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The Geometry of Contact Pseudo-Slant Submanifolds of a Sasakian Manifold

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Abstract. In this paper, we study the geometry of the contact pseudo-slant submanifolds of a Sasakian manifold. We verify some properties of the components of the tensor field acting on that kind of submanifold and find out the necessary and sufficient conditions for them to be parallel. Also, necessary and sufficient conditions are given for a submanifold to be a pseudo-slant submanifold, contact pseudo-slant product, D^θ, $D^\perp$ and mixed-geodesic in Sasakian manifold.

1. Introduction

B.Y. Chen [5, 6] in 1990 initiated the study of slant submanifolds of an almost Hermitian manifold as a natural generalization of both invariant and anti-invariant submanifolds. In 1994, N. Papaghiuc [12] introduced semi-slant submanifolds in an almost Hermitian manifold, which includes the class of proper CR-submanifolds and slant submanifolds. A. Lotta [3] extended the idea of slant immersions in the setting of almost contact metric manifold in 1996. Then several works have been done on these submanifolds in various known spaces.

As a special case of bi-slant submanifolds, A. Carriazo [2] introduced the notion of pseudo-slant submanifolds of an almost Hermitian manifold. Then in 2007, the contact version of pseudo-slant submanifolds was defined and studied by V.A. Khan and M.A. Khan [17]. The idea of such submanifolds in (LCS)n-manifolds was elaborated by M. Atceken and S.K. Hui [10] in 2013. Recently M. Atceken with S. Dirik has worked on the geometry of contact pseudo-slant submanifolds of Kenmotsu manifold and Cosymplectic manifold [9–11, 13].

2. Preliminaries

Given an odd-dimensional Riemannian manifold $(\widetilde{M}, g)$, let φ be a $(1,1)$-type tensor field, ξ is a unit vector field and η is a 1-form on $\widetilde{M}$. If we have

$$\varphi^2 U = -U + \eta(U)\xi, \quad g(U,\xi) = \eta(U) \tag{1}$$

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and

$$g(\varphi U, \varphi Y) = g(U, Y) - \eta(U)\eta(Y) \tag{2}$$

for any vector fields on $\widetilde{M}$, then $\widetilde{M}$ is said to be have an almost contact metric structure (φ, ξ, η, g) and it is called an almost contact metric manifold.

Let Φ denote the fundamental form 2-form in $\widetilde{M}$, given by $\Phi(U, Y) = g(U, \varphi Y)$, for any vector fields U, Y on $\widetilde{M}$. If $\Phi = d\eta$, then $\widetilde{M}$ is said to be a contact metric manifold. Furthermore, the contact metric structure is called a K-contact structure if ξ is a Killing vector field, that is, $\widetilde{\nabla}_U \xi = -\varphi U$, for any vector field U on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Levi-Civita connection on $\widetilde{M}$.

The structure (φ, ξ, η, g) is said to be normal if $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A normal contact metric manifold is called a Sasakian manifold. So every Sasakian manifold is a K-contact manifold. It is well-know that an almost contact metric manifold is a Sasakian if and only if

$$(\widetilde{\nabla}_U \varphi)Y = g(U, Y)\xi - \eta(Y)U, \tag{3}$$

for any vector fields U, Y on $\widetilde{M}$.

Now, let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ with the induced metric g. Also, let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then the Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{5}$$

where h and A_N are, respectively, the second fundamental form and the shape operator (corresponding to the normal vector field N) for the submanifold of M into $\widetilde{M}$. The second fundamental form h and shape operator A_N are related by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{6}$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$. If $h(X, Y) = 0$, for each $X, Y \in \Gamma(TM)$ then M is said to be totally geodesic submanifold.

Now, let M be a submanifold of an almost contact metric manifold $\widetilde{M}$, then for any $X \in \Gamma(TM)$, we can write

$$\varphi X = TX + FX, \tag{7}$$

where TX is the tangential component and FX is the normal component of φX. Similarly for $N \in \Gamma(T^{\perp}M)$, we can write

$$\varphi N = BN + CN, \tag{8}$$

where BN is the tangential component and CN is also the normal component of φN. Thus by using (1), (7) and (8), we obtain

$$T^2 = -I + \eta \otimes \xi - BF, \quad FT + CF = 0 \tag{9}$$

and

$$C^2 = -I - FB, \quad TB + BC = 0. \tag{10}$$

Furthermore, the covariant derivatives of the tensor field T, F, B and C are, respectively, defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y \tag{11}$$

$$(\nabla_X F)Y = \nabla^\perp_X FY - F\nabla_X Y \tag{12}$$

$$(\nabla_X B)N = \nabla_X BN - B\nabla^\perp_X N \tag{13}$$

and

$$(\nabla_X C)N = \nabla^\perp_X CN - C\nabla^\perp_X N. \tag{14}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have

$$g(TX, Y) = -g(X, TY), \tag{15}$$

$N, U \in \Gamma(T^\perp M)$, we get

$$g(U, CN) = -g(CU, N). \tag{16}$$

These show that T and C are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, we have

$$g(FX, N) = -g(X, BN), \tag{17}$$

which gives the relation between F and B.

A submanifold M is said to be *invariant* if F is identically zero, that is, $\varphi X \in \Gamma(TM)$ for all $X \in \Gamma(TM)$. On the other hand, M is said to be *anti- invariant* if T is identically zero, that is, $\varphi X \in \Gamma(T^\perp M)$ for all $X \in \Gamma(TM)$. By direct calculations, we obtain the following formulas;

$$(\nabla_X T)Y = A_{FY}X + Bh(X, Y) + g(X, Y)\xi - \eta(Y)X \tag{18}$$

and

$$(\nabla_X F)Y = Ch(X, Y) - h(X, TY). \tag{19}$$

Similarly, for any $N \in \Gamma(T^\perp M)$ and $X \in \Gamma(TM)$, we obtain

$$(\nabla_X B)N = A_{CN}X - TA_N X \tag{20}$$

and

$$(\nabla_X C)N = -h(BN, X) - FA_N X. \tag{21}$$

Since ξ tangent to M, making use of $\widetilde{\nabla}_X \xi = -\varphi X$, (4), (5) and (6) we obtain

$$\nabla_X \xi = -TX, \quad h(X, \xi) = -FX, \quad A_N \xi = BN, \tag{22}$$

for all $N \in \Gamma(T^\perp M)$ and $X \in \Gamma(TM)$.

In contact geometry, A. Lotta introduced slant submanifolds as follows:
Let M be a submanifold of an almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$. Then M is said to be a contact slant submanifold if the angle $\theta(X)$ between φX and $T_M(x)$ is constant at any point $x \in M$ for any X linearly independent of ξ. Thus the invariant and anti-invariant submanifolds are special class of slant submanifolds with slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. If the slant angle θ is neither zero nor $\frac{\pi}{2}$, then slant submanifold is said to be proper contact slant submanifold. The slant submanifolds of an almost contact metric manifold, the following theorem is well known [3].

For slant submanifolds of contact manifolds J.L. Cabrerizo et al. proved the following theorem.

Theorem 2.1. *[7]. Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM)$. Then M is slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi), \tag{23}$$

furthermore, if θ is the slant angle of M, then $\lambda = \cos^2 \theta$.

Corollary 2.2. *[7]. Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ with slant angle θ. Then for any $X, Y \in \Gamma(TM)$, we have*

$$g(TX, TY) = \cos^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\} \tag{24}$$

and

$$g(FX, FY) = \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\}. \tag{25}$$

3. Contact Pseudo-Slant Submanifolds of a Sasakian Manifold

In this section, we study on some geometric properties of contact pseudo-slant submanifolds of a Sasakian manifold. Necessary and sufficient conditions are given for a submanifold to be a contact pseudo-slant submanifold, contact pseudo-slant product, mixed geodesic, D^θ and $D^\perp$- geodesic in Sasakian manifolds.

Definition 3.1. *[17]. Let M be a submanifold of a Sasakian manifold $\widetilde{M}$. M is said to be* contact pseudo-slant submanifold *of $\widetilde{M}$ if there exist two orthogonal distributions $D^\perp$ and D^θ on M such that:*
(i) TM has the orthogonal direct decomposition $TM = D^\perp \oplus D^\theta$, $\xi \in \Gamma(D_\theta)$.
(ii) The distribution $D^\perp$ is an anti-invariant i.e., $\varphi(D^\perp) \subset T^\perp M$.
(iii) The distribution D^θ is a slant, that is, the slant angle between of D^θ and $\varphi(D^\theta)$ is a constant.

Let p =dim($D^\perp$) and q=dim(D^θ. Then we have the following cases: .
(i) If $q = 0$, then M is an anti-invariant submanifold.
(ii) If $p = 0$ and $\theta = 0$, then M is invariant submanifold.
(iii) If $p = 0$ and $0 < \theta < \frac{\pi}{2}$, then M is a proper slant submanifold.
(iv) If $\theta = \frac{\pi}{2}$ then, M is an anti-invariant submanifold.
(v) If $pq \neq 0$ and $\theta = 0$, then M is a semi-invariant submanifold.
(vi) If $pq \neq 0$ and $0 < \theta < \frac{\pi}{2}$, then M is a contact pseudo-slant submanifold.

For a contact pseudo-slant submanifold M of a Sasakian manifold $\bar{M}$, the normal bundle $T^\perp M$ of a contact pseudo-slant submanifold M is decomposable as

$$T^\perp M = F(D^\perp) \oplus F(D^\theta) \oplus \mu, \quad F(D^\perp) \perp F(D^\theta). \tag{26}$$

Moreover, for any $Z, Y \in \Gamma(D^\perp)$ and $U \in \Gamma(TM)$, also by using (1), (3), (4) and (6), we have

$$\begin{aligned} g(A_{FZ}Y - A_{FY}Z, U) &= g(h(Y, U), FZ) - g(h(Z, U), FY) \\ &= g(\widetilde{\nabla}_U Y, \varphi Z) - g(\widetilde{\nabla}_U Z, \varphi Y) \\ &= g(\phi\widetilde{\nabla}_U Z, Y) - g(\widetilde{\nabla}_U \varphi Z, Y) \\ &= -g((\widetilde{\nabla}_U \varphi)Z, Y) \\ &= -g(U, Z)\eta(Y) + \eta(Z)g(U, Y) = 0. \end{aligned}$$

It follows that

$$A_{FZ}Y = A_{FY}Z, \tag{27}$$

for any $Z, Y \in \Gamma(D^\perp)$.

Theorem 3.2. *Let M be a contact pseudo-slant submanifold in Sasakian manifold $\widetilde{M}$ such that $\xi \in \Gamma(D^\theta)$. Then we have*

$$g(A_{F(D^\perp)}D^\theta - T\nabla_{D^\theta}D^\perp - Bh(D^\theta, D^\perp), D^\theta) = 0. \tag{28}$$

Proof. For any $Z \in \Gamma(D^\perp)$ and $X, U \in \Gamma(D^\theta)$ from (1), (3), (4), (5), (15) and (17), we obtain

$$\begin{aligned} g(A_{\varphi Z}U, X) &= -g(\widetilde{\nabla}_U \varphi Z, X) = -g((\widetilde{\nabla}_U \varphi)Z + \varphi\widetilde{\nabla}_U Z, X) \\ &= -g(g(U, Z)\xi + \eta(Z)U, X) - g(\widetilde{\nabla}_U Z, \varphi X) \\ &= -g(\nabla_U Z + h(U, Z), \varphi X) \\ &= -g(\nabla_U Z, TX) - g(h(U, Z), FX) \\ &= g(T\nabla_U Z, X) + g(Bh(U, Z), X). \end{aligned}$$

This proves our assertion. □

Theorem 3.3. *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\widetilde{M}$. Then the tensor F is parallel iff B is parallel.*

Proof. For any $X, Y \in \Gamma(TM)$ from (19)

$$(\nabla_X F)Y = Ch(X, Y) - h(X, TY).$$

Taking inner product with respect to $N \in \Gamma(T^\perp M)$. On both sides, we get from (1),(6),(15),(16), (19) and (20)

$$\begin{aligned} g((\nabla_X F)Y, N) &= g(Ch(X, Y) - h(X, TY), N) \\ &= g(Ch(X, Y), N) - g(h(X, TY), N) \\ &= -g(h(X, Y), CN) - g(A_N X, TY) \\ &= g(TA_N X, Y) - g(A_{CN} X, Y) \\ &= g(TA_N X - A_{CN} X, Y) \\ &= -g((\nabla_X B)N, Y) \end{aligned} \tag{29}$$

this proves our assertion. □

Theorem 3.4. *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\widetilde{M}$. Then the covariant derivation of C is skew-symmetric.*

Proof. We know from (21)

$$g((\nabla_X C)U, N) = g(-h(BU, X) - FA_U X, N). \tag{30}$$

Again from (6), we have

$$\begin{aligned} g((\nabla_X C)U, N) &= -g(A_N X, BU) + g(A_U X, N) \\ &= g(BA_N X, U) + g(h(X, BN), U) \\ &= g(BA_N X + h(X, BN), U) \\ &= -g((\nabla_X C)N, U) \end{aligned} \tag{31}$$

for any $X \in \Gamma(TM)$ and $U, N \in \Gamma(T^\perp M)$. So we can say covariant derivation of C is skew-symmetric. □

Definition 3.5. *A contact pseudo-slant submanifold M of Sasakian manifold $\widetilde{M}$ is said to be D_θ*-geodesic *$h(X, Y) = 0$ for all $X, Y \in \Gamma(D^\theta)$. M is said to be $D^\perp$-geodesic when $h(U, W) = 0$ for all $U, W \in \Gamma(D^\perp)$. We call M as mixed-geodesic submanifold $h(X, Z) = 0$ for all $X \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$.*

Theorem 3.6. *Let M be a proper contact pseudo slant submanifold a Sasakian manifold $\widetilde{M}$. If B is parallel, then either M is a mixed geodesic or an anti-invariant submanifold.*

Proof. By theorem 3.3, if B is parallel then F parallel, from (19),(20), we obtain

$$Ch(X,Z) = 0 \tag{32}$$

for any $X \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$. Replacing X and Z in (19) and taking into account of F being parallel, we have

$$Ch(X,Z) - h(Z,TX) = 0.$$

Thus we have

$$h(Z,TX) = 0. \tag{33}$$

Replacing TX in place of X in (33) we get

$$h(Z,T^2X) = -cos^2\theta h(X,Z) = 0.$$

So, $cos^2\theta = 0$ or $h(X,Z) = 0$. If $cos^2\theta = 0$, then $\theta = \frac{\Pi}{2}$, so M is anti invariant submanifold. If $h(X,Z) = 0$, then M is a mixed geodesic submanifold. □

Theorem 3.7. *Let M be a proper contact pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. If B is parallel, then either M is a $D^\perp$ -geodesic or an anti-invariant submanifold of $\widetilde{M}$.*

Proof. If B is parallel, then making use of (20), we obtain

$$A_{CFY}Z - TA_{FY}Z = 0,$$

for any $Y, Z \in \Gamma(D^\perp)$, which implies that

$$TA_{FY}Z = 0.$$

This tell us that M is either anti-invariant or $A_{FY}Z = 0$. So we obtain

$$g(h(Z,W),FY) = 0,$$

for any $W \in \Gamma(D^\perp)$. Also by using (20), we conclude that

$$g(A_{CN}Z,Y) - g(TA_NZ,Y) = g(h(Y,Z),CN) = 0,$$

for any $N \in \Gamma(T^\perp M)$. This tells us that M is either $D^\perp$-geodesic or it is an anti-invariant submanifold. □

Theorem 3.8. *Let M be a proper contact pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. If F is parallel on D^θ, then either M is a $D^\theta-$ geodesic submanifold or $h(X,Y)$ is an eigenvector of C^2 with eigenvalue $-\cos^2\theta$, for any $X,Y\in \Gamma(D^\theta)$.*

Proof. If F is parallel, for any $X, Y \in \Gamma(D^\theta)$, from (19), we have

$$Ch(X,Y) - h(X,TY) = 0. \tag{34}$$

On the other hand, since D^θ is a slant distribution, we obtain

$$Ch(X,Y - \eta(Y)\xi) - h(X,T(Y - \eta(Y)\xi)) = 0$$

that is,

$$Ch(X,Y - \eta(Y)\xi) = h(X,TY). \tag{35}$$

Now, applying C to (35), we have

$$C^2h(X,Y - \eta(Y)\xi) = Ch(X,TY).$$

On the other hand, by interchanging of Y and TY in (34), we have

$$Ch(X,TY) = h(X,T^2Y).$$

Hence, by using (23), we obtain

$$C^2h(X,Y-\eta(Y)\xi) = Ch(X,TY) = h(X,T^2Y) = -\cos^2\theta h(X,Y-\eta(Y)\xi).$$

This implies that either h vanishes on D^θ or h is an eigenvector of C^2 with eigenvalue $-\cos^2\theta$. □

Theorem 3.9. *Let M be a proper contact pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. If F is parallel, then either M is a mixed-geodesic or an anti-invariant submanifold.*

Proof. From (19) we obtain

$$Ch(X,Y) = 0,$$

for any $X \in \Gamma(D^\theta)$ and $Y \in \Gamma(D^\perp)$. Replacing X by Y in (19) and taking into account of F being parallel, we have

$$Ch(X,Y) - h(Y,TX) = 0.$$

Thus we have

$$h(Y,TX) = 0,$$

which is equivalent to

$$h(Y,T^2X) = -\cos^2\theta h(X,Y) = 0.$$

This proves our assertion. □

Theorem 3.10. *Let M be a contact pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. Then the anti-invariant distribution $D^\perp$ defines totally geodesic foliation in M if and only if*

$$-A_{FZ}TX + A_{FTX}Z \in \Gamma(D^\theta), \tag{36}$$

for any $X \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$.

Proof. For any $X \in \Gamma(D^\theta)$ and $Y, Z \in \Gamma(D^\perp)$, we have

$$\begin{aligned}
g(\nabla_Y Z, X) &= g(\varphi\nabla_Y Z, \varphi X) + \eta(\nabla_Y Z)\eta(X)\\
&= g(\varphi\nabla_Y Z, \varphi X) - g(\nabla_Y \xi, Z)\eta(X)\\
&= g(\varphi\nabla_Y Z, \varphi X) + g(TY, Z)\eta(X)\\
&= g(\varphi(\widetilde{\nabla}_Y Z - h(Y,Z)), \varphi X)\\
&= g(\varphi\widetilde{\nabla}_Y Z, \varphi X) - g(\varphi h(Y,Z)), \varphi X)\\
&= g(\varphi\widetilde{\nabla}_Y Z, \varphi X) = g(\widetilde{\nabla}_Y \varphi Z - (\widetilde{\nabla}_Y \varphi)Z, \varphi X)\\
&= g(\widetilde{\nabla}_Y \varphi Z, \varphi X) - g(g(Y,Z)\xi - \eta(Z)Y, \varphi X)\\
&= g(\widetilde{\nabla}_Y \varphi Z, \varphi X).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
g(\nabla_Y Z, X) &= g(\widetilde{\nabla}_Y \varphi Z, \varphi X) = g(\widetilde{\nabla}_Y \varphi Z, FX) + g(\widetilde{\nabla}_Y \varphi Z, TX)\\
&= -g(A_{\varphi Z}TX, Y) - g(\widetilde{\nabla}_Y Z, BFX) - g(\widetilde{\nabla}_Y Z, CFX)\\
&= -g(A_{\varphi Z}TX, Y) + \sin^2\theta g(\nabla_Y Z, X - \eta(X)\xi) + g(A_{FTX}Y, Z),
\end{aligned}$$

that is,

$$\cos^2\theta g(\nabla_Y Z, X) = g(A_{FTX}Z - A_{\varphi Z}TX, Y).$$

This proves our assertion. □

Theorem 3.11. *Let M be a contact pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. The slant distribution D^θ defines totally geodesic foliation in M is if and only if*

$$A_{\varphi Z}TY - A_{FTY}Z \in \Gamma(D^\perp), \tag{37}$$

for any $Y \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$.

Proof. By using, (2), (4) and (5), we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\varphi\nabla_X Y, \varphi Z) + \eta(\nabla_X Y)\eta(Z) \\ &= g(\varphi\nabla_X Y, \varphi Z) \\ &= g(\varphi\widetilde{\nabla}_X Y, \varphi Z) - g(\varphi h(X, Y)), \varphi Z) \\ &= g(\widetilde{\nabla}_X \varphi Y - (\widetilde{\nabla}_X \varphi)Y, \varphi Z) \\ &= g(\widetilde{\nabla}_X \varphi Y, \varphi Z) - g(g(X, Y)\xi - \eta(Y)X, \varphi Z) \\ &= g(\widetilde{\nabla}_X \varphi Y, \varphi Z) = -g(\widetilde{\nabla}_X \varphi Z, \varphi Y). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= -g(\widetilde{\nabla}_X \varphi Z, \varphi Y) \\ &= -g(\widetilde{\nabla}_X \varphi Z, TY) - g(\widetilde{\nabla}_X \varphi Z, FY) \\ &= g(A_{\varphi Z}TY, X) + g(\widetilde{\nabla}_X Z, BFY) + g(\widetilde{\nabla}_X Z, CFY) \\ &= g(A_{\varphi Z}TY, X) - \sin^2\theta g(\widetilde{\nabla}_X Z, Y - \eta(Y)\xi) + g(A_{CFY}X, Z), \end{aligned}$$

for any $X, Y \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$. This implies that

$$\cos^2\theta g(\nabla_X Y, Z) = g(A_{\varphi Z}TY - A_{FTY}Z, X).$$

□

Definition 3.12. *Let M be a contact pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. M is said to be contact pseudo-slant product if the distributions $D^\perp$ and D^θ are totally geodesic in M [10].*

From Theorems 3.10 and 3.11, we have the following statement.

Proposition 3.13. *Let M be a contact proper pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. Then M is a contact pseudo-slant product if and only if and only if the shape operator of M satisfies*

$$A_{\varphi(D^\perp)}T(D^\theta) = A_{FT(D^\theta)}D^\perp. \tag{38}$$

Theorem 3.14. *Let M be a contact pseudo-slant submanifold of a Sasakian manifold $\widetilde{M}$. Then M is a contact pseudo-slant product if and only if the shape operator of M satisfies*

$$A_{F(D^\perp)}T(D^\theta) = A_{FT(D^\theta)}D^\perp. \tag{39}$$

Proof. By using (18), we have

$$\nabla_X TY - T\nabla_X Y = A_{FY}X + Bh(X,Y) + g(X,Y)\xi - \eta(Y)X,$$

for any $X, Y \in \Gamma(D_\theta)$. This implies that

$$g(\nabla_X TY, W) = g(A_{FY}X, W) + g(Bh(X,Y), W), \tag{40}$$

for any $W \in \Gamma(D^\perp)$. Replacing Y by PY in (40) and taking into account of (24), we obtain

$$\cos^2\theta g(\nabla_X Y, W) = g(A_{FW}TY - A_{FTY}W, X). \tag{41}$$

Also, from (11) and (18) we have

$$-T\nabla_W U = A_{FU}W + Bh(W,U) + g(W,U)\xi,$$

for any $U, W \in \Gamma(D^\perp)$, from which

$$-g(T\nabla_W U, TX) = g(A_{FU}W, TX) + g(Bh(W,U), TX),$$

that is,

$$-\cos^2\theta g(\nabla_W U, X) = g(A_{FU}TX - A_{FTX}U, W), \tag{42}$$

for any $X \in \Gamma(D_\theta)$. (41) and (42) imply that (39). □

Theorem 3.15. *Let M be a proper contact pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. If the tensor field B is parallel, then M is a contact pseudo-slant product.*

Proof. Since B is parallel, from Theorem 3.3., (19) and (20), we have

$$TA_{FZ}U = 0, \quad U \in \Gamma(TM), \quad Z \in \Gamma(D^\perp).$$

This implies that $A_{\varphi Z}U \in \Gamma(D^\perp)$ and $Bh(U,Z) = 0$. The proof is completes. □

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