. [25], Santos et al. [26], Truong et al. [28], for many interesting results and further references. In these works, the results concerning local existence, global existence, asymptotic expansion, asymptotic behavior, decay and blow-up of solutions have been examined.

Recently, Gongwei Liu [13] studied the damped wave equation of Kirchhoff type with initial and Dirichlet boundary condition

$$\begin{cases} u_{tt} - M\left(\|\nabla u(t)\|^2\right)\Delta u + u_t = g(u) \text{ in } \Omega \times (0, \infty), \\ u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ x \in \Omega, \\ u(x, t) = 0 \text{ on } \partial\Omega \times (0, \infty), \end{cases} \tag{1.7}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, g is a source term with exponential growth at the infinity to be specified later. Here $M(s)$ is a positive C^1 function on $[0, \infty)$ and $M(s) \geq 1$, $|M'(s)| \leq s^\alpha$ for all $s > 1$, $\alpha \geq 0$ and for suitably chosen initial data, (1.7) possesses a global weak solution which decays exponentially. On the other hand, if conditions of M, g and initial data are suitable, the solution u of (1.7) blows-up at a finite T^*.

In [15] Miranda and Jutuca dealt with the existence and uniqueness of solutions and exponential decay of solutions of an initial-homogeneous boundary value problem for the Kirchhoff equation.

In [1], [2], Cavalcanti also studied the existence and uniform decay of solutions of the Kirchhoff-Carrier equation.

In [28], the global existence and regularity of weak solutions for the linear wave equation

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f(x, t),\ 0 < x < 1,\ t > 0 \tag{1.8}$$

with the initial conditions as in (1.3) and two-point boundary conditions. The exponential decay of solutions was given there by using Lyapunov method.

Motivated by the above work, we intend to study the existence and uniqueness, the blow-up and exponential decay of solutions for problem (1.1-1.3) under suitable conditions on f, μ and initial data. Our paper is organized as follows.

First, we present preliminaries in Section 2, with the notations, definitions, list of appropriate spaces and required lemmas. We prove the existence and uniqueness a weak solution in Section 3 by using Faedo-Galerkin method, the linearization method and the weak compact method. Next, in Section 4, Prob. (1.1)-(1.3) is considered in case $\mu = \mu\left(\|u_x(t)\|_0^2\right)$, $f = -\lambda u_t + f(u)$, $g_0(t) = g_1(t) \equiv 0$, with $\lambda > 0$ is constant. If some auxiliary conditions are satisfied, we imply that the weak solution u of Prob. (1.1)-(1.3) blows-up at finite time.

Finally, in Section 5 with the case $\mu = \mu\left(\|u_x(t)\|_0^2\right)$, $f = -\lambda u_t + f(u) + F(x,t)$, $g_0(t) = g_1(t) \equiv 0$, if $\int_0^{\|\tilde{u}_{0x}\|_0^2} \mu(z)dz - p\int_\rho^1 xdx \int_0^{\tilde{u}_0(x)} f(z)dz > 0$ and the initial energy and $\|F(t)\|_0$ are small enough, we verify that the energy of the solution decays exponentially as $t \to +\infty$. In the proofs, to obtain the blow-up and the exponential decay, we use the multiplier technique combined with a suitable Lyapunov functionals. Our results can be regarded as an extension and improvement of the corresponding results of [7], [10]-[12], [18]-[22], [28].

2. Preliminaries

First, we put $\Omega = (\rho, 1)$, $Q_T = \Omega \times (0,T)$, $T > 0$, and denote the usual function spaces used throughout the paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. We denote the usual norm in L^2 by $\|\cdot\|$ and we denote $\|\cdot\|_X$ for the norm in the Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p\, dt\right) & ,\text{if } \quad 1 \le p < \infty, \\ ess\sup_{0<t<T} \|u(t)\|_X & ,\text{if } \quad p = \infty. \end{cases}$$

With $f \in C^k\left([\rho,1] \times [0,T^*] \times \mathbb{R}^3 \times \mathbb{R}_+^2\right)$, $f = f(x,t,y_1,y_2,y_3,y_4,y_5)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2} f = \frac{\partial f}{\partial y_i}$ with $i = 1,\ldots,5$, and $D^\alpha f = D_1^{\alpha_1} \ldots D_7^{\alpha_7} f$, $\alpha = (\alpha_1,\ldots,\alpha_7) \in \mathbf{Z}_+^7$, $|\alpha| = \alpha_1 + \ldots + \alpha_7 = k$, $D^{(0,\ldots,0)} f = f$.

With $\mu \in C^k\left([0,T^*] \times \mathbb{R}_+^2\right)$, $\mu = \mu(t,y,z)$, we put $D_1\mu = \frac{\partial \mu}{\partial t}$, $D_2\mu = \frac{\partial \mu}{\partial y}$, $D_3\mu = \frac{\partial \mu}{\partial z}$ and $D^\beta \mu = D_1^{\beta_1} \ldots D_3^{\beta_3} \mu$, $\beta = (\beta_1,\ldots,\beta_3) \in \mathbf{Z}_+^3$, $|\beta| = \beta_1 + \ldots + \beta_3 = k$, $D^{(0,\ldots,0)}\mu = \mu$.

On H^1, H^2, we use the following norms

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{\frac{1}{2}}, \tag{2.1}$$

and

$$\|v\|_{H^2} = \left(\|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2\right)^{\frac{1}{2}}, \tag{2.2}$$

respectively.

We remark that L^2, H^1, H^2 are the Hilbert spaces with respect to the corresponding scalar products

$$\langle u, v\rangle = \int_\rho^1 xu(x)v(x)dx,\ \langle u,v\rangle + \langle u_x, v_x\rangle,\ \langle u,v\rangle + \langle u_x,v_x\rangle + \langle u_{xx}, v_{xx}\rangle. \tag{2.3}$$

The norms in L^2, H^1 and H^2 induced by the corresponding scalar products (2.3) are denoted by $\|\cdot\|_0$, $\|\cdot\|_1$ and $\|\cdot\|_2$.

We then have the following lemmas.

Lemma 2.1. *The following inequalities are fulfilled* *(i)* $\sqrt{\rho}\|v\| \leq \|v\|_0 \leq \|v\|$ *for all* $v \in L^2$,

(ii) $\sqrt{\rho}\|v\|_{H^1} \leq \|v\|_1 \leq \|v\|_{H^1}$ *for all* $v \in H^1$.

Lemma 2.2. *The embedding* $H_0^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$ *is compact and for all* $v \in H_0^1$, *we have*

(i) $\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{1-\rho}\|v_x\|$,

(ii) $\|v\| \leq \frac{1-\rho}{\sqrt{2}}\|v_x\|$,

(iii) $\|v\|_0 \leq \frac{1-\rho}{\sqrt{2}}\|v_x\|_0$.

Proofs of Lemma 2.1 and Lemma 2.2 are straightforward, so we omit the details.

Remark 2.3. *On* L^2, *two norms* $v \mapsto \|v\|$ *and* $v \mapsto \|v\|_0$ *are equivalent. So are two norms* $v \mapsto \|v\|_{H^1}$ *and* $v \mapsto \|v\|_1$ *on* H^1, *and four norms* $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_1$, $v \mapsto \|v_x\|$, *and* $v \mapsto \|v_x\|_0$ *on* H_0^1.

Lemma 2.4. *We have*

$$\|v\|_{C^0(\overline{\Omega})} \leq \alpha_0\|v\|_{H^1} \text{ for all } v \in H^1 \tag{2.4}$$

where $\alpha_0 = \frac{1}{\sqrt{2(1-\rho)}}\sqrt{1+\sqrt{1+16(1-\rho)^2}}$.

Proof of Lemma 2.4. It is well known that the embedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact. Since $C^1\left(\overline{\Omega}\right)$ is dense in H^1, we only show that (2.4) holds for all $v \in C^1\left(\overline{\Omega}\right)$. For all $v \in C^1\left(\overline{\Omega}\right)$, and $x, y \in \overline{\Omega}$, we have

$$v^2(x) = v^2(y) + 2\int_y^x v(z)v_x(z)dz.$$

Integrating over y from ρ to 1 to get

$$\begin{aligned}
(1-\rho)v^2(x) &= \|v\|^2 + 2\int_\rho^1 dy \int_y^x v(z)v_x(z)dz \\
&= \|v\|^2 + 2\int_\rho^1 dy \int_\rho^x v(z)v_x(z)dz - 2\int_\rho^1 dy \int_\rho^y v(z)v_x(z)dz \\
&\leq \|v\|^2 + 2(1-\rho)\int_\rho^x |v(z)v_x(z)|dz + 2\int_\rho^1 (1-z)|v(z)v_x(z)|dz \\
&\leq \|v\|^2 + 4(1-\rho)\int_\rho^1 (1-z)|v(z)v_x(z)|dz.
\end{aligned} \tag{2.5}$$

Note that $\alpha_0^2 = \frac{1+\sqrt{1+16(1-\rho)^2}}{2(1-\rho)}$ satisfies $1 + 4(1-\rho)\frac{1}{\alpha_0^2} = (1-\rho)\alpha_0^2$, applying the inequality $2ab \leq \frac{2}{\alpha_0^2}a^2 + \frac{\alpha_0^2}{2}b^2$, for all $a, b \in \mathbb{R}$, we deduce from (2.5), that

$$\begin{aligned}
(1-\rho)v^2(x) &\leq \|v\|^2 + 2(1-\rho)\left(\frac{2}{\alpha_0^2}\|v\|^2 + \frac{\alpha_0^2}{2}\right) \\
&= \left(1 + 4(1-\rho)\frac{1}{\alpha_0^2}\right)\|v\|^2 + (1-\rho)\alpha_0^2\|v_x\|^2 \\
&= (1-\rho)\alpha_0^2\|v\|_{H^1}^2.
\end{aligned}$$

Hence (2.4) holds. Lemma 2.4 is complete. □

Now, we define the following bilinear form

$$a(u,v) = \int_\rho^1 xu_x(x)v_x(x)dx, \text{ for all } u, v \in H_0^1. \tag{2.6}$$

Lemma 2.5. *The symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.6) is continuous on $H_0^1 \times H_0^1$ and coercive on H_0^1, i.e.,*

(i) $|a(u,v)| \le \|u_x\|_0 \|v_x\|_0$,

(ii) $a(v,v) \ge \|v_x\|_0^2$,

for all $u, v \in H_0^1$.

Lemma 2.6. *There exists the Hilbert orthonormal base $\{w_j\}$ of the space L^2 consisting of eigenfunctions w_j corresponding to eigenvalues λ_j such that*

(i) $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \lambda_{j+1} \le \cdots$, $\lim_{j\to+\infty} \lambda_j = +\infty$,

(ii) $a(w_j, v) = \lambda_j \langle w_j, v\rangle$ *for all* $v \in H_0^1$, $j = 1, 2, \ldots$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is also the Hilbert orthonormal base of H_0^1 with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have w_j satisfying the following boundary value problem

$$\begin{cases} Aw_j \equiv -\left(w_{jxx} + \frac{1}{x} w_{jx}\right) = -\frac{1}{x}\frac{\partial}{\partial x}\left(x w_{jx}\right) = \lambda_j w_j, \text{ in } \Omega, \\ w_j(\rho) = w_j(1) = 0, w_j \in C^\infty([\rho, 1]). \end{cases} \tag{2.7}$$

The proof of Lemma 2.6 can be found in [[27], p.87, Theorem 7.7], with $H = L^2$, and $a(\cdot,\cdot)$ as defined by (2.6). □

We also note that the operator $A : H_0^1 \longrightarrow (H_0^1)'$ in (2.7) is uniquely defined by the Lax-Milgram's lemma, i.e.

$$a(u,v) = \langle Au, v\rangle \text{ for all } u,\ v \in H_0^1. \tag{2.8}$$

Lemma 2.7. *On $H_0^1 \cap H^2$, two norms $v \mapsto \|v\|_{H_0^1\cap H^2} = \sqrt{\|v_x\|_0^2 + \|v_{xx}\|_0^2}$ and $v \mapsto \|v\|_{2*} = \sqrt{\|v_x\|_0^2 + \|Av\|_0^2}$ are equivalent and*

$$C_{1\rho}\|v\|_{H_0^1\cap H^2} \le \|v\|_{2*} \le C_{2\rho}\|v\|_{H_0^1\cap H^2} \text{ for all } v \in H_0^1 \cap H^2, \tag{2.9}$$

where

$$C_{1\rho} = \sqrt{(1-\rho)\rho^3},\ C_{2\rho} = \sqrt{1 + \frac{2}{\rho^2}}. \tag{2.10}$$

Proof of Lemma 2.7. For all $v \in H_0^1 \cap H^2$, we have

$$\begin{aligned} \|Av\|_0 &\le \|v_{xx}\|_0 + \left\|\tfrac{1}{x} v_x\right\|_0 \\ &\le \|v_{xx}\|_0 + \tfrac{1}{\rho}\|v_x\|_0. \end{aligned}$$

Hence

$$\begin{aligned} \|v\|_{2*}^2 &= \|v_x\|_0^2 + \|Av\|_0^2 \\ &\le \|v_x\|_0^2 + 2\|v_{xx}\|_0^2 + \tfrac{2}{\rho^2}\|v_x\|_0^2 \\ &\le C_{2\rho}^2 \|v\|_{H_0^1\cap H^2}^2. \end{aligned}$$

On the other hand

$$\begin{aligned} \|Av\|_0^2 &= \|v_{xx}\|_0^2 + \left\|\frac{1}{x} v_x\right\|_0^2 + 2\int_\rho^1 v_x(x) v_{xx}(x) dx \\ &\ge \|v_{xx}\|_0^2 + \|v_x\|_0^2 + 2\int_\rho^1 v_x(x) v_{xx}(x) dx. \end{aligned}$$

Therefore

$$\begin{aligned}\|v_{xx}\|_0^2+\|v_x\|_0^2 &\le \|Av\|_0^2-2\int_\rho^1 v_x(x)v_{xx}(x)dx\\ &\le \|Av\|_0^2+\frac{2}{\rho}\|v_x\|_0\|v_{xx}\|_0\\ &\le \|Av\|_0^2+\frac{1}{\rho}\left(\frac{1}{\rho^2}\|v_x\|_0^2+\rho^2\|v_{xx}\|_0^2\right),\end{aligned}$$

this implies that

$$\begin{aligned}\|v_{xx}\|_0^2+\|v_x\|_0^2 &\le \frac{1}{1-\rho}\left(\|Av\|_0^2+\frac{1-\rho^4}{\rho^3}\|v_x\|_0^2\right)\\ &\le \frac{1}{C_{1\rho}^2}\|v\|_{2*}^2.\end{aligned}$$

Lemma 2.7 is complete. □

3. The existence and uniqueness theorem

First, we make the following assumptions:

(H_1) : $g_0, g_1 \in C^3([0,T^*])$;

(H_2) : $\tilde{u}_0 \in H^2$, $\tilde{u}_1 \in H^1$, $\tilde{u}_0(\rho)-g_0(0)=\tilde{u}_0(1)-g_1(0)=0$;

(H_3) : $\mu \in C^1\left([0,T^*]\times\mathbb{R}_+^2\right)$, $\mu(t,y,z)\ge\mu_*>0$, $\forall(t,y,z)\in[0,T^*]\times\mathbb{R}_+^2$;

(H_4) : $f\in C^1\left([\rho,1]\times[0,T^*]\times\mathbb{R}^3\times\mathbb{R}_+^2\right)$, satisfying the following conditions

$$f(\rho,t,y_1,y_2,y_3,y,z)=f(1,t,y_1,y_2,y_3,y,z)=0,$$

$\forall(t,y_1,y_2,y_3,y,z)\in[0,T^*]\times\mathbb{R}^3\times\mathbb{R}_+^2$.

Put

$$\varphi(x,t)=\frac{1}{1-\rho}\left\{[g_1(t)-g_0(t)]x+g_0(t)-\rho g_1(t)\right\}.$$

By the transformation $v(x,t)=u(x,t)-\varphi(x,t)$, Prob. (1.1)-(1.3) reduces to the following problem with the homogeneous boundary conditions

$$\begin{cases} v_{tt}+\mu[v](t)Av=\tilde{f}[v](x,t),\ \rho<x<1,\ 0<t<T,\\ v(\rho,t)=v(1,t)=0,\\ v(x,0)=\tilde{v}_0(x),\ v_t(x,0)=\tilde{v}_1(x),\end{cases} \tag{3.1}$$

in which

$$\begin{cases} \mu[v](t)=\mu\left(t,\left\|v+\varphi\right\|_0^2,\left\|v_x+\varphi_x\right\|_0^2\right),\\ \tilde{f}[v](x,t)=f\left(x,t,v+\varphi,v_x+\varphi_x,v_t+\varphi_t,\left\|v+\varphi\right\|_0^2,\left\|v_x+\varphi_x\right\|_0^2\right)\\ \qquad\qquad -\varphi_{tt}+\frac{1}{x}\tilde{\varphi}(t)\mu[v](t),\\ \tilde{\varphi}(t)=\varphi_x(x,t)=\frac{g_1(t)-g_0(t)}{1-\rho},\\ \tilde{v}_0(x)=\tilde{u}_0(x)-\varphi(x,0),\ \tilde{v}_1(x)=\tilde{u}_1(x)-\varphi_t(x,0),\\ (\tilde{v}_0,\tilde{v}_1)\in(H_0^1\cap H^2)\times H_0^1.\end{cases} \tag{3.2}$$

Consider $T^* > 0$ fixed, let $M > 0$, we put

$$\begin{aligned} K_M(f) &= \|f\|_{C^1(A_M)} = \|f\|_{C^0(A_M)} + \sum_{i=1}^{7} \|D_i f\|_{C^0(A_M)}, \\ \tilde{K}_M(\mu) &= \|\mu\|_{C^1(\tilde{A}_M)} = \sum_{i=1}^{3} \|D_i \mu\|_{C^0(\tilde{A}_M)}, \\ \|f\|_{C^0(A_M)} &= \sup\left\{ |f(x,t,y_1,y_2,y_3,y,z)| : (x,t,y_1,y_2,y_3,y,z) \in A_M \right\}, \\ \|\mu\|_{C^0(\tilde{A}_M)} &= \sup_{(t,y,z)\in \tilde{A}_M} |\mu(t,y,z)|, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} A_M = \Bigg\{ &(x,t,y_1,y_2,y_3,y,z) \in [\rho,1]\times[0,T^*]\times\mathbb{R}^3\times\mathbb{R}_+^2 : |y_1| \le \sqrt{\frac{1-\rho}{\rho\mu_*}}M + M^*, \\ &|y_2| \le \frac{\alpha_0}{C_{1\rho}\sqrt{\rho\mu_*}}M + M^*, \\ &|y_3| \le \sqrt{\frac{1-\rho}{\rho}}M + M^*, \\ &0 \le y \le \left(\frac{(1-\rho)M}{\sqrt{2\rho\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^* \right)^2, \\ &0 \le z \le \left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^* \right)^2 \Bigg\}, \\ \tilde{A}_M = \Bigg\{ &(t,y,z) \in [0,T^*]\times\mathbb{R}_+^2 : 0 \le y \le \left(\frac{(1-\rho)M}{\sqrt{2\rho\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^* \right)^2, \\ &\qquad 0 \le z \le \left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^* \right)^2 \Bigg\}, \\ M^* = \|&\varphi\|_{C^1(\overline{\Omega}\times[0,T^*])}. \end{aligned} \tag{3.4}$$

Now, for each $M > 0$ and $T \in (0, T^*]$, we consider the sets

$$\begin{aligned} W(M,T) = \{ &v \in L^\infty\left(0,T;H_0^1 \cap H^2\right) : v' \in L^\infty\left(0,T;H_0^1\right),\ v'' \in L^2(Q_T), \\ &\|v\|_{L^\infty(0,T;H_0^1\cap H^2)} \le \frac{M}{\sqrt{\mu_*}C_{1\rho}}, \|v'\|_{L^\infty(0,T;H_0^1)} \le M, \|v''\|_{L^2(Q_T)} \le M \}, \\ W_1(M,T) = \{ &u \in W(M,T) : u_{tt} \in L^\infty\left(0,T;L^2\right) \}, \end{aligned} \tag{3.5}$$

and we establish the linear recurrent sequence $\{v_m\}$ as follows.

We shall choose as first initial term $v_0 \equiv \tilde{v}_0$, suppose that

$$v_{m-1} \in W_1(M,T), \tag{3.6}$$

and associate with the problem (3.1) the following variational problem:

Find $v_m \in W_1(M,T)$ $(m \ge 1)$ so that

$$\begin{cases} \langle v_m''(t), w \rangle + \mu_m(t)\, a(v_m(t), w) = \langle F_m(t), w \rangle,\ \forall w \in H_0^1, \\ v_m(0) = \tilde{v}_0,\ v_m'(0) = \tilde{v}_1, \end{cases} \tag{3.7}$$

where

$$\begin{cases} \mu_m(t) = \mu[v_{m-1}](t) = \mu\left(t, \|v_{m-1}(t) + \varphi(t)\|_0^2, \|\nabla v_{m-1} + \varphi_x\|_0^2\right), \\ F_m(t) = \tilde{f}[v_{m-1}](x,t). \end{cases} \tag{3.8}$$

The existence of a sequence $\{v_m\}$ defined by (3.6) - (3.8) is established by our following theorem.

Theorem 3.1. *Let the assumptions* $(H_1) - (H_4)$ *hold. Then there exist positive constants* M, T *such that the problem* (3.7), (3.8) *has a solution* $v_m \in W_1(M, T)$.

Proof of Theorem 3.1. The proof consists of three steps.

Step 1. The Faedo - Galerkin approximation (introduced by Lions [9]). Consider the basis $\{w_j\}$ for H_0^1 as in Lemma 2.6. Put

$$v_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.9}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\begin{cases} \left\langle \ddot{v}_m^{(k)}(t), w_j \right\rangle + \mu_m(t)\, a(v_m^{(k)}(t), w_j) = \left\langle F_m(t), w_j \right\rangle, \\ v_m^{(k)}(0) = \tilde{v}_{0k},\ \dot{v}_m^{(k)}(0) = \tilde{v}_{1k},\ j = 1, ..., k, \end{cases} \tag{3.10}$$

with

$$\begin{cases} v_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{v}_0 \text{ strongly in } H_0^1 \cap H^2, \\ v_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{v}_1 \text{ strongly in } H_0^1. \end{cases} \tag{3.11}$$

The system (3.10) can be rewritten in form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda_j \mu_m(t) c_{mj}^{(k)}(t) = F_{mj}(t), \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)},\ \ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)},\ 1 \le j \le k, \end{cases} \tag{3.12}$$

in which

$$F_{mj}(t) = \langle F_m(t), w_j \rangle,\ 1 \le j \le k. \tag{3.13}$$

Note that by (3.6), it is not difficult to prove that the system (3.12) has a unique solution $c_{mj}^{(k)}(t)$, $1 \le j \le k$ on interval $[0, T]$, so let us omit the details.

Step 2. A priori estimates.

We put

$$\begin{aligned} S_m^{(k)}(t) &= \left\| \dot{v}_m^{(k)}(t) \right\|_0^2 + \left\| \dot{v}_{mx}^{(k)}(t) \right\|_0^2 \\ &\quad + \mu_m(t) \left(\left\| v_{mx}^{(k)}(t) \right\|_0^2 + \left\| A v_m^{(k)}(t) \right\|_0^2 \right) + \int_0^t \left\| \ddot{v}_m^{(k)}(s) \right\|_0^2 ds, \end{aligned} \tag{3.14}$$

and

$$\sigma_{1m}(t) = g_1''(t) - \tilde{\varphi}(t)\, \mu_m(t),\ \ \sigma_{\rho m}(t) = -\rho g_0''(t) + \tilde{\varphi}(t)\, \mu_m(t). \tag{3.15}$$

Then, it follows from (3.10), (3.14), (3.15), that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + \int_0^t \mu_m'(s) \left[\left\| v_{mx}^{(k)}(s) \right\|_0^2 + \left\| A v_m^{(k)}(s) \right\|_0^2 \right] ds \\ &\quad + 2 \int_0^t \left\langle F_m(s), \dot{v}_m^{(k)}(s) \right\rangle ds + 2 \int_0^t \left\langle F_{mx}(s), \dot{v}_{mx}^{(k)}(s) \right\rangle ds \\ &\quad + 2 \int_0^t \sigma_{1m}(s) \dot{v}_{mx}^{(k)}(1, s) ds + 2 \int_0^t \sigma_{\rho m}(s) \dot{v}_{mx}^{(k)}(\rho, s) ds \\ &\quad + \int_0^t \left\| \ddot{v}_m^{(k)}(s) \right\|_0^2 ds \\ &\equiv S_m^{(k)}(0) + \sum\nolimits_{j=1}^{6} I_j. \end{aligned} \tag{3.16}$$

In order to estimate the terms I_j we need the following lemma

Lemma 3.2. *We have the following estimates*

(i) $\left|\mu_m'(t)\right| \le \bar{\mu}_M$, a.e. $t \in (0,T)$,

(ii) $|F_m(x,t)| \le \bar{F}_1(M)$, a.e. $(x,t) \in Q_T = (\rho,1)\times(0,T)$,

(iii) $\|F_{mx}(t)\|_0 \le \bar{F}_2(M)$, a.e. $t \in (0,T)$,

(iv) $\left|\sigma_{1m}'(t)\right| \le \bar{\sigma}_{1M}$, a.e. $t \in (0,T)$,

(v) $\left|\sigma_{\rho m}'(t)\right| \le \bar{\sigma}_{\rho M}$, a.e. $t \in (0,T)$,

where

$$\bar{\mu}_M = \tilde{K}_M(\mu)\left[1 + 4\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\left(M + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\right], \tag{3.17}$$

$$\bar{F}_1(M) = K_M(f) + \left(\|g_0\|_{C^2([0,T^*])} + \|g_1\|_{C^2([0,T^*])}\right)\left(1 + \frac{\tilde{K}_M(\mu)}{\rho(1-\rho)}\right),$$

$$\bar{F}_2(M) = 2K_M(f)\left[\frac{1+M}{2} + \frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right] + \|\tilde{\varphi}\|_{C^2([0,T^*])}\left(1 + \frac{1}{\rho^2}\tilde{K}_M(\mu)\right)\sqrt{\frac{1-\rho^2}{2}},$$

$$\bar{\sigma}_{1M} = \|g_1'''\|_{C^0([0,T^*])} + \|\tilde{\varphi}'\|_{C^0([0,T^*])}\tilde{K}_M(\mu) + \|\tilde{\varphi}\|_{C^0([0,T^*])}\bar{\mu}_M,$$

$$\bar{\sigma}_{\rho M} = \rho\|g_0'''\|_{C^0([0,T^*])} + \|\tilde{\varphi}'\|_{C^0([0,T^*])}\tilde{K}_M(\mu) + \|\tilde{\varphi}\|_{C^0([0,T^*])}\bar{\mu}_M.$$

Proof of Lemma 3.2.

Proof (i). Note that

$$\mu_m'(t) = D_1\mu[v_{m-1}](t) + 2D_2\mu[v_{m-1}](t)\langle v_{m-1}(t) + \varphi(t), v_{m-1}'(t) + \varphi'(t)\rangle + 2D_3\mu[v_{m-1}](t)\langle \nabla v_{m-1}(t) + \varphi_x(t), \nabla v_{m-1}'(t) + \varphi_x'(t)\rangle,$$

with $D_i\mu[v_{m-1}](t) = D_i\mu(t, \|v_{m-1}(t) + \varphi(t)\|_0^2, \|\nabla v_{m-1}(t) + \varphi_x(t)\|_0^2)$, $i = 1,2,3$, it implies from (3.6) that $\left|\mu_m'(t)\right| \le \bar{\mu}_M$.

Proof (ii). By $(3.2)_2$ and $(3.8)_2$, we have

$$\begin{aligned}|F_m(x,t)| &\le K_M(f) + |\varphi_{tt}| + \frac{1}{x}\left|\tilde{\varphi}(t)\right|\tilde{K}_M(\mu)\\ &\le K_M(f) + \left|g_0''(t)\right| + \left|g_1''(t)\right| + \frac{1}{\rho}\frac{\left|g_0(t)\right| + \left|g_1(t)\right|}{1-\rho}\tilde{K}_M(\mu)\\ &\le \bar{F}_1(M).\end{aligned}$$

Proof (iii). Note that

$$\begin{aligned}F_{mx}(t) &= D_1 f[v_{m-1}](t) + D_3 f[v_{m-1}](t)(\nabla v_{m-1}(t) + \varphi_x)\\ &\quad + D_4 f[v_{m-1}](t)(\Delta v_{m-1}(t) + \varphi_{xx}) + D_5 f[v_{m-1}](t)\left(\nabla v_{m-1}'(t) + \varphi_x'\right)\\ &\quad - \tilde{\varphi}''(t) - \frac{1}{x^2}\tilde{\varphi}(t)\mu_m(t),\end{aligned}$$

where $D_i f[v_{m-1}] = D_i f\left(x, t, v_{m-1} + \varphi, \nabla v_{m-1} + \varphi_x, v_{m-1}' + \varphi', \|v_{m-1} + \varphi\|_0^2, \|\nabla v_{m-1} + \varphi_x\|_0^2\right)$, $i = 1, \cdots, 7$.

It implies from (3.6) that

$$\begin{aligned}\|F_{mx}(t)\|_0 &\leq K_M(f)\left[1+\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}}+\sqrt{\frac{1-\rho^2}{2}}M^*\right)+\frac{M}{\sqrt{\mu_*}C_{1\rho}}+\left(M+\sqrt{\frac{1-\rho^2}{2}}M^*\right)\right]\\ &+\left(\left\|\tilde{\varphi}''\right\|_{C^0([0,T^*])}+\frac{1}{\rho^2}\left\|\tilde{\varphi}\right\|_{C^0([0,T^*])}\tilde{K}_M(\mu)\right)\sqrt{\frac{1-\rho^2}{2}}\\ &\leq 2K_M(f)\left[\frac{1+M}{2}+\frac{M}{\sqrt{\mu_*}C_{1\rho}}+\sqrt{\frac{1-\rho^2}{2}}M^*\right]\\ &+\left\|\tilde{\varphi}\right\|_{C^2([0,T^*])}\left(1+\frac{1}{\rho^2}\tilde{K}_M(\mu)\right)\sqrt{\frac{1-\rho^2}{2}}\\ &\leq \bar{F}_2(M).\end{aligned}$$

Proof (iv). We have

$$\sigma_{1m}'(t)=g_1'''(t)-\tilde{\varphi}'(t)\,\mu_m(t)-\tilde{\varphi}(t)\,\mu_m'(t),$$

so

$$\left|\sigma_{1m}'(t)\right|\leq\left|g_1'''(t)\right|+\left|\tilde{\varphi}'(t)\right|\mu_m(t)+\left|\tilde{\varphi}(t)\right|\left|\mu_m'(t)\right|\leq\bar{\sigma}_{1M}.$$

Proof (v). It is similarly to the proof of (iv).
The proof of Lemma 3.2 is complete. □

Applying Lemma 3.2, we now estimate the terms I_j on the right - hand side of (3.16) as follows. By Lemma 3.2 (i)-(iii) and the following inequality

$$S_m^{(k)}(t)\geq\left\|\dot{v}_m^{(k)}(t)\right\|_0^2+\left\|\dot{v}_{mx}^{(k)}(t)\right\|_0^2+\mu_*\left[\left\|v_{mx}^{(k)}(t)\right\|_0^2+\left\|Av_m^{(k)}(t)\right\|_0^2\right],$$

we obtain

$$\begin{aligned}I_1&=\int_0^t\mu_m'(s)\left[\left\|v_{mx}^{(k)}(s)\right\|_0^2+\left\|Av_m^{(k)}(s)\right\|_0^2\right]ds\leq\frac{\bar{\mu}_M}{\mu_*}\int_0^t S_m^{(k)}(s)ds,\\ I_2&=2\int_0^t\left\langle F_m(s),\dot{v}_m^{(k)}(s)\right\rangle ds\leq\frac{1-\rho^2}{2}T\bar{F}_1^2(M)+\int_0^t S_m^{(k)}(s)ds,\\ I_3&=2\int_0^t\left\langle F_{mx}(s),\dot{v}_{mx}^{(k)}(s)\right\rangle ds\leq T\bar{F}_2^2(M)+\int_0^t S_m^{(k)}(s)ds.\end{aligned}\tag{3.18}$$

Using integration by parts leads to

$$\begin{aligned}I_4&=2\int_0^t\sigma_{1m}(s)\dot{v}_{mx}^{(k)}(1,s)ds\\ &=-2\sigma_{1m}(0)\tilde{v}_{0kx}(1)+2\sigma_{1m}(0)v_{mx}^{(k)}(1,t)+2v_{mx}^{(k)}(1,t)\int_0^t\sigma_{1m}'(s)ds\\ &-2\int_0^t\sigma_{1m}'(s)v_{mx}^{(k)}(1,s)ds\\ &=-2\sigma_{1m}(0)\tilde{v}_{0kx}(1)+I_4^{(1)}+I_4^{(2)}+I_4^{(3)}.\end{aligned}\tag{3.19}$$

By Lemma 3.2 (iv) and the following inequality

$$\left|v_{mx}^{(k)}(1,t)\right| \leq \left\|v_{mx}^{(k)}(t)\right\|_{C^0(\bar{\Omega})} \leq \alpha_0 \left\|v_{mx}^{(k)}(t)\right\|_{H^1} \leq \frac{\alpha_0}{\sqrt{\rho\mu_*}C_{1\rho}} \sqrt{S_m^{(k)}(t)},$$

it is not difficult to estimate the following terms $I_4^{(1)}, I_4^{(2)}, I_4^{(3)}$

$$\begin{aligned}
\left|I_4^{(1)}\right| &= \left|2\sigma_{1m}(0)v_{mx}^{(k)}(1,t)\right| \\
&\leq \frac{2\left|\sigma_{1m}(0)\right|\alpha_0}{\sqrt{\rho\mu_*}C_{1\rho}} \sqrt{S_m^{(k)}(t)} \leq \frac{1}{8}S_m^{(k)}(t) + \frac{8\alpha_0^2\sigma_{1m}^2(0)}{\rho\mu_*C_{1\rho}^2}, \\
\left|I_4^{(2)}\right| &= 2\left|v_{mx}^{(k)}(1,t)\int_0^t \sigma_{1m}'(s)ds\right| \\
&\leq \frac{2\alpha_0 T\bar{\sigma}_{1M}}{\sqrt{\rho\mu_*}C_{1\rho}} \sqrt{S_m^{(k)}(t)} \leq \frac{1}{8}S_m^{(k)}(t) + \frac{8T^2\bar{\sigma}_{1M}^2\alpha_0^2}{\rho\mu_*C_{1\rho}^2}, \\
\left|I_4^{(3)}\right| &= \left|-2\int_0^t \sigma_{1m}'(s)v_{mx}^{(k)}(1,s)ds\right| \\
&\leq \frac{2\bar{\sigma}_{1M}\alpha_0}{\sqrt{\rho\mu_*}C_{1\rho}} \int_0^t \sqrt{S_m^{(k)}(s)}ds \leq \frac{T\bar{\sigma}_{1M}^2\alpha_0^2}{\rho\mu_*C_{1\rho}^2} + \int_0^t S_m^{(k)}(s)ds.
\end{aligned} \tag{3.20}$$

Hence, we deduce from (3.19) and (3.20) that

$$\begin{aligned}
I_4 &\leq 2\left|\sigma_{1m}(0)\tilde{v}_{0kx}(1)\right| + \frac{8\alpha_0^2\sigma_{1m}^2(0)}{\rho\mu_*C_{1\rho}^2} \\
&+ \frac{1}{4}S_m^{(k)}(t) + T(1+8T)\frac{\alpha_0^2\bar{\sigma}_{1M}^2}{\rho\mu_*C_{1\rho}^2} + \int_0^t S_m^{(k)}(s)ds.
\end{aligned} \tag{3.21}$$

Similarly, by using integration by parts

$$\begin{aligned}
I_5 &= 2\int_0^t \sigma_{\rho m}(s)\dot{v}_{mx}^{(k)}(\rho,s)ds \\
&= -2\sigma_{\rho m}(0)\tilde{v}_{0kx}(\rho) + 2\sigma_{\rho m}(0)v_{mx}^{(k)}(\rho,t) \\
&+ 2v_{mx}^{(k)}(\rho,t)\int_0^t \sigma_{\rho m}'(s)ds - 2\int_0^t \sigma_{\rho m}'(s)v_{mx}^{(k)}(\rho,s)ds,
\end{aligned}$$

from Lemma 3.2 (v) and the following inequality

$$\begin{aligned}
\left|v_{mx}^{(k)}(\rho,t)\right| &\leq \left\|v_{mx}^{(k)}(t)\right\|_{C^0(\bar{\Omega})} \leq \alpha_0 \left\|v_{mx}^{(k)}(t)\right\|_{H^1} \\
&\leq \frac{\alpha_0}{\sqrt{\rho\mu_*}C_{1\rho}} \sqrt{S_m^{(k)}(t)},
\end{aligned}$$

we also have

$$\begin{aligned}
I_5 &\leq 2\left|\sigma_{\rho m}(0)\tilde{v}_{0kx}(\rho)\right| + \frac{8\alpha_0^2\sigma_{\rho m}^2(0)}{\rho\mu_*C_{1\rho}^2} \\
&+ \frac{1}{4}S_m^{(k)}(t) + T(1+8T)\frac{\alpha_0^2\bar{\sigma}_{\rho M}^2}{\rho\mu_*C_{1\rho}^2} + \int_0^t S_m^{(k)}(s)ds.
\end{aligned} \tag{3.22}$$

Eq. $(3.10)_1$ can be rewritten as follows

$$\left\langle \ddot{v}_m^{(k)}(t), w_j \right\rangle + \mu_m(t) \left\langle A v_m^{(k)}(t), w_j \right\rangle = \left\langle F_m(t), w_j \right\rangle, \ j = 1, \cdots, k. \tag{3.23}$$

Hence, it follows after replacing w_j with $\ddot{v}_m^{(k)}(t)$, that

$$\begin{aligned} \left\| \ddot{v}_m^{(k)}(t) \right\|_0^2 &= -\mu_m(t) \left\langle A v_m^{(k)}(t), \ddot{v}_m^{(k)}(t) \right\rangle + \left\langle F_m(t), \ddot{v}_m^{(k)}(t) \right\rangle \\ &\leq \left(\mu_m(t) \left\| A v_m^{(k)}(t) \right\|_0 + \| F_m(t) \|_0 \right) \left\| \ddot{v}_m^{(k)}(t) \right\|_0 \\ &\leq \left(\mu_m(t) \left\| A v_m^{(k)}(t) \right\|_0 + \| F_m(t) \|_0 \right)^2 \\ &\leq 2\mu_m^2(t) \left\| A v_m^{(k)}(t) \right\|_0^2 + 2 \| F_m(t) \|_0^2 \\ &\leq 2\tilde{K}_M(\mu) S_m^{(k)}(t) + \left(1 - \rho^2\right) \bar{F}_1^2(M). \end{aligned} \tag{3.24}$$

Integrating in t to get

$$I_6 = \int_0^t \left\| \ddot{v}_m^{(k)}(s) \right\|_0^2 ds \leq T \left(1 - \rho^2\right) \bar{F}_1^2(M) + 2\tilde{K}_M(\mu) \int_0^t S_m^{(k)}(s) ds. \tag{3.25}$$

It follows from (3.16), (3.18), (3.21), (3.22) and (3.25), that

$$S_m^{(k)}(t) \leq \bar{S}_{0m}^{(k)} + T D_1(M) + D_2(M) \int_0^t S_m^{(k)}(s) ds, \tag{3.26}$$

where

$$\begin{aligned} \bar{S}_{0m}^{(k)} &= 2S_m^{(k)}(0) + 4 \left(|\sigma_{1m}(0) \tilde{v}_{0kx}(1)| + \left| \sigma_{\rho m}(0) \tilde{v}_{0kx}(\rho) \right| \right) \\ &\quad + \frac{16 \alpha_0^2 \left(\sigma_{1m}^2(0) + \sigma_{\rho m}^2(0) \right)}{\rho \mu_* C_{1\rho}^2}, \end{aligned} \tag{3.27}$$

$$D_1(M) = 3 \left(1 - \rho^2\right) \bar{F}_1^2(M) + 2\bar{F}_2^2(M) + 2(1 + 8T^*) \frac{\alpha_0^2}{\rho \mu_* C_{1\rho}^2} \left(\bar{\sigma}_{1M}^2 + \bar{\sigma}_{\rho M}^2 \right),$$

$$D_2(M) = 2 \left(4 + \frac{\bar{\mu}_M}{\mu_*} + 2\tilde{K}_M(\mu) \right).$$

It remains to estimate the term $\bar{S}_{0m}^{(k)}$.

Notice that $\mu_m(0) = \mu(0, \left\| \tilde{v}_0 + \varphi(0) \right\|_0^2, \left\| \tilde{v}_{0x} + \varphi_x(0) \right\|_0^2)$ is independent of m, so $S_m^{(k)}(0)$, $\sigma_{1m}(0)$, $\sigma_{\rho m}(0)$ are also independent of m, because of

$$\begin{aligned} S_m^{(k)}(0) &= \|\tilde{v}_{1k}\|_0^2 + \|\tilde{v}_{1kx}\|_0^2 + \mu_m(0) \left(\|\tilde{v}_{0kx}\|_0^2 + \|A \tilde{v}_{0k}\|_0^2 \right), \\ \sigma_{1m}(0) &= g_1''(0) - \tilde{\varphi}(0) \mu_m(0), \\ \sigma_{\rho m}(0) &= -\rho g_0''(0) + \tilde{\varphi}(0) \mu_m(0). \end{aligned}$$

By means of the convergences in (3.11), we can deduce the existence of a constant $M > 0$, it is independent of k and m such that

$$\begin{aligned} \bar{S}_{0m}^{(k)} &= 2S_m^{(k)}(0) + 4 \left(|\sigma_{1m}(0) \tilde{v}_{0kx}(1)| + \left| \sigma_{\rho m}(0) \tilde{v}_{0kx}(\rho) \right| \right) \\ &\quad + \frac{16 \alpha_0^2 \left(\sigma_{1m}^2(0) + \sigma_{\rho m}^2(0) \right)}{\rho \mu_* C_{1\rho}^2} \\ &\leq \frac{1}{2} M^2, \text{ for all } m \text{ and } k. \end{aligned} \tag{3.28}$$

Therefore, we can choose $T \in (0, T^*]$, such that

$$\left(\frac{1}{2}M^2 + TD_1(M)\right)\exp\left(TD_2(M)\right) \leq M^2, \tag{3.29}$$

and

$$k_T = \left(1 + \frac{1}{\sqrt{\mu_*}}\right)\sqrt{T\left(\tilde{\mu}_M + F_M^2\right)e^{T\left(2+\frac{\tilde{\mu}_M}{\mu_*}\right)}} < 1, \tag{3.30}$$

where

$$\tilde{\mu}_M = \frac{8}{\mu_*}\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)^2 M^2\tilde{K}_M^2(\mu), \tag{3.31}$$

$$F_M = \left(1 + \sqrt{2}\right)K_M(f) + 2\sqrt{2}\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\left(K_M(f) + \frac{\|\tilde{\varphi}\|_{C^0([0,T^*])}}{\rho}\tilde{K}_M(\mu)\right)\sqrt{\frac{1-\rho^2}{2}}.$$

Combining (3.26), (3.28) and (3.29), we get

$$S_m^{(k)}(t) \leq M^2 e^{-TD_2(M)} + D_2(M)\int_0^t S_m^{(k)}(s)ds. \tag{3.32}$$

By using Gronwall's Lemma, (3.32) yields

$$S_m^{(k)}(t) \leq M^2 e^{-TD_2(M)} e^{tD_2(M)} \leq M^2, \tag{3.33}$$

for all $t \in [0, T]$, for all m and k. It implies that

$$v_m^{(k)} \in W(M, T), \text{ for all } m \text{ and } k. \tag{3.34}$$

Step 3. Limiting process. From (3.34), there exists a subsequence of $\{u_m^{(k)}\}$, still so denoted, such that

$$\begin{cases} v_m^{(k)} \to v_m & \text{in} \quad L^\infty(0, T; H_0^1 \cap H^2) \text{ weakly*}, \\ \dot{v}_m^{(k)} \to v_m' & \text{in} \quad L^\infty(0, T; H_0^1) \text{ weakly*}, \\ \ddot{v}_m^{(k)} \to v_m'' & \text{in} \quad L^2(Q_T) \text{ weakly}, \\ v_m \in W(M, T). \end{cases} \tag{3.35}$$

Passing to limit in (3.10), we have v_m satisfying (3.7), (3.8) in $L^2(0, T)$. On the other hand, it follows from $(3.7)_1$ and $(3.35)_4$ that $v_m'' = -\mu_m(t)Av_m + F_m \in L^\infty(0, T; L^2)$, hence $v_m \in W_1(M, T)$ and the proof of Theorem 3.1 is complete. □

In order to get the existence and uniqueness, we shall use the following Banach space (see [9])

$$W_1(T) = \{v \in L^\infty(0, T; H_0^1) : v' \in L^\infty(0, T; L^2)\},$$

with respect to the norm $\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H_0^1)} + \|v'\|_{L^\infty(0,T;L^2)}$.

By the result given in Theorem 3.1 and by the compact imbedding theorems, we now prove the main results in this section as follows.

Theorem 3.3. *Let* $(H_1) - (H_4)$ *hold. Then, there exist positive constants* M, T *satisfying* (3.28)-(3.30) *such that Prob.* (3.1) - (3.2) *has a unique weak solution* $v \in W_1(M, T)$. *Furthermore, the linear recurrent sequence* $\{v_m\}$ *defined by* (3.7), (3.8) *converges to the solution* v *strongly in the space* $W_1(T)$ *with the estimate*

$$\|v_m - v\|_{W_1(T)} \leq \frac{2M}{1-k_T}k_T^m, \text{ for all } m \in \mathbb{N}. \tag{3.36}$$

Proof of Theorem 3.3.

(a) *The existence.* First, we shall prove that $\{v_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = v_{m+1} - v_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + \mu_{m+1}(t)\, a(w_m(t), w) = -\left[\mu_{m+1}(t) - \mu_m(t)\right]\langle Av_m(t), w\rangle \\ \qquad\qquad + \langle F_{m+1}(t) - F_m(t), w\rangle, \ \forall w \in H_0^1, \\ w_m(0) = w_m'(0) = 0, \end{cases} \tag{3.37}$$

where

$$\mu_m(t) = \mu[v_{m-1}](t) = \mu\left(t, \left\|v_{m-1}(t) + \varphi(t)\right\|_0^2, \left\|\nabla v_{m-1} + \varphi_x\right\|_0^2\right), \tag{3.38}$$
$$F_m(t) = \tilde{f}[v_{m-1}](x,t).$$

Taking $w = w_m'$ in $(3.37)_1$, after integrating in t, we get

$$\begin{aligned} Z_m(t) &= \int_0^t \mu_{m+1}'(s)\, \|w_{mx}(s)\|_0^2\, ds - 2\int_0^t \left[\mu_{m+1}(s) - \mu_m(s)\right]\langle Av_m(s), w_m'(s)\rangle ds \\ &\quad + 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle\, ds \\ &\equiv J_1 + J_2 + J_3, \end{aligned} \tag{3.39}$$

where

$$Z_m(t) = \left\|w_m'(t)\right\|_0^2 + \mu_{m+1}(t)\, \|w_{mx}(t)\|_0^2 \geq \left\|w_m'(t)\right\|_0^2 + \mu_*\, \|w_{mx}(t)\|_0^2. \tag{3.40}$$

All integrals on the right - hand side of (3.39) will be estimated as below.

The integral J_1. By (3.40), we have

$$|J_1| \leq \int_0^t \left|\mu_{m+1}'(s)\right| \|w_{mx}(s)\|_0^2\, ds \leq \frac{\bar{\mu}_M}{\mu_*}\int_0^t Z_m(s) ds, \tag{3.41}$$

where $\bar{\mu}_M$ as in Lemma 3.2 (i).

The integral J_2. By (H_3), it is clear to see that

$$\begin{aligned} &\left|\mu_{m+1}(t) - \mu_m(t)\right| \\ &\leq 2\tilde{K}_M(\mu)\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\left(\|w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0\right) \\ &\leq 2\sqrt{2}\tilde{K}_M(\mu)\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\|w_{m-1}\|_{W_1(T)}. \end{aligned} \tag{3.42}$$

Hence

$$\begin{aligned} |J_2| &= 2\left|\int_0^t \left[\mu_{m+1}(s) - \mu_m(s)\right]\langle Av_m(s), w_m'(s)\rangle ds\right| \\ &\leq 2\int_0^t \left|\mu_{m+1}(s) - \mu_m(s)\right| \|Av_m(s)\|_0 \left\|w_m'(s)\right\|_0 ds \\ &\leq 4\sqrt{2}\tilde{K}_M(\mu)\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\frac{M}{\sqrt{\mu_*}}\|w_{m-1}\|_{W_1(T)}\int_0^t \left\|w_m'(s)\right\|_0 ds \\ &\leq T\tilde{\mu}_M\, \|w_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s) ds, \end{aligned} \tag{3.43}$$

where $\tilde{\mu}_M = \frac{8}{\mu_*}\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)^2 M^2 \tilde{K}_M^2(\mu)$.

The integral J_3. By (H_4) it yields

$$\begin{aligned}
&\|F_{m+1}(t) - F_m(t)\|_0 \\
&\le K_M(f)\left(\|w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0 + \left\|w'_{m-1}(t)\right\|_0\right) \\
&+ 2K_M(f)\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\sqrt{\frac{1-\rho^2}{2}}\left(\|w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0\right) \\
&+ \frac{1}{\rho}\left\|\tilde{\varphi}\right\|_{C^0([0,T^*])}\sqrt{\frac{1-\rho^2}{2}}\left|\mu_{m+1}(t) - \mu_m(t)\right| \\
&\le K_M(f)\left(\sqrt{2}\|w_{m-1}(t)\|_1 + \left\|w'_{m-1}(t)\right\|_0\right) \\
&+ 2\sqrt{2}K_M(f)\sqrt{\frac{1-\rho^2}{2}}\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\|w_{m-1}(t)\|_1 \\
&+ \frac{1}{\rho}\left\|\tilde{\varphi}\right\|_{C^0([0,T^*])}\sqrt{\frac{1-\rho^2}{2}}2\sqrt{2}\tilde{K}_M(\mu)\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\|w_{m-1}\|_{W_1(T)} \\
&\le F_M\|w_{m-1}\|_{W_1(T)},
\end{aligned} \tag{3.44}$$

where F_M as in (3.31). Hence

$$\begin{aligned}
|J_3| &= 2\left|\int_0^t \langle F_{m+1}(s) - F_m(s), w'_m(s)\rangle\, ds\right| \\
&\le TF_M^2\|w_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)ds.
\end{aligned} \tag{3.45}$$

Combining (3.39), (3.41), (3.43) and (3.45), we obtain

$$Z_m(t) \le T\left(\tilde{\mu}_M + F_M^2\right)\|w_{m-1}\|_{W_1(T)}^2 + \left(2 + \frac{\bar{\mu}_M}{\mu_*}\right)\int_0^t Z_m(s)ds. \tag{3.46}$$

Using Gronwall's Lemma, we deduce from (3.46) that

$$\|w_m\|_{W_1(T)} \le k_T\|w_{m-1}\|_{W_1(T)} \quad \forall m \in \mathbb{N}, \tag{3.47}$$

where k_T as in (3.30).

It implies that

$$\begin{aligned}
\left\|v_m - v_{m+p}\right\|_{W_1(T)} &\le \|\tilde{v}_0 - v_1\|_{W_1(T)}(1-k_T)^{-1}k_T^m \\
&\le \frac{2M}{1-k_T}k_T^m \quad \forall m, p \in \mathbb{N}.
\end{aligned} \tag{3.48}$$

It follows that $\{v_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $v \in W_1(T)$ such that

$$v_m \to v \text{ strongly in } W_1(T). \tag{3.49}$$

Note that $v_m \in W_1(M,T)$, then there exists a subsequence $\{v_{m_j}\}$ of $\{v_m\}$ such that

$$\begin{cases}
v_{m_j} \to v & \text{in} \quad L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\
v'_{m_j} \to v' & \text{in} \quad L^\infty(0,T;H_0^1) \text{ weakly*}, \\
v''_{m_j} \to v'' & \text{in} \quad L^2(Q_T) \text{ weakly}, \\
v \in W(M,T).
\end{cases} \tag{3.50}$$

We also note that

$$\left|\mu_m(t) - \mu[v](t)\right| \leq 2\sqrt{2}\tilde{K}_M(\mu)\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\|v_{m-1} - v\|_{W_1(T)}. \tag{3.51}$$

Hence, it follows from (3.49) and (3.51) that

$$\mu_m \to \mu[v] \text{ strongly in } L^\infty(0,T). \tag{3.52}$$

On the other hand, we have

$$\left\|F_m - \tilde{f}[v]\right\|_{L^\infty(0,T;L^2)} \leq F_M \|v_{m-1} - v\|_{W_1(T)}. \tag{3.53}$$

Hence, from (3.49) and (3.53), we obtain

$$F_m \to \tilde{f}[v] \text{ strongly in } L^\infty(0,T;L^2). \tag{3.54}$$

Finally, passing to limit in (3.7), (3.8) as $m = m_j \to \infty$, it implies from (3.49), $(3.50)_{1,3}$ (3.52) and (3.54) that there exists $u \in W(M,T)$ satisfying the equation

$$\langle v''(t), w\rangle + \mu[v](t)a(v(t), w) = \left\langle \tilde{f}[v](t), w\right\rangle, \tag{3.55}$$

for all $w \in H_0^1$ and the initial conditions

$$v(0) = \tilde{v}_0,\ v'(0) = \tilde{v}_1. \tag{3.56}$$

Furthermore, from the assumptions (H_3), (H_4) we obtain from $(3.50)_4$, (3.52), (3.54) and (3.55), that

$$v'' = -\mu[v](t)Av(t) + \tilde{f}[v](t) \in L^\infty(0,T;L^2), \tag{3.57}$$

thus we have $v \in W_1(M,T)$. The existence of a weak solution of Prob. (3.1) - (3.2) is proved.

(b) *The uniqueness.* Let v_1, $v_2 \in W_1(M,T)$ be two weak solutions of Prob. (3.1) - (3.2). Then $v = v_1 - v_2$ satisfies the variational problem

$$\begin{cases} \langle v''(t), w\rangle + \mu_1(t)a(v(t), w) = -[\mu_1(t) - \mu_2(t)]\langle Av_2(t), w\rangle \\ \qquad\qquad\qquad\qquad + \langle \bar{F}_1(t) - \bar{F}_2(t), w\rangle, \forall w \in H_0^1, \\ v(0) = v'(0) = 0, \end{cases} \tag{3.58}$$

where $\bar{F}_i(x,t) = \tilde{f}[v_i](t)$, $\mu_i(t) = \mu[v_i](t)$, $i = 1, 2$.

We take $w = v'$ in $(3.58)_1$ and integrate in t to get

$$\begin{aligned} Z(t) = &\int_0^t \mu_1'(s)\|v_x(s)\|_0^2 ds - 2\int_0^t [\mu_1(s) - \mu_2(s)]\langle Av_2(s), v'(s)\rangle ds \\ &+ 2\int_0^t \langle \bar{F}_1(s) - \bar{F}_2(s), v'(s)\rangle ds, \end{aligned} \tag{3.59}$$

with $Z(t) = \|v'(t)\|_0^2 + \mu_1(t)\|v_x(t)\|_0^2$. Put $K_M^* = \dfrac{\bar{\mu}_M}{\mu_*} + \dfrac{2M}{\sqrt{\mu_*}}R_M\tilde{K}_M(\mu) + 2K_M(f,\mu)$, where

$$\begin{aligned} R_M &= \frac{2}{\sqrt{\mu_*}}\left(\frac{M}{\sqrt{\mu_*}C_{1\rho}} + \sqrt{\frac{1-\rho^2}{2}}M^*\right)\left(\frac{1-\rho}{\sqrt{2\rho}} + 1\right), \\ K_M(f,\mu) &= K_M(f)\left[\left(\frac{1-\rho}{\sqrt{2\rho}} + 1\right)\frac{1}{\sqrt{\mu_*}} + R_M + 1\right] \\ &\quad + \frac{1}{\rho}\left\|\tilde{\varphi}\right\|_{C^0([0,T^*])}\sqrt{\frac{1-\rho^2}{2}}\tilde{K}_M(\mu)R_M, \end{aligned}$$

it follows from (3.59) that

$$Z(t) \leq K_M^* \int_0^t Z(s)ds, \text{ for all } t \in [0,T].$$

Using Gronwall's Lemma, it follows that $Z(t) \equiv 0$, ie., $v_1 \equiv v_2$.
Therefore, Theorem 3.3 is proved. □

4. Blow-up result

In this section, Prob. (1.1) – (1.3) is considered with $f = -\lambda u_t + f(u)$, $\mu = \mu(\|u_x(t)\|_0^2)$, $g_0(t) = g_1(t) \equiv 0$, as follows

$$\begin{cases} u_{tt} - \mu(\|u_x(t)\|_0^2)(u_{xx} + \frac{1}{x}u_x) + \lambda u_t = f(u), \ \rho < x < 1, \ 0 < t < T, \\ u(\rho,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{4.1}$$

where $\lambda > 0, 0 < \rho < 1$ are given constants and $\tilde{u}_0$, $\tilde{u}_1$, μ, f are given functions satisfying conditions specified later. First, we assume that

(H_2^*) $\mu \in C^1(\mathbb{R}_+)$ and there exists the constant $\mu_* > 0$ such that $\mu(y) \geq \mu_* > 0$, $\forall y \in \mathbb{R}_+$;

(H_3^*) $f \in C^1(\mathbb{R})$, $f(0) = 0$.

Then we obtain the following theorem about the existence of a weak solution.

Theorem 4.1. *Suppose that* (H_1), (H_2^*) and (H_3^*) *hold. Then Prob.* (4.1) *has a unique local solution*

$$\begin{aligned} &u \in C\left([0,T_*];H_0^1\right) \cap C^1\left([0,T_*];L^2\right) \cap L^\infty\left(0,T_*;H_0^1 \cap H^2\right), \\ &u' \in L^\infty\left(0,T_*;H_0^1\right), \ u'' \in L^\infty\left(0,T_*;L^2\right), \end{aligned} \tag{4.2}$$

for $T_* > 0$ *small enough.*

Proof of Theorem 4.1. The proof is similar to the Theorem 3.1 and Theorem 3.3. □

Next, in order to obtain a blow-up result in Theorem 4.2 below, we make more the following assumptions.

($\hat{H}_2^*$) $\mu \in C^1(\mathbb{R}_+)$, and there exist the constants $\mu_* > 0$, $\bar{\mu}_1 > 0$ such that
(i) $\mu(y) \geq \mu_* > 0, \forall y \geq 0$,
(ii) $y\mu(y) \leq \bar{\mu}_1 \int_0^y \mu(z)dz, \ \forall y \geq 0$;

($\hat{H}_3^*$) $f \in C^1(\mathbb{R})$, $f(0) = 0$ and there exist the constants $p > 2$, $d_1 > 2$, $\bar{d}_1 > 0$ such that
(i) $yf(y) \geq d_1 \int_0^y f(z)dz, \ \forall y \in \mathbb{R}$,
(ii) $\int_0^y f(z)dz \geq \bar{d}_1 \left|y\right|^p, \ \forall y \in \mathbb{R}$;

($\hat{H}_4^*$) $d_1 > 2\bar{\mu}_1$, with d_1, $\bar{\mu}_1$ as in ($\hat{H}_2^*$)(*ii*), ($\hat{H}_3^*$)(*i*).

Note that we can give an example of two following functions f and μ which satisfy ($\hat{H}_2^*$) and ($\hat{H}_3^*$).

Example. Consider the function $\mu \in C^1(\mathbb{R}_+)$ with

$$\mu(z) = \mu_* + z^q, \ \forall z \geq 0,$$

where $\mu_* > 0$ and $q > 1$ are constants. It is obvious that $(\hat{H}_2^*)(i)$ holds. On the other hand, we have

$$\int_0^y \mu(z)dz = \int_0^y (\mu_* + z^q)\,dz = y\left(\mu_* + \frac{y^q}{q+1}\right) \geq \frac{1}{q+1} y\mu(y), \ \forall y \geq 0.$$

Hence, $(\hat{H}_2^*)(i)$ holds with $\bar{\mu}_1 = q + 1$.

Let $\bar{\beta} > 0$, $p > 2$ and $k > 1$, we define the function $f \in C^1(\mathbb{R})$ with

$$f(z) = \bar{\beta}\,|z|^{p-2} z \ln^k\left(z^2 + e\right), \ \forall z \in \mathbb{R}.$$

It is clearly that $f(0) = 0$. By integration by parts, we obtain

$$\int_0^y f(z)dz = \frac{1}{p} y f(y) - \frac{2k\bar{\beta}}{p}\int_0^y \frac{|z|^p z}{e+z^2}\ln^{k-1}\left(z^2+e\right)dz \leq \frac{1}{p} y f(y), \ \forall y \in \mathbb{R},$$

since $\int_0^y \frac{|z|^p z}{e+z^2}\ln^{k-1}\left(z^2+e\right)dz \geq 0$ for all $y \in \mathbb{R}$.

Thus, $(\hat{H}_3^*)(i)$ and $(\hat{H}_4^*)$ hold when $d_1 = p > 2(q+1) > 2$.

For $y \geq 0$: $\int_0^y f(z)dz = \bar{\beta}\int_0^y |z|^{p-2} z \ln^k\left(z^2+e\right)dz \geq \bar{\beta}\int_0^y |z|^{p-2} z dz = \frac{\bar{\beta}}{p}\left|y\right|^p.$

For $y < 0$: $\int_0^y f(z)dz = \bar{\beta}\int_0^y |z|^{p-2} z \ln^k\left(z^2+e\right)dz = \bar{\beta}\int_0^{-y} |z|^{p-2} z \ln^k\left(z^2+e\right)dz$
$\geq \bar{\beta}\int_0^{-y} |z|^{p-2} z dz = \frac{\bar{\beta}}{p}\left|-y\right|^p = \frac{\bar{\beta}}{p}\left|y\right|^p.$

Therefore

$$\int_0^y f(z)dz \geq \bar{d}_1\left|y\right|^p \text{ for all } y \in \mathbb{R},$$

where $\bar{d}_1 = \frac{\bar{\beta}}{p} > 0$. Thus, $(\hat{H}_3^*)(ii)$ is true.

Put

$$H(0) = -\frac{1}{2}\|\tilde{u}_1\|_0^2 - \frac{1}{2}\int_0^{\|\tilde{u}_{0x}\|_0^2}\mu(z)dz + \int_\rho^1 x dx\int_0^{\tilde{u}_0(x)} f(z)dz. \tag{4.3}$$

Theorem 4.2. *Let $(\hat{H}_2^*) - (\hat{H}_4^*)$ hold. Then, for any $(\tilde{u}_0, \tilde{u}_1) \in (H_0^1 \cap H^2) \times H_0^1$ such that $H(0) > 0$, the weak solution $u = u(x,t)$ of Prob. (4.1) blows-up in finite time.*

Proof of Theorem 4.2. It consists of two steps, in which the Lyapunov functional $L(t)$ is constructed in step 1 and then the blow-up is proved in step 2.

Step 1. We define the energy associated with (4.1) by

$$E(t) = \frac{1}{2}\|u'(t)\|_0^2 + \frac{1}{2}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz - \int_\rho^1 x dx\int_0^{u(x,t)} f(z)dz, \tag{4.4}$$

and we put $H(t) = -E(t)$, $\forall t \in [0, T_*)$. Multiplying $(4.1)_1$ by $xu'(x,t)$ and integrating the resulting equation over $(\rho, 1)$, we have

$$H'(t) = \lambda\|u'(t)\|_0^2 \geq 0. \tag{4.5}$$

Hence, we can deduce from (4.5) and $H(0) > 0$ that

$$H(t) \geq H(0) > 0, \ \forall t \in [0, T_*), \tag{4.6}$$

so

$$\begin{cases} 0 < H(0) \leq H(t) \leq \int_\rho^1 x dx \int_0^{u(x,t)} f(z) dz; \\ \|u'(t)\|_0^2 + \int_0^{\|u_x(t)\|_0^2} \mu(z) dz \leq 2 \int_\rho^1 x dx \int_0^{u(x,t)} f(z) dz, \ \forall t \in [0, T_*). \end{cases} \tag{4.7}$$

We define the functional

$$L(t) = H^{1-\eta}(t) + \varepsilon \Psi(t), \tag{4.8}$$

where

$$\Psi(t) = \langle u'(t), u(t) \rangle + \frac{\lambda}{2} \|u(t)\|_0^2, \tag{4.9}$$

for ε small enough and

$$0 < 2\eta < 1, \ 2/(1 - 2\eta) \leq p. \tag{4.10}$$

In what follows, we show that there exists a constant $\bar{\lambda}_1 > 0$ such that

$$L'(t) \geq \bar{\lambda}_1 \left[H(t) + \|u'(t)\|_0^2 + \|u(t)\|_{L^p}^p + \|u_x(t)\|_0^2 \right]. \tag{4.11}$$

Multiplying $(4.1)_1$ by $xu(x, t)$ and integrating over $[\rho, 1]$, it leads to

$$\Psi'(t) = \|u'(t)\|_0^2 - \|u_x(t)\|_0^2 \mu \left(\|u_x(t)\|_0^2 \right) + \langle f(u(t)), u(t) \rangle. \tag{4.12}$$

Therefore

$$L'(t) = (1 - \eta) H^{-\eta}(t) H'(t) + \varepsilon \Psi'(t) \geq \varepsilon \Psi'(t). \tag{4.13}$$

By $(\hat{H}_2^*)$, $(\hat{H}_3^*)$, we obtain

$$\begin{aligned} \|u_x(t)\|_0^2 \mu \left(\|u_x(t)\|_0^2 \right) &\leq \bar{\mu}_1 \int_0^{\|u_x(t)\|_0^2} \mu(z) dz, \\ \langle f(u(t)), u(t) \rangle &\geq d_1 \int_\rho^1 x dx \int_0^{u(x,t)} f(z) dz, \\ \int_\rho^1 x dx \int_0^{u(x,t)} f(z) dz &\geq \bar{d}_1 \rho \|u(t)\|_{L^p}^p. \end{aligned} \tag{4.14}$$

Hence, combining (4.4), (4.12) and (4.14) give

$$\begin{aligned}
\Psi'(t) &= \|u'(t)\|_0^2 - \|u_x(t)\|_0^2\,\mu\left(\|u_x(t)\|_0^2\right) + \langle f\left(u(t)\right), u(t)\rangle \qquad (4.15)\\
&\geq \|u'(t)\|_0^2 - \bar{\mu}_1 \int_0^{\|u_x(t)\|_0^2} \mu(z)dz + d_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \\
&= \|u'(t)\|_0^2 - \bar{\mu}_1 \int_0^{\|u_x(t)\|_0^2} \mu(z)dz + d_1\delta_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \\
&+ d_1(1-\delta_1) \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \\
&= \|u'(t)\|_0^2 - \bar{\mu}_1 \int_0^{\|u_x(t)\|_0^2} \mu(z)dz + d_1\delta_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \\
&+ d_1(1-\delta_1)\left[H(t) + \frac{1}{2}\|u'(t)\|_0^2 + \frac{1}{2}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz\right] \\
&= d_1(1-\delta_1)H(t) + \left(1 + \frac{d_1}{2}(1-\delta_1)\right)\|u'(t)\|_0^2 \\
&+ d_1\delta_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz + \frac{1}{2}\left[d_1 - 2\bar{\mu}_1 - \delta_1 d_1\right] \int_0^{\|u_x(t)\|_0^2} \mu(z)dz \\
&\geq d_1(1-\delta_1)H(t) + \left(1 + \frac{d_1}{2}(1-\delta_1)\right)\|u'(t)\|_0^2 \\
&+ d_1\delta_1\bar{d}_1\rho\,\|u(t)\|_{L^p}^p + \frac{1}{2}\left[d_1 - 2\bar{\mu}_1 - \delta_1 d_1\right] \int_0^{\|u_x(t)\|_0^2} \mu(z)dz.
\end{aligned}$$

By $d_1 > 2\bar{\mu}_1$, we can choose $\delta_1 \in (0,1)$ such that

$$d_1 - 2\bar{\mu}_1 - \delta_1 d_1 > 0. \qquad (4.16)$$

By using the inequalities (4.13), (4.15), (4.16), we obtain (4.11) with choosing $\bar{\lambda}_1 > 0$ small enough.
From the formula of $L(t)$ and (4.6), we can choose ε small enough such that

$$L(t) \geq L(0) > 0, \ \forall t \in [0, T_*). \qquad (4.17)$$

Using the inequality $\left(\sum_{i=1}^3 x_i\right)^r \leq 3^{r-1}\sum_{i=1}^3 x_i^r$, for all $r > 1$ and $x_1, ..., x_3 \geq 0$, we deduce from (4.8) - (4.10) that

$$L^{1/(1-\eta)}(t) \leq Const\left(H(t) + |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} + \|u(t)\|_0^{2/(1-\eta)}\right). \qquad (4.18)$$

Using Young's inequality, we have

$$|\langle u(t), u'(t)\rangle|^{1/(1-\eta)} \leq Const\left(\|u(t)\|_{L^p}^s + \|u'(t)\|_0^2\right), \qquad (4.19)$$

where $s = 2/(1-2\eta) \leq p$ by (4.10).
Now, we shall need the following lemma
Lemma 4.3. *Let* $s = 2/(1-2\eta) \leq p$, *we obtain*

$$\|v\|_{L^p}^s + \|v\|_0^{2/(1-\eta)} \leq \frac{2}{\rho}\left(\|v_x\|_0^2 + \|v\|_{L^p}^p\right), \text{ for any } v \in H_0^1. \qquad (4.20)$$

Proof of Lemma 4.3 is straightforward, so we omit the details.

Step 2. Blow-up.

It follows from (4.18) - (4.20) that

$$L^{1/(1-\eta)}(t) \leq Const \left(H(t) + \|u'(t)\|_0^2 + \|u_x(t)\|_0^2 + \|u(t)\|_{L^p}^p \right), \ \forall t \in [0, T_*). \tag{4.21}$$

Using (4.11), (4.21) it yields

$$L'(t) \geq \bar{\lambda}_2 L^{1/(1-\eta)}(t), \ \forall t \in [0, T_*), \tag{4.22}$$

where $\bar{\lambda}_2$ is a positive constant. By integrating (4.22) over $(0, t)$, it gives

$$L^{\eta/(1-\eta)}(t) \geq \frac{1}{L^{-\eta/(1-\eta)}(0) - \frac{\bar{\lambda}_2 \eta}{1-\eta} t}, \ 0 \leq t < \frac{1}{\bar{\lambda}_2 \eta} (1-\eta) L^{-\eta/(1-\eta)}(0). \tag{4.23}$$

Therefore, $L(t)$ blows-up in a finite time given by $T_* = \frac{1}{\bar{\lambda}_2 \eta} (1-\eta) L^{-\eta/(1-\eta)}(0)$.

Theorem 4.2 is proved completely. □

5. Exponential decay of solutions

This section investigates the decay of the solution of Prob. (1.1) – (1.3) corresponding to $f = -\lambda u_t + f(u) + F(x,t)$, $\mu = \mu(\|u_x(t)\|_0^2)$ and $g_0(t) = g_1(t) \equiv 0$, as follows

$$\begin{cases} u_{tt} - \mu(\|u_x(t)\|_0^2)(u_{xx} + \frac{1}{x} u_x) + \lambda u_t = f(u) + F(x,t), \ \rho < x < 1, \ t > 0, \\ u(\rho, t) = u(1, t) = 0, \\ u(x, 0) = \tilde{u}_0(x), \ u_t(x, 0) = \tilde{u}_1(x), \end{cases} \tag{5.1}$$

where μ, f, F, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $\lambda > 0$, $0 < \rho < 1$ are the given constants.

We prove that if $\int_0^{\|\tilde{u}_{0x}\|_0^2} \mu(z)dz - p \int_\rho^1 x dx \int_0^{\tilde{u}_0(x)} f(z)dz > 0$ and if the initial energy, $\|F(t)\|_0$ are small enough, then the energy of the solution decays exponentially as $t \to +\infty$. For this purpose, we make the following assumptions

$(\overline{H}_3)$ $f \in C^1(\mathbb{R})$, $f(0) = 0$ and there exist the constants $\alpha, \beta > 2$; d_2, $\bar{d}_2 > 0$, such that

(i) $y f(y) \leq d_2 \int_0^y f(z)dz$, for all $y \in \mathbb{R}$,

(ii) $\int_0^y f(z)dz \leq \bar{d}_2 \left(|y|^\alpha + |y|^\beta \right)$, for all $y \in \mathbb{R}$;

$(\overline{H}_4)$ $F \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2)$, $F' \in L^2(\mathbb{R}_+; L^2)$ and there exist two constants $\bar{C}_0 > 0$, $\bar{\gamma}_0 > 0$ such that $\|F(t)\|_0 \leq \bar{C}_0 e^{-\bar{\gamma}_0 t}$, for all $t \geq 0$.

We will show that the example of f in Section 4 also satisfies $(\overline{H}_3)$. Let $\bar{\beta} > 0$, $p > 2k$ and $k > 1$, we set

$$f(z) = \bar{\beta} |z|^{p-2} z \ln^k \left(z^2 + e \right), \ \forall z \in \mathbb{R}.$$

We know that $f \in C^1(\mathbb{R})$ and $f(0) = 0$. For $y \geq 0$,

$$\begin{aligned} \int_0^y f(z)dz &= \frac{1}{p} y f(y) - \frac{2k\overline{\beta}}{p} \int_0^y \frac{|z|^p z}{e + z^2} \ln^{k-1} \left(z^2 + e \right) dz \\ &\geq \frac{1}{p} y f(y) - \frac{2k\overline{\beta}}{p} \ln^{k-1} \left(y^2 + e \right) \int_0^y \frac{|z|^p z}{e + z^2} dz \\ &\geq \frac{1}{p} y f(y) - \frac{2k\overline{\beta}}{p^2} |y|^p \ln^{k-1} \left(y^2 + e \right) \\ &\geq \frac{1}{p} y f(y) - \frac{2k\overline{\beta}}{p^2} |y|^p \ln^k \left(y^2 + e \right) = \frac{p-2k}{p^2} y f(y). \end{aligned}$$

Note that

$$\int_0^y \frac{|z|^p z}{e+z^2} \ln^{k-1}\left(z^2+e\right) dz = \int_0^{-y} \frac{|z|^p z}{e+z^2} \ln^{k-1}\left(z^2+e\right) dz \geq 0 \text{ for all } y \in \mathbb{R}.$$

By same argument in case $y \geq 0$, we also have

$$\int_0^y f(z)\,dz \geq \frac{p-2k}{p^2} y f(y) \ \text{ for } y < 0.$$

Hence, $(\overline{H}_3)\,(i)$ holds when $d_2 = \frac{p^2}{p-2k} > 0$.
By the inequality $\ln(1+x) \leq x$ for all $x \geq 0$,

$$\begin{aligned}\int_0^y f(z)\,dz &\leq \frac{1}{p} y f(y) = \frac{1}{p} |y|^p \ln^k\left(y^2+e\right) \\ &= \frac{1}{p}|y|^p\left[1+\ln\left(1+\frac{y^2}{e}\right)\right]^k \leq \frac{1}{p}|y|^p\left(1+\frac{y^2}{e}\right)^k.\end{aligned}$$

Using the inquality $(a+b)^r \leq 2^{r-1}(a^r+b^r)$, for all $r>1$ and $a, b \geq 0$, we deduce that

$$\begin{aligned}\int_0^y f(z)\,dz &\leq \frac{1}{p}|y|^p\left(1+\frac{y^2}{e}\right)^k \leq \frac{2^{k-1}}{p}|y|^p\left(1+\frac{y^{2k}}{e^k}\right) \\ &\leq \frac{2^{k-1}}{p}\left(|y|^p + |y|^{p+2k}\right) \text{ for all } y \in \mathbb{R}.\end{aligned}$$

Thus, $(\overline{H}_3)$ holds when $\alpha = p > 2$, $\beta = p+2k > 2$ and $\bar{d}_2 = \frac{2^{k-1}}{p} > 0$.
First, we construct the following Lyapunov functional

$$\mathcal{L}(t) = E(t) + \delta\Psi(t), \tag{5.2}$$

where $\delta > 0$ is chosen later and

$$\begin{aligned}E(t) &= \frac{1}{2}\|u'(t)\|_0^2 + \frac{1}{2}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz - \int_\rho^1 x dx \int_0^{u(x,t)} f(z)dz \\ &= \frac{1}{2}\|u'(t)\|_0^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2} \mu(z)dz + \frac{1}{p}I(t),\end{aligned} \tag{5.3}$$

$$\Psi(t) = \langle u'(t), u(t)\rangle + \frac{\lambda}{2}\|u(t)\|_0^2, \tag{5.4}$$

where

$$I(t) = I(u(t)) = \int_0^{\|u_x(t)\|_0^2} \mu(z)dz - p\int_\rho^1 x dx \int_0^{u(x,t)} f(z)dz. \tag{5.5}$$

Then we have the following theorem.

Theorem 5.1. *Assume that* (H_2^*), $(\overline{H}_3)$, $(\overline{H}_4)$ *hold. Let* $(\tilde{u}_0, \tilde{u}_1) \in (H_0^1 \cap H^2) \times H_0^1$ *such that* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfy*

$$\eta^* = \mu_* - p\bar{d}_2(1-\rho)\left[\sqrt{\left(\frac{1-\rho}{\rho}\right)^\alpha} R_*^{\alpha-2} + \sqrt{\left(\frac{1-\rho}{\rho}\right)^\beta} R_*^{\beta-2}\right] > 0, \tag{5.6}$$

where

$$R_* = \left(\frac{2pE_*}{(p-2)\mu_*}\right)^{1/2}, \; E_* = \left(E(0) + \frac{1}{2}\rho_*\right)\exp\left(\rho_*\right), \; \rho_* = \|F\|_{L^1(\mathbb{R}_+;L^2)}.$$

Let $\mu^*_{\max} = \max\limits_{0\le z\le R_*^2} \mu(z) < \frac{p\mu_*}{d_2} + \eta^*$, *with* μ_*, d_2 *as in* (H_2^*), $(\overline{H}_3)(i)$.
Then, there exist positive constants $\bar{C}$, $\bar{\gamma}$ *such that*

$$\|u'(t)\|_0^2 + \|u_x(t)\|_0^2 \le \bar{C}\exp(-\bar{\gamma}t), \text{ for all } t \ge 0. \tag{5.7}$$

Proof of Theorem 5.1.
First, we need the following lemmas.
Lemma 5.2. *The energy functional* $E(t)$ *defined by* (5.3) *satisfies*

$$\begin{aligned} &\text{(i)} \;\; E'(t) \le \frac{1}{2}\|F(t)\|_0 + \frac{1}{2}\|F(t)\|_0\|u'(t)\|_0^2, \\ &\text{(ii)} \;\; E'(t) \le -\left(\lambda - \frac{\varepsilon_1}{2}\right)\|u'(t)\|_0^2 + \frac{1}{2\varepsilon_1}\|F(t)\|_0^2, \end{aligned} \tag{5.8}$$

for all $\varepsilon_1 > 0$.
Proof of Lemma 5.2. Multiplying (5.1) by $xu'(x,t)$ and integrating over $[\rho, 1]$, we get

$$E'(t) = -\lambda\|u'(t)\|_0^2 + \langle F(t), u'(t)\rangle. \tag{5.9}$$

On the other hand

$$\langle F(t), u'(t)\rangle \le \frac{1}{2}\|F(t)\|_0 + \frac{1}{2}\|F(t)\|_0\|u'(t)\|_0^2. \tag{5.10}$$

It follows from (5.9) and (5.10) that $(5.8)_{(i)}$ holds.
Similarly,

$$\langle F(t), u'(t)\rangle \le \frac{1}{2\varepsilon_1}\|F(t)\|_0^2 + \frac{\varepsilon_1}{2}\|u'(t)\|_0^2, \text{ for all } \varepsilon_1 > 0. \tag{5.11}$$

It follows from (5.9) and (5.11) that $(5.8)_{(ii)}$ holds.
Lemma 5.2 is proved completely. □
Lemma 5.3. *Assume that* (H_2^*), $(\overline{H}_3)$, $(\overline{H}_4)$ *hold. Let* $I(0) > 0$ *and* (5.6) *hold. Then* $I(t) > 0$, $\forall t \ge 0$.
Proof of Lemma 5.3. By the continuity of $I(t)$ and $I(0) > 0$, there exists $\tilde{T}_1 > 0$ such that

$$I(t) = I(u(t)) \ge 0, \; \forall t \in [0, \tilde{T}_1], \tag{5.12}$$

this implies

$$\begin{aligned} E(t) &\ge \frac{1}{2}\|u'(t)\|_0^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \\ &\ge \frac{1}{2}\|u'(t)\|_0^2 + \frac{(p-2)\mu_*}{2p}\|u_x(t)\|_0^2, \; \forall t \in [0, \tilde{T}_1]. \end{aligned} \tag{5.13}$$

Combining $(5.8)_i$, (5.13) and using Gronwall's inequality to obtain

$$\|u_x(t)\|_0^2 \le \frac{2p}{(p-2)\mu_*}E(t) \le \frac{2pE_*}{(p-2)\mu_*} \equiv R_*^2, \; \forall t \in [0, \tilde{T}_1], \tag{5.14}$$

where E_* as in (5.6).

Hence, it follows from $(\overline{H}_3)(ii)$ and (5.14) that

$$\begin{aligned} & p\int_{\rho}^{1} xdx \int_{0}^{u(x,t)} f(z)dz \\ & \leq p\bar{d}_2 \left(\|u(t)\|_{L^{\alpha}}^{\alpha} + \|u(t)\|_{L^{\beta}}^{\beta} \right) \\ & \leq p\bar{d}_2 (1-\rho) \left(\sqrt{\left(\frac{1-\rho}{\rho}\right)^{\alpha}} \|u_x(t)\|_0^{\alpha} + \sqrt{\left(\frac{1-\rho}{\rho}\right)^{\beta}} \|u_x(t)\|_0^{\beta} \right) \\ & \leq p\bar{d}_2 (1-\rho) \left[\sqrt{\left(\frac{1-\rho}{\rho}\right)^{\alpha}} R_*^{\alpha-2} + \sqrt{\left(\frac{1-\rho}{\rho}\right)^{\beta}} R_*^{\beta-2} \right] \|u_x(t)\|_0^2 . \end{aligned} \tag{5.15}$$

Therefore, $I(t) \geq \eta^* \|u_x(t)\|_0^2 > 0$, $\forall t \in [0, \tilde{T}_1]$, where η^* as in (5.6).

Now, we put $T_\infty = \sup\{T > 0 : I(t) > 0,\ \forall t \in [0, T]\}$. If $T_\infty < +\infty$, then by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$. By the same arguments as above, we can deduce that there exists $\tilde{T}_\infty > T_\infty$ such that $I(t) > 0$, $\forall t \in [0, \tilde{T}_\infty]$. Hence, we conclude that $I(t) > 0$, $\forall t \geq 0$.

Lemma 5.3 is proved completely. □

Lemma 5.4. *Assume that* (H_2^*), $(\overline{H}_3)$, $(\overline{H}_4)$ *hold. Let* $I(0) > 0$ *and* (5.6) *hold. Put*

$$E_1(t) = \|u'(t)\|_0^2 + \int_0^{\|u_x(t)\|_0^2} \mu(z)dz + I(t). \tag{5.16}$$

Then there exist the positive constants $\bar{\beta}_1, \bar{\beta}_2$ *such that*

$$\bar{\beta}_1 E_1(t) \leq \mathcal{L}(t) \leq \bar{\beta}_2 E_1(t),\ \forall t \geq 0, \tag{5.17}$$

for δ *is small enough.*

Proof of Lemma 5.4. It is easy to see that

$$\begin{aligned} \mathcal{L}(t) = & \frac{1}{2}\|u'(t)\|_0^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2} \mu(z)dz \\ & + \frac{1}{p}I(t) + \delta\langle u'(t), u(t)\rangle + \frac{\delta\lambda}{2}\|u(t)\|_0^2 . \end{aligned} \tag{5.18}$$

From the following inequalities

$$\begin{aligned} \delta\langle u'(t), u(t)\rangle & \leq \frac{1}{2}\delta\|u'(t)\|_0^2 + \frac{1}{2}\delta\|u(t)\|_0^2 \\ & \leq \frac{1}{2}\delta\|u'(t)\|_0^2 + \delta\frac{(1-\rho)^2}{4\rho}\|u_x(t)\|_0^2 , \end{aligned} \tag{5.19}$$

$$\int_0^{\|u_x(t)\|_0^2} \mu(z)dz \geq \mu_*\|u_x(t)\|_0^2 ,$$

we deduce that

$$
\begin{aligned}
\mathcal{L}(t) &\geq \frac{1}{2}\left\|u'(t)\right\|_0^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz + \frac{1}{p}I(t) + \delta\langle u'(t), u(t)\rangle \\
&\geq \frac{1}{2}\left\|u'(t)\right\|_0^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \\
&+ \frac{1}{p}I(t) - \frac{1}{2}\delta\left\|u'(t)\right\|_0^2 - \delta\frac{(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2 \\
&\geq \frac{1}{2}\left\|u'(t)\right\|_0^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \\
&+ \frac{1}{p}I(t) - \frac{1}{2}\delta\left\|u'(t)\right\|_0^2 - \delta\frac{(1-\rho)^2}{4\rho\mu_*}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \\
&= \frac{1-\delta}{2}\left\|u'(t)\right\|_0^2 + \left(\frac{1}{2}-\frac{1}{p}-\delta\frac{(1-\rho)^2}{4\rho\mu_*}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz + \frac{1}{p}I(t) \\
&\geq \bar{\beta}_1 E_1(t),
\end{aligned} \tag{5.20}
$$

where we choose $\bar{\beta}_1 = \min\{\frac{1-\delta}{2}, \frac{1}{2}-\frac{1}{p}-\delta\frac{(1-\rho)^2}{4\rho\mu_*}, \frac{1}{p}\}$, with δ is small enough, $0 < \delta < \min\{1; \dfrac{4\rho\mu_*}{(1-\rho)^2}\left(\frac{1}{2}-\frac{1}{p}\right)\}$.

Similarly, we can prove that

$$
\begin{aligned}
\mathcal{L}(t) &\leq \frac{1}{2}\left\|u'(t)\right\|_0^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz + \frac{1}{p}I(t) \\
&+ \frac{1}{2}\delta\left\|u'(t)\right\|_0^2 + \delta\frac{(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2 + \delta\lambda\frac{(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2 \\
&\leq \frac{1+\delta}{2}\left\|u'(t)\right\|_0^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz + \frac{1}{p}I(t) \\
&+ \frac{\delta(1+\lambda)(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2 \\
&\leq \frac{1+\delta}{2}\left\|u'(t)\right\|_0^2 + \left(\frac{1}{2}-\frac{1}{p}+\frac{\delta(1+\lambda)(1-\rho)^2}{4\rho\mu_*}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz + \frac{1}{p}I(t) \\
&\leq \bar{\beta}_2 E_1(t),
\end{aligned} \tag{5.21}
$$

where $\bar{\beta}_2 = \max\left\{\frac{1+\delta}{2}, \frac{1}{2}-\frac{1}{p}+\frac{\delta(1+\lambda)(1-\rho)^2}{4\rho\mu_*}\right\}$.

Lemma 5.4 is proved completely. □

Lemma 5.5. *Assume that* (H_2^*), $(\overline{H}_3)$, $(\overline{H}_4)$ *hold. Let* $I(0) > 0$ *and* (5.6) *hold. The functional* $\Psi(t)$ *defined by* (5.4) *satisfies*

$$
\begin{aligned}
\Psi'(t) &\leq \left\|u'(t)\right\|_0^2 - \frac{\delta_1 d_2}{p}I(t) + \frac{1}{2\varepsilon_2}\left\|F(t)\right\|_0^2 \\
&- \frac{\delta}{\mu_{\max}^*}\left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2}+\eta^*-\mu_{\max}^*\right) - \frac{\delta_1 d_2\eta^*}{p} - \varepsilon_2\frac{(1-\rho)^2}{4\rho}\right]\int_0^{\|u_x(t)\|_0^2}\mu(z)dz,
\end{aligned} \tag{5.22}
$$

for all $\varepsilon_2 > 0$, $\delta_1 \in (0,1)$.

Proof of Lemma 5.5. By multiplying (5.1) by $xu(x,t)$ and integrating over $[\rho, 1]$, we obtain

$$\Psi'(t) = \|u'(t)\|_0^2 - \|u_x(t)\|_0^2 \, \mu\left(\|u_x(t)\|_0^2\right) + \langle f(u(t)), u(t)\rangle + \langle F(t), u(t)\rangle. \tag{5.23}$$

By the following inequalities

$$\begin{aligned}
- \|u_x(t)\|_0^2 \, \mu\left(\|u_x(t)\|_0^2\right) &\leq -\mu_* \|u_x(t)\|_0^2, \qquad (5.24)\\
\langle f(u(t)), u(t)\rangle &\leq d_2 \int_\rho^1 x dx \int_0^{u(x,t)} f(z)dz\\
&= \frac{d_2}{p}\left[\int_0^{\|u_x(t)\|_0^2} \mu(z)dz - I(t)\right],\\
I(t) &\geq \eta^* \|u_x(t)\|_0^2,\\
\langle F(t), u(t)\rangle &\leq \frac{\varepsilon_2}{2}\|u(t)\|_0^2 + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2\\
&\leq \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2, \ \forall \varepsilon_2 > 0,
\end{aligned}$$

we deduce that

$$\begin{aligned}
\Psi'(t) &= \|u'(t)\|_0^2 - \|u_x(t)\|_0^2 \, \mu\left(\|u_x(t)\|_0^2\right) + \langle f(u(t)), u(t)\rangle + \langle F(t), u(t)\rangle \qquad (5.25)\\
&\leq \|u'(t)\|_0^2 - \mu_* \|u_x(t)\|_0^2 + \frac{d_2}{p}\left[\int_0^{\|u_x(t)\|_0^2} \mu(z)dz - I(t)\right]\\
&+ \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2\\
&= \|u'(t)\|_0^2 - \left(\mu_* - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right)\|u_x(t)\|_0^2 + \frac{d_2}{p}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz\\
&- \frac{\delta_1 d_2}{p} I(t) - \frac{(1-\delta_1)d_2}{p} I(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2\\
&\leq \|u'(t)\|_0^2 - \left(\mu_* - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right)\|u_x(t)\|_0^2 + \frac{d_2}{p}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz\\
&- \frac{\delta_1 d_2}{p} I(t) - \frac{(1-\delta_1)d_2}{p}\eta^* \|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2\\
&= \|u'(t)\|_0^2 - \left(\mu_* + \frac{(1-\delta_1)d_2}{p}\eta^* - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right)\|u_x(t)\|_0^2\\
&+ \frac{d_2}{p}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz - \frac{\delta_1 d_2}{p} I(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2\\
&\leq \|u'(t)\|_0^2 - \left(\mu_* + \frac{(1-\delta_1)d_2}{p}\eta^* - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right)\frac{1}{\mu_{\max}^*}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz\\
&+ \frac{d_2}{p}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz - \frac{\delta_1 d_2}{p} I(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2\\
&= \|u'(t)\|_0^2 - \frac{1}{\mu_{\max}^*}\left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu_{\max}^*\right) - \frac{\delta_1 d_2 \eta^*}{p} - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right]\int_0^{\|u_x(t)\|_0^2} \mu(z)dz\\
&- \frac{\delta_1 d_2}{p} I(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2.
\end{aligned}$$

Hence, the lemma 5.5 is proved by using some simple estimates. □

Now we continue to prove Theorem 5.1.

Then, we deduce from (5.2), (5.8)(*ii*) and (5.22) that

$$\begin{aligned} \mathcal{L}'(t) \le & -\left(\lambda - \frac{\varepsilon_1}{2} - \delta\right) \|u'(t)\|_0^2 - \frac{\delta\delta_1 d_2}{p} I(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|_0^2 \\ & - \frac{\delta}{\mu^*_{\max}}\left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu^*_{\max}\right) - \frac{\delta_1 d_2 \eta^*}{p} - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right]\int_0^{\|u_x(t)\|_0^2} \mu(z)dz \end{aligned} \tag{5.26}$$

for all $\delta, \varepsilon_1, \varepsilon_2 > 0$, $\delta_1 \in (0,1)$.

Because of $\mu^*_{\max} < \dfrac{p\mu_*}{d_2} + \eta^*$ and

$$\begin{aligned} & \lim_{\delta_1 \to 0_+,\, \varepsilon_2 \to 0_+} \left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu^*_{\max}\right) - \frac{\delta_1 d_2 \eta^*}{p} - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right] \\ & = \frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu^*_{\max}\right) > 0, \end{aligned}$$

we can choose $\delta_1 \in (0,1)$ and $\varepsilon_2 > 0$ such that

$$\theta_1 = \frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu^*_{\max}\right) - \frac{\delta_1 d_2 \eta^*}{p} - \varepsilon_2 \frac{(1-\rho)^2}{4\rho} > 0. \tag{5.27}$$

Then, for ε_1 small enough such that $0 < \dfrac{\varepsilon_1}{2} < \lambda$ and if $\delta > 0$ such that

$$\theta_2 = \lambda - \frac{\varepsilon_1}{2} - \delta > 0,\ 0 < \delta < \min\{1; \frac{4\rho\mu_*}{(1-\rho)^2}\left(\frac{1}{2} - \frac{1}{p}\right)\}, \tag{5.28}$$

it follows from (5.17), (5.26)-(5.28) that

$$\begin{aligned} \mathcal{L}'(t) & \le -\bar{\beta}_3 E_1(t) + \tilde{C}_0 e^{-2\bar{\gamma}_0 t} \\ & \le -\frac{\bar{\beta}_3}{\bar{\beta}_2}\mathcal{L}(t) + \tilde{C}_0 e^{-2\bar{\gamma}_0 t} \le -\bar{\gamma}\mathcal{L}(t) + \tilde{C}_0 e^{-2\bar{\gamma}_0 t}, \end{aligned} \tag{5.29}$$

where $\bar{\beta}_3 = \min\{\dfrac{\delta\theta_1}{\mu^*_{\max}}, \theta_2, \dfrac{\delta\delta_1 d_2}{p}\}$, $0 < \bar{\gamma} < \min\{\frac{\bar{\beta}_3}{\bar{\beta}_2}, 2\bar{\gamma}_0\}$, $\tilde{C}_0 = \dfrac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\bar{C}_0^2$.

On the other hand, we have

$$\mathcal{L}(t) \ge \bar{\beta}_1 \min\{1, \mu_*\}\left(\|u'(t)\|_0^2 + \|u_x(t)\|_0^2\right). \tag{5.30}$$

Combining (5.29) and (5.30) we get (5.7). Theorem 5.1 is proved completely. □

Acknowledgements. The authors wish to express their sincere thanks to the referees and the Editor for their valuable comments. This research is funded by Vietnam National University HoChiMinh City (VNU-HCM) under Grant no. **B2017-18-04**.

References

[1] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation, Adv. Differential Equations 6(6) (2001) 701–730.

[2] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation, Commun. Contemp. Math. 6 (5) (2004) 705–731.

[3] G. F. Carrier, On the nonlinear vibrations problem of elastic string, Quart. J. Appl. Math. 3 (1945) 157–165.

[4] Y. Ebihara, L. A. Medeiros, M. M. Miranda, Local solutions for a nonlinear degenerate hyperbolic equation, Nonlinear Anal. 10 (1986) 27–40.
[5] M. Hosoya, Y. Yamada, On some nonlinear wave equation I : Local existence and regularity of solutions, J. Fac. Sci. Univ. Tokyo. Sect. IA, Math. 38 (1991) 225–238.
[6] G. R. Kirchhoff, Vorlesungen ber Mathematische Physik: Mechanik, Teuber, Leipzig, 1876, Section 29.7.
[7] N. A. Larkin, Global regular solutions for the nonhomogeneous Carrier equation, Mathematical Problems in Engineering, 8 (2002) 15–31.
[8] Lasiecka, I, Ong, J, Global solvability and uniform decays of solutions to quasilinear equation with nonlinear boundary dissipation, Comm. Partial Differential Equations, 24 (11-12) (1999) 2069–2108.
[9] J. L. Lions, Quelques mthodes de rsolution des problems aux limites non-linares, Dunod; Gauthier - Villars, Paris, 1969.
[10] N. T. Long, On the nonlinear wave equation $u_{tt} - B\left(t, \|u\|^2, \|u_x\|^2\right) u_{xx} = f\left(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2\right)$ associated with the mixed homogeneous conditions, J. Math. Anal. Appl. 306 (1) (2005) 243–268.
[11] N. T. Long, A. P. N. Dinh, T. N. Diem, Linear recursive schemes and asymptotic expansion associated with the Kirchhoff-Carrier operator, J. Math. Anal. Appl. 267 (1) (2002) 116–134.
[12] N. T. Long, L. T. P. Ngoc, On a nonlinear wave equation with boundary conditions of two-point type, J. Math. Anal. Appl. 385 (2) (2012) 1070–1093.
[13] Gongwei Liu, On global solution, energy decay and blow-up for 2-D Kirchhoff equation with exponential terms, Boundary Value Problems 2014, 2014:230.
[14] L. A. Medeiros, On some nonlinear perturbation of Kirchhoff-Carrier operator, Comp. Appl. Math. 13 (1994) 225–233.
[15] Miranda, M. Milla, Jutuca, L. P. San Gil, Existence and boundary stabilization of solutions for the Kirchhoff equation, Comm. Partial Differential Equations, 24 (9-10) (1999) 1759–1800.
[16] G. P. Menzala, On global classical solutions of a nonlinear wave equation, Appl. Anal. 10 (1980) 179–195.
[17] Messaoudi S. A, Blow up and global existence in a nonlinear viscoelastic wave equation, Mathematische Nachrichten, 260 (2003) 58–66.
[18] L. T. P. Ngoc, N. A. Triet, N. T. Long, On a nonlinear wave equation involving the term $-\frac{\partial}{\partial x}\left(\mu\left(x, t, u, \|u_x\|^2\right) u_x\right)$: Linear approximation and asymptotic expansion of solution in many small parameters, Nonlinear Anal. RWA. 11 (4) (2010) 2479–2510.
[19] L. T. P. Ngoc, N. T. Long, Linear approximation and asymptotic expansion of solutions in many small parameters for a nonlinear Kirchhoff wave equation with mixed nonhomogeneous conditions, Acta Applicanda Mathematicae, 112 (2) (2010) 137–169.
[20] L. T. P. Ngoc, N. T. Long, Existence, blow-up and exponential decay for a nonlinear Love equation associated with Dirichlet conditions, Applications of Mathematics, 61 (2) (2016) 165–196.
[21] L. T. P. Ngoc, L. H. K. Son, T. M. Thuyet, N. T. Long, Linear approximation and asymptotic expansion of solutions for a nonlinear Carrier wave equation in an annular membrane with Robin-Dirichlet conditions, Mathematical Problems in Engineering, Vol. 2016 (2016), Article ID 8031638, 18 pages.
[22] L. T. P. Ngoc, N. A. Triet, A. P. N. Dinh, N. T. Long, Existence and exponential decay of solutions for a wave equation with integral nonlocal boundary conditions of memory type, Numerical Functional Analysis and Optimization, 38 (9) (2017) 1173–1207.
[23] J. Y. Park, J. J. Bae, I. H. Jung, Uniform decay of solution for wave equation of Kirchhoff type with nonlinear boundary damping and memory term, Nonlinear Anal. TMA. 50 (2002) 871–884.
[24] J. Y. Park, J. J. Bae, On coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term, Applied Math. Comput. 129 (2002) 87–105.
[25] T. N. Rabello, M. C. C. Vieira, C. L. Frota, L. A. Medeiros, Small vertical vibrations of strings with moving ends, Rev. Mat. Complutent 16 (2003) 179–206.
[26] M. L. Santos, J. Ferreira, D. C. Pereira, C. A. Raposo, Global existence and stability for wave equation of Kirchhoff type with memory condition at the boundary, Nonlinear Anal. TMA. 54 (2003) 959–976.
[27] R. E. Showater, Hilbert space methods for partial differential equations, Electronic J. Differential Equations, Monograph 01, 1994.
[28] L. X. Truong, L. T. P. Ngoc, A. P. N. Dinh, N. T. Long, Existence, blow-up and exponential decay estimates for a nonlinear wave equation with boundary conditions of two-point type, Nonlinear Anal. TMA. 74 (18) (2011) 6933–6949.