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Solvability of Infinite Systems of Fractional Differential Equations in the Spaces of Tempered Sequences

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Abstract. In this paper we discuss the existence of solution of infinite systems of fractional differential equations with the help of Hausdorff measure of noncompactness and Meir–Keeler fixed point theorem in the tempered sequence spaces. We provide examples to established the applicability of our results.

1. Introduction and Definitions

The fractional differential equations describe many phenomena in the fields of engineering, physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetic and rheology etc. The fractional differential equations have important tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations. For different types of applications of fractional differential equations we refer [2, 3, 12] and references therein.

The theory of infinite systems of ordinary differential equations is a very important branch of the theory of differential equations in Banach spaces. Infinite systems of ordinary differential equations describes many real life problems which can found in the theory of neural nets, the theory of branching processes and mechanics etc (see [9, 11, 19]).

In functional analysis the measure of noncompactness play important role which was introduced by Kuratowski [13]. The idea of measure of noncompactness has been used by many authors in obtaining the existence of solutions of infinite systems of integral equations and differential equations (see [8]). Mursaleen and Mohiuddine [16] proved existence theorems for the infinite systems of differential equations in the space ℓ_p. On the other hand, existence theorems for the infinite systems of linear equations in ℓ_1 and ℓ_p was discussed by Alotaibi et al. [5]. Mursaleen and Alotaibi [18] proved existence theorems for the infinite systems of differential equations in some BK-spaces. Mursaleen et al. [15] proved the existence of infinite systems of fractional differential equations in the spaces c_0 and ℓ_p. Srivastava et al. [20] studied the existence

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of solutions of infinite systems of n^{th} order differential equations in the spaces c_0 and ℓ_1 via the measure of noncompactness.

Definition 1.1. *[13] Let (X, d) be a metric space and Q a bounded subset of X. Then the Kuratowski measure of noncompactness (α-measure or set measure of non-compactness) of Q, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $\epsilon > 0$, that is,*

$$\alpha(Q) = \inf\left\{\epsilon > 0 : Q \subset \cup_{i=1}^{n} S_i, S_i \subset X, diam(S_i) < \epsilon \ (i = 1, 2, ..., n) \ , n \in \mathbb{N}\right\}$$

The function α is called Kuratowski measure of noncompactness. It was introduced by Kuratowski [13].

Clearly

$$\alpha(Q) \leq \text{diam}(Q) \text{ for each bounded subset } Q \text{ of } X.$$

Suppose E is a real Banach space with the norm $\| \, . \, \|$. Let $B(x_0, r)$ be a closed ball in E centered at x_0 and with radius r. If X is a nonempty subset of E then by $\bar{X}$ and Conv(X) we denote the closure and convex closure of X. Moreover let $\mathcal{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.

Definition 1.2. *[8] A function $\mu : \mathcal{M}_E \to [0, \infty)$ is called a measure of noncompactness if it satisfies the following conditions:*

(i) *the family* $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker \mu \subset \mathcal{N}_E$.
(ii) $X \subset Y \implies \mu(X) \leq \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(X)$.
(iv) $\mu(ConvX) = \mu(X)$.
(v) $\mu(\lambda X + (1 - \lambda) Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y)$ *for* $\lambda \in [0, 1]$.
(vi) *if* $X_n \in \mathcal{M}_E$, $X_n = \bar{X}_n$, $X_{n+1} \subset X_n$ *for* $n = 1, 2, 3, ...$ *and* $\lim_{n\to\infty} \mu(X_n) = 0$ *then* $\bigcap_{n=1}^{\infty} X_n \neq \phi$.

The family $\ker\mu$ is said to be the *kernel of measure* μ. A measure μ is said to be the sublinear if it satisfies the following conditions:

(1) $\mu(\lambda X) = |\lambda| \mu(X)$ for $\lambda \in \mathbb{R}$.
(2) $\mu(X + Y) \leq \mu(Y) + \mu(Y)$.

A sublinear measure of noncompactness μ satisfying the condition:

$$\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$$

and $ker\mu = \mathcal{N}_E$ is said to be regular.

Definition 1.3. *[8] Let (X, d) be a metric space, Q be a bounded subset of X and $B(x, r) = \{y \in X : d(x, y) < r\}$. Then the Hausdorff measure of noncompactness $\chi(Q)$ of Q is defined by*

$$\chi(Q) := \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon \quad (i = 1, 2, \ldots, n), n \in \mathbb{N}\right\}.$$

The definition of the Hausdorff measure of noncompactness of the set Q it is not supposed that centers of the balls that cover Q belong to Q. Hence it can equivalently be stated as follows:

$$\chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon - \text{net in } X\}.$$

Consider the following sequence spaces, which are Banach spaces with their respective norms

$$c_0 = \left\{x \in \omega : \lim_{k\to\infty} x_k = 0, \ \| x \|_{c_0} = \sup_k |x_k|\right\}$$

the space of all null sequences and

$$c = \left\{ x \in \omega : \lim_{k\to\infty} x_k = l,\ l \in \mathbb{C}\ , \| x \|_c = \sup_k |x_k| \right\}$$

the space of all convergent sequences.
In [8], the Hausdorff measure of noncompactness χ in the Banach space $(c_0, \| \,.\, \|_{c_0})$ is defined by

$$\chi(B) = \lim_{n\to\infty} \left[\sup_{u\in B} \left(\max_{k\geq n} | u_k | \right) \right], \text{ where } B \in \mathcal{M}_{c_0}. \tag{1}$$

In [17], the most convenient measure of noncompactness μ for the Banach space $(c, \| \,.\, \|_c)$ is defined by

$$\mu(B) = \lim_{p\to\infty} \left[\sup_{u\in B} \left\{ \sup_{k\geq p} | u_k - \lim_{m\to\infty} u_m | \right\} \right], \tag{2}$$

where $B \in \mathcal{M}_c$. The measure μ is regular.
Recently Banaś and Krajewska [7] have introduced tempering sequence and space of tempered sequences. Let us fix a positive non increasing real sequence $\beta = (\beta_n)_{n=1}^{\infty}$, such a sequence is called the tempering sequence.
Let the set X consisting of all real (or complex) sequences $x = (x_n)_{n=1}^{\infty}$ such that $\beta_n x_n \to 0$ as $n \to \infty$. It is obvious that X forms a linear space over the field of real (or complex) numbers. We denote the space by c_0^β. It is easy to see that c_0^β is a Banach space with the norm

$$\| x \|_{c_0^\beta} = \sup_{n\in\mathbb{N}} \{\beta_n |x_n|\}.$$

Similarly, let the set X consisting of all real (or complex) sequences $x = (x_n)_{n=1}^{\infty}$ such that $(\beta_n x_n)$ converges to a finite limit. It is obvious that X forms a linear space over the field of real (or complex) numbers. We denote the space by c^β. It is easy to see that c^β is a Banach space with the norm

$$\| x \|_{c^\beta} = \sup_{n\in\mathbb{N}} \{\beta_n |x_n|\}.$$

Also there is a isometry between the spaces c_0^β and c_0 and between the spaces c^β and c.
In [7], the Hausdorff measure of noncompactness $\chi(B)$ for $B \in \mathcal{M}_{c_0^\beta}$ is defined by

$$\chi(B) = \lim_{n\to\infty} \left\{ \sup_{x\in B} \left[\sup_{k\geq n} (\beta_k |x_k|) \right] \right\}.$$

Similarly the analogue of the measure of noncompactness μ on c^β defined by formula (2) has the form

$$\mu_{c^\beta}(B) = \lim_{p\to\infty} \left[\sup_{x\in B} \left\{ \sup_{k\geq p} | \beta_k x_k - \lim_{m\to\infty} (\beta_m x_m) | \right\} \right], \text{ where } B \in \mathcal{M}_{c^\beta}.$$

Let us consider the function spaces $C(I, c_0^\beta)$ and $C(I, c^\beta)$ where $I = [0, T], T > 0$ the spaces of all continuous functions on I with values in c_0^β and the spaces of all continuous functions on I with values in c^β respectively. Then $C(I, c_0^\beta)$ and $C(I, c^\beta)$ are Banach spaces with respect to the norm

$$\| u \|_{C(I,c_0^\beta)} = \max \left\{ \| u(t) \|_{c_0^\beta} : t \in I \right\},\ u \in C(I, c_0^\beta)$$

and

$$\| u \|_{C(I,c^{\beta})} = \max \{ \| u(t) \|_{c^{\beta}} : t \in I \}, \ u \in C(I, c^{\beta})$$

respectively.

For any non-empty, closed, bounded and convex subset X of $C(I, c_0^{\beta})$ or $C(I, c^{\beta})$ and $t \in I$, let

$$X(t) = \{x(t) : x \in X\},$$

$$\chi_{C(I,c_0^{\beta})}(X) = \sup \{\chi(X(t)) : t \in I\}$$

and

$$\mu_{C(I,c^{\beta})}(X) = \sup \{\mu_{c^{\beta}}(X(t)) : t \in I\}.$$

It was proved in [6] that for a bounded closed and convex $X \subset C(I, E)$ where E is a Banach space the measure of noncompactness is given by

$$\mu_{C(I,E)}(X) = \sup_{t \in I} \mu_E \{X(t)\}.$$

Thus $\chi_{C(I,c_0^{\beta})}$ and $\mu_{C(I,c^{\beta})}$ will satisfy all the axioms of measure of noncompactness on $C(I, c_0^{\beta})$ and $C(I, c^{\beta})$ respectively.

Definition 1.4. *[4] Let E_1 and E_2 be two Banach spaces and let μ_1 and μ_2 be arbitrary measure of noncompactness on E_1 and E_2, respectively. An operator f from E_1 to E_2 is called a (μ_1, μ_2)-condensing operator if it is continuous and $\mu_2(f(D)) < \mu_1(D)$ for every set $D \subset E_1$ with compact closure.*

Remark 1.5. *If $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$, then f is called a μ-condensing operator.*

Theorem 1.6. *[10] Let Ω be a nonempty, closed, bounded and convex subset of a Banach space E and let $f : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property $\mu_2(f(\Omega)) < k\mu_1(\Omega)$. Then f has a fixed point in Ω.*

Definition 1.7. *[14] Let (X, d) be a metric space. Then a mapping T on X is said to be a Meir–Keeler contraction if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta \implies d(Tx, Ty) < \epsilon, \forall x, y \in X.$$

Theorem 1.8. *[14] Let (X, d) be a complete metric space. If $T : X \to X$ is a Meir-Keeler contraction, then T has a unique fixed point.*

Definition 1.9. *[1] Let C be a nonempty subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. We say that an operator $T : C \to C$ is a Meir–Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq \mu(X) < \epsilon + \delta \implies \mu(T(X)) < \epsilon$$

for any bounded subset X of C.

Theorem 1.10. *[1] Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Meir–Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.*

2. Main Results

For a function $f : (0, \infty) \to \mathbb{R}$, the fractional integral of order α is defined as follows

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\alpha > 0$, provided the integral exists. Similarly the fractional derivative of order α for a function f is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{1}{(t-s)^{\alpha-n+1}} f(s) ds,$$

where $n = [\alpha] + 1 = N + 1$.
We mention the following properties of the operator I and D for $\alpha, \beta > 0$

$$I^{\alpha+\beta} f(t) = I^{\alpha} I^{\beta} f(t), \ D^{\alpha} I^{\alpha} f(t) = f(t).$$

For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha} y(t) = 0$ with $y(t) \in C(0, T) \cap L^1_{loc}(0, \infty)$ is given by

$$y(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_N t^{\alpha-N},$$

where $C_i \in \mathbb{R}, \ i = 1, 2, \ldots, N$.
We discuss the infinite systems of fractional differential equations by transforming the system into an infinite systems of integral equations with the help of Green's function.
Consider the infinite systems of fractional differential equations

$$D^{\alpha} y_i(t) + h_i (t, y(t)) = 0, \ 0 < t < T, \ h_i \in C[0, T] \tag{3}$$

with $y_i(0) = y_i(T) = 0$, where $y(t) = (y_i(t))_{i=1}^{\infty} \in \mathbb{R}^{\infty}$ and $i = 1, 2, 3, \ldots$.
If $y_i(t) \in C[0, T]$ and $1 < \alpha < 2$, the unique solution of (3) is given by

$$y_i(t) = \int_0^T G(t, s) h_i(s, y(s)) ds, \tag{4}$$

where $i = 1, 2, 3, ...$ and $t \in I$ and the Green's function associated to (3) is given by

$$G(t, s) = \begin{cases} \frac{1}{T^{\alpha-1}\Gamma(\alpha)} \left[t^{\alpha-1} (T-s)^{\alpha-1} - T^{\alpha-1} (t-s)^{\alpha-1} \right], & 0 \le s \le t \le T, \\ \frac{1}{T^{\alpha-1}\Gamma(\alpha)} t^{\alpha-1} (T-s)^{\alpha-1}, & 0 \le t \le s \le T. \end{cases} \tag{5}$$

In this article we establish the existence of solution of the infinite systems (3) for the sequence spaces $C(I, c_0^{\beta})$ and $C(I, c^{\beta})$.

3. Solvability of infinite systems of fractional differential equations in $C(I, c_0^{\beta})$

Suppose that

(i) The functions h_i are defined on the set $I \times \mathbb{R}^{\infty}$, where $I = [0, T]$ and take real values. The operator h defined on the space $I \times c_0^{\beta}$ into c_0^{β} as

$$(t, y(t)) \to (hy)(t) = (h_i(t, y(t)))_{i=1}^{\infty}$$

is the class of all functions $((hy)(t))_{t \in I}$ which is equicontinuous at every point of the space c_0^{β}.

(ii) For every $y(t) \in c_0^\beta,\ t \in I,\ i \in \mathbb{N}$ we have

$$\left|h_i\left(t, y(t)\right)\right| \leq a_i(t) + b_i(t)\left|y_i(t)\right|,$$

where for all $i \in \mathbb{N}$ and both $a_i(t), b_i(t)$ are real continuous functions defined on I such that the sequence $(\beta_i a_i(t))$ converges uniformly to zero on I and the sequence $(b_i(t))$ is equibounded on I.

Let us assume

$$b(t) = \sup_{i\in\mathbb{N}}\{b_i(t)\}$$
$$B = \sup_{t\in I}\{b(t)\}$$
$$A = \sup_{i\in\mathbb{N}, t\in I}\{\beta_i a_i(t)\}$$

and $\frac{2BT^\alpha}{\Gamma(\alpha)} < 1$.

Theorem 3.1. *Under the hypothesis (i)-(ii), infinite systems (3) has at least one solution $y(t) = (y_i(t)) \in C(I, c_0^\beta)$ for all $t \in I$.*

Proof. We have $\sup_{i\in\mathbb{N}}\{\beta_i \mid y_i(t) \mid\} \leq L$ for all $y(t) = (y_i(t))_{i=1}^\infty \in C(I, c_0^\beta)$ and $t \in I$, where L is a finite positive real number.
By using (4) and (ii), for arbitrary fixed $t \in I$, we have

$$\begin{aligned}
\parallel y(t) \parallel_{c_0^\beta} &= \sup_{i\geq1}\left[\beta_i\left|\int_0^T G(t,s)h_i(s,y(s))ds\right|\right] \\
&\leq \sup_{i\geq1}\left[\beta_i\int_0^T |G(t,s)|\left|h_i(s,y(s))\right|ds\right] \\
&\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)}\sup_{i\geq1}\left[\beta_i\int_0^T\{a_i(s) + b_i(s) \mid y_i(s) \mid\}ds\right] \\
&\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)}\sup_{i\geq1}\left[\int_0^T (A+BL)ds\right] \\
&= \frac{2(A+BL)T^\alpha}{\Gamma(\alpha)} = d(say) \\
&i.e.\ \parallel y(t) \parallel_{c_0^\beta} \leq d.
\end{aligned}$$

Thus

$$\max_{t\in I}\parallel y(t) \parallel_{c_0^\beta} \leq d \implies \parallel y \parallel_{C(I,c_0^\beta)} \leq d.$$

Let $y^0(t) = \left(y_i^0(t)\right)_{i=1}^\infty$, where $y_i^0(t) = 0\ \forall\ t \in I, i \in \mathbb{N}$.
Consider $B = B\left(y^0(t), d\right)$ the closed ball centered at $y^0(t)$ and radius d, thus B is a non-empty, bounded, closed and convex subset of $C(I, c_0^\beta)$.

For arbitrary fixed $t \in I$, define the operator $S = (S_i)_{i=1}^\infty$ on from $C(I, c_0^\beta)$ to $C(I, c_0^\beta)$ defined as follows

$$(Sy)(t) = \{(S_i y)(t)\}_{i=1}^\infty = \left\{\int_0^T G(t,s)h_i(s,y(s))ds\right\}_{i=1}^\infty,$$

where $y(t) = (y_i(t))_{i=1}^{\infty} \in C(I, c_0^{\beta})$ and $y_i(t) \in C(I, \mathbb{R})$.
As $(h_i(t, y(t))) \in c_0^{\beta}$ for each $t \in I$, we have

$$\begin{aligned}\lim_{i\to\infty}[\beta_i (S_i y)(t)] &= \lim_{i\to\infty}\left[\beta_i \int_0^T G(t,s)h_i(s,y(s))ds\right]\\ &= \int_0^T G(t,s)\lim_{i\to\infty}[\beta_i h_i(s,y(s))] = 0.\end{aligned}$$

Hence $(Sy)(t) \in C(I, c_0^{\beta})$.
Also $(S_i y)(t)$ satisfies boundary conditions i.e.

$$\begin{aligned}(S_i y)(0) &= \int_0^T G(0,s)h_i(s,y(s))ds = \int_0^T 0.f_i(s,y(s))ds = 0,\\ (S_i y)(T) &= \int_0^T G(T,s)h_i(s,y(s))ds = \int_0^T 0.f_i(s,y(s))ds = 0.\end{aligned}$$

For fixed $t \in I$ and $y(t) \in B$ we have $\| (Sy)(t) - y^0(t) \|_{c_0^{\beta}} \leq d$ gives $\max_{t\in I} \| (Sy)(t) - y^0(t) \|_{c_0^{\beta}} \leq d \implies \| (Sy)(t) - y^0(t) \|_{C(I,c_0^{\beta})} \leq d$ thus S is self mapping on B.
By assumption (i) we can assume $z(t) = (z_i(t))_{i=1}^{\infty} \in B$ and there exists $\epsilon > 0$ for each $\delta > 0$ such that $\| (hy)(t) - (hz)(t) \|_{c_0^{\beta}} < \frac{\epsilon\Gamma(\alpha)}{2T^{\alpha}}$ for each $y(t), z(t) \in B$, whenever $\| y(t) - z(t) \| \leq \delta$, where $t \in I$.
For arbitrary fixed $t \in I$

$$\begin{aligned}\| (Sy)(t) - (Sz)(t) \|_{c_0^{\beta}} &= \sup_{i\geq 1}\left\{\beta_i \left|(S_i y)(t) - (S_i z)(t)\right|\right\}\\ &\leq \sup_{i\geq 1}\left\{\beta_i \int_0^T |G(t,s)| \left|h_i(s,y(s)) - h_i(s,z(s))\right| ds\right\}\\ &\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)} \sup_{i\geq 1}\left\{\int_0^T \beta_i \left|h_i(s,y(s)) - h_i(s,z(s))\right| ds\right\}\\ &< \frac{2T^{\alpha-1}}{\Gamma(\alpha)}.\frac{\epsilon\Gamma(\alpha)}{2T^{\alpha}}.T < \epsilon.\end{aligned}$$

Thus S is continuous on $B \subset C(I, c_0^{\beta})$. Since t is arbitrarily fixed therefore S is continuous on B for all $t \in I$.
We have

$$\begin{aligned}\chi(SB) &= \lim_{i\to\infty}\left[\sup_{y(t)\in B}\sup_{k\geq i}\left\{\beta_k \left|\int_0^T G(t,s)h_k(s,y(s))ds\right|\right\}\right]\\ &\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)}\lim_{i\to\infty}\left[\sup_{y(t)\in B}\sup_{k\geq i}\left\{\int_0^T \left(\beta_k a_k(s) + \beta_k b_k(s)\left|y_k(s)\right|\right)\right\}\right]\\ &\leq \frac{2BT^{\alpha}}{\Gamma(\alpha)}\chi(B)\end{aligned}$$

i.e. $\chi(SB) \leq \frac{2BT^{\alpha}}{\Gamma(\alpha)}\chi(B)$.
Thus $\sup_{t\in I}\chi(SB) \leq \frac{2BT^{\alpha}}{\Gamma(\alpha)}\sup_{t\in I}\chi(B) \implies \chi_{C(I,c_0^{\beta})}(SB) \leq \frac{2BT^{\alpha}}{\Gamma(\alpha)}\chi_{C(I,c_0^{\beta})}(B)$.

Hence $\chi_{C(I,c_0^{\beta})}(SB) \leq \frac{2BT^{\alpha}}{\Gamma(\alpha)}\chi_{C(I,c_0^{\beta})}(B) < \epsilon \implies \chi_{C(I,c_0^{\beta})}(B) < \frac{\epsilon\Gamma(\alpha)}{2BT^{\alpha}}$.
Taking $\delta = \frac{\epsilon}{2BT^{\alpha}}(\Gamma(\alpha) - 2BT^{\alpha})$ we get $\epsilon \leq \chi_{C(I,c_0^{\beta})}(B) < \epsilon + \delta$. Therefore S is a Meir–Keeler condensing operator defined on the set $B \subset C(I, c_0^{\beta})$. Since t is arbitrarily fixed, thus for all $t \in I$, S satisfies all the conditions of Theorem 3 which implies S has a fixed point in B. Thus the systems (3) has a solution in $C(I, c_0^{\beta})$. □

4. Examples

Let us consider the following systems of differential equations

$$D^{\frac{3}{2}} y_i(t) = -\frac{e^{-it^2}}{i^2} - \sum_{j=i}^{\infty} \frac{y_i(t)}{4j^2} \tag{6}$$

with $y_i(0) = y_i(1) = 0$, where $h_i\,(t, y(t)) = \frac{e^{-it^2}}{i^2} + \sum_{j=i}^{\infty} \frac{y_i(t)}{4j^2},\ \forall\, i \in \mathbb{N},\ t \in (0,1)$.
Here $T = 1,\ \alpha = \frac{3}{2}$. Let $\beta_i = \frac{1}{i^2}$ for all $i \in \mathbb{N}$.
If $y(t) \in C(I, c_0^{\beta})$ then for any $t \in [0,1]$ we have

$$\lim_{i\to\infty} \beta_i h_i\,(t, y(t)) = \lim_{i\to\infty} \left(\frac{e^{-it^2}}{i^4} + \frac{1}{i^2} \sum_{j=i}^{\infty} \frac{y_i(t)}{4j^2} \right) = 0.$$

Thus if $y(t) = (y_i(t)) \in C(I, c_0^{\beta})$ i.e. $(h_i\,(t, y(t))) \in c_0^{\beta}$.
Let $t \in [0,1]$ and $z(t) \in C(I, c_0^{\beta})$ be arbitrary, where $z(t) = (z_i(t))_{i=1}^{\infty}$. For $\epsilon > 0$, we have

$$\begin{aligned} \|\,(hy)(t) - (hz)(t)\,\|_{c_0^{\beta}} &= \sup_{i\geq 1} \left\{ \beta_i \left| h_i(t, y(t)) - h_i(t, z(t)) \right| \right\} \\ &= \sup_{i\geq 1} \left\{ \frac{1}{i^2} \left| \sum_{j=i}^{\infty} \left\{ \frac{y_i(t)}{4j^2} - \frac{z_i(t)}{4j^2} \right\} \right| \right\} \\ &\leq \sup_{i\geq 1} \left\{ \frac{1}{4i^2} \left| y_i(t) - z_i(t) \right| \sum_{j=i}^{\infty} \frac{1}{j^2} \right\} \\ &\leq \frac{\pi^2}{24} \,\|\, y(t) - z(t)\,\|_{c_0^{\beta}} < \epsilon \end{aligned}$$

whenever $\|\, y(t) - z(t)\,\|_{c_0^{\beta}} < \delta = \frac{24\epsilon}{\pi^2}$, which implies the equicontinuity of $((hy)(t))_{t\in I}$ on c_0^{β}, where $I = [0,1]$.
Moreover for all $i \in \mathbb{N}$ and $t \in I$, we have

$$\left| h_i(t, y(t)) \right| \leq \frac{e^{-it^2}}{i^2} + \left| y_i(t) \right| \sum_{j=i}^{\infty} \frac{1}{4j^2} \leq \frac{e^{-it^2}}{i^2} + \frac{\pi^2}{24} \left| y_i(t) \right|,$$

where $a_i(t) = \frac{e^{-it^2}}{i^2},\ b_i(t) = \frac{\pi^2}{24}$ are real continuous functions on I and $B = \frac{\pi^2}{24}$. We observe that $(\beta_i a_i(t)) = \left(\frac{e^{-it^2}}{i^4} \right)$ converges uniformly to zero on I and the sequence $\{b_i(t)\}$ is equibounded on I. Also $\frac{2BT^{\alpha}}{\Gamma(\alpha)} = 2.\frac{\pi^2}{24}.\frac{2}{\sqrt{\pi}} \approx \frac{9.86}{10.63} < 1$.
Thus by Theorem 3.1 the systems (6) has unique solution in $C(I, c_0^{\beta})$.

5. Solvability of infinite systems of fractional differential equations in $C(I, c^{\beta})$

Suppose that

(i) The functions h_i are defined on the set $I \times \mathbb{R}^{\infty}$, where $I = [0, T]$ and take real values. The operator h defined on the space $I \times c^{\beta}$ into c^{β} as

$$(t, y(t)) \to (hy)\,(t) = (h_i(t, y(t)))_{i=1}^{\infty}$$

is the class of all functions $((hy)\,(t))_{t\in I}$ which is equicontinuous at every point of the space c^{β}.

(ii) For every $y(t) \in c^{\beta},\ t \in I,\ i \in \mathbb{N}$ we have

$$h_i\left(t, y(t)\right) = \hat{a}_i(t) + \hat{b}_i(t) y_i(t),$$

where for all $i \in \mathbb{N}$ and both $\hat{a}_i(t), \hat{b}_i(t)$ are nonnegative continuous functions defined on I, the sequence $(\beta_i \hat{a}_i(t))$ converges uniformly to zero on I and the sequence $\left(\hat{b}_i(t)\right)$ is convergent on I.

Let us consider

$$\hat{b}(t) = \sup_{i \in \mathbb{N}} \left\{\hat{b}_i(t)\right\},$$
$$\hat{B} = \sup_{t \in I} \left\{\hat{b}(t)\right\},$$
$$\hat{A} = \sup_{i \in \mathbb{N}, t \in I} \left\{\beta_i \hat{a}_i(t)\right\}$$

and $\frac{2\hat{B}T^{\alpha}}{\Gamma(\alpha)} < 1$.

Theorem 5.1. *Under the hypothesis (i)-(ii), infinite systems (3) has at least one solution $y(t) = (y_i(t))_{i=1}^{\infty} \in C(I, c^{\beta})$ for all $t \in I$.*

Proof. We have $\sup_{i \in \mathbb{N}} \{\beta_i \mid y_i(t) \mid\} \leq L_1$ for all $y(t) = (y_i(t))_{i=1}^{\infty} \in C(I, c^{\beta})$ and $t \in I$, where L_1 is a finite positive real number.
By using (4) and (ii), we have for arbitrary fixed $t \in I$,

$$\begin{aligned}
\| y(t) \|_{c^{\beta}} &= \sup_{i \geq 1} \left[\beta_i \left| \int_0^T G(t,s) h_i(s, y(s)) ds \right|\right] \\
&\leq \sup_{i \geq 1} \left[\beta_i \int_0^T |G(t,s)| \left|h_i(s, y(s))\right| ds\right] \\
&\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)} \sup_{i \geq 1} \left[\beta_i \int_0^T \left\{\hat{a}_i(s) + \hat{b}_i(s) \mid y_i(s) \mid\right\} ds\right] \\
&\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)} \sup_{i \geq 1} \left[\int_0^T (\hat{A} + \hat{B} L_1) ds\right] \\
&= \frac{2(\hat{A} + \hat{B} L_1) T^{\alpha}}{\Gamma(\alpha)} = d_1 (say) \\
& i.e.\ \| y(t) \|_{c^{\beta}} \leq d_1.
\end{aligned}$$

Thus

$$\max_{t \in I} \| y(t) \|_{c^{\beta}} \leq d_1 \implies \| y \|_{C(I, c^{\beta})} \leq d_1.$$

Let $y^0(t) = \left(y_i^0(t)\right)_{i=1}^{\infty}$, where $y_i^0(t) = 0\ \forall\ t \in I, i \in \mathbb{N}$.
Consider $B_1 = B_1\left(y^0(t), d_1\right)$, the closed ball centered at $y^0(t)$ and radius d_1, thus B_1 is a non-empty, bounded, closed and convex subset of $C(I, c^{\beta})$.
For arbitrarily fixed $t \in I$, define the operator $S = (S_i)_{i=1}^{\infty}$ from $C(I, c^{\beta})$ to $C(I, c^{\beta})$ as follows

$$(Sy)(t) = \{(S_i y)(t)\}_{i=1}^{\infty} = \left\{\int_0^T G(t,s) h_i(s, y(s)) ds\right\}_{i=1}^{\infty},$$

where $y(t) = (y_i(t))_{i=1}^{\infty} \in C(I, c^{\beta})$ and $y_i(t) \in C(I, \mathbb{R})$.
Now let $j \in \mathbb{N}$ and

$$\begin{aligned}\left|\beta_i(S_iy)(t) - \beta_j(S_iy)(t)\right| &= \left|\beta_i \int_0^T G(t,s)h_i(s,y(s))ds - \beta_j \int_0^T G(t,s)h_j(s,y(s))ds\right| \\ &\leq \left|\beta_i \int_0^T G(t,s)\left(\hat{a}_i(s) + \hat{b}_i(s)y_i(s)\right)ds - \beta_j \int_0^T G(t,s)\left(\hat{a}_j(s) + \hat{b}_j(s)y_j(s)\right)ds\right| \\ &\leq \int_0^T |G(t,s)| \left|\beta_i\hat{a}_i(s) - \beta_j\hat{a}_j(s)\right| ds + \int_0^T |G(t,s)| \left|\beta_i\hat{b}_i(s)y_i(s) - \beta_j\hat{b}_j(s)y_j(s)\right| ds.\end{aligned}$$

Also

$$\left|\beta_i\hat{b}_i(s)y_i(s) - \beta_j\hat{b}_j(s)y_j(s)\right| \leq \beta_i \left|y_i(s)\right| \left|\hat{b}_i(s) - \hat{b}_j(s)\right| + \left|\hat{b}_j(s)\right| \left|\beta_i(s)y_i(s) - \beta_j(s)y_j(s)\right|.$$

As $i, j \to \infty$ we get $\left|\hat{b}_i(s) - \hat{b}_j(s)\right| \to 0$, $\left|\beta_i(s)y_i(s) - \beta_j(s)y_j(s)\right| \to 0$ and $\left|\beta_i\hat{a}_i(s) - \beta_j\hat{a}_j(s)\right| \to 0$ because $(\beta_i\hat{a}_i(t))$, $\left(\hat{b}_i(t)\right)$ are convergent on I and $y(t) \in C(I, c^{\beta})$ for all $t \in I$.
Thus as $i, j \to \infty$ we get

$$\left|\beta_i(S_iy)(t) - \beta_j(S_iy)(t)\right| \to 0.$$

Hence $(Sy)(t) \in C(I, c^{\beta})$.

For fixed $t \in I$ and $y(t) \in B_1$ we have $\| (Sy)(t) - y^0(t) \|_{c^{\beta}} \leq d_1$ gives $\max_{t \in I} \| (Sy)(t) - y^0(t) \|_{c^{\beta}} \leq d_1 \implies \| (Sy)(t) - y^0(t) \|_{C(I,c^{\beta})} \leq d_1$ thus S is self mapping on B_1. Also $(S_iy)(t)$ satisfies boundary conditions i.e.

$$\begin{aligned}(S_iy)(0) &= \int_0^T G(0,s)h_i(s,y(s))ds = \int_0^T 0.f_i(s,y(s))ds = 0, \\ (S_iy)(T) &= \int_0^T G(T,s)h_i(s,y(s))ds = \int_0^T 0.f_i(s,y(s))ds = 0.\end{aligned}$$

By assumption (i) we can assume $\bar{z}(t) = (\bar{z}_i(t))_{i=1}^{\infty} \in B_1$ and there exists $\epsilon > 0$ for each $\delta > 0$ such that $\| (hy)(t) - (h\bar{z})(t) \|_{c^{\beta}} < \frac{\epsilon\Gamma(\alpha)}{2T^{\alpha}}$ for each $y(t), z(t) \in B_1$, whenever $\| y(t) - \bar{z}(t) \|_{c^{\beta}} \leq \delta$, where $t \in I$.
For arbitrarily fixed $t \in I$,

$$\begin{aligned}\| (Sy)(t) - (S\bar{z})(t) \|_{c^{\beta}} &= \sup_{i \geq 1} \left\{\beta_i \left|(S_iy)(t) - (S_i\bar{z})(t)\right|\right\} \\ &\leq \sup_{i \geq 1} \left\{\beta_i \int_0^T |G(t,s)| \left|h_i(s,y(s)) - h_i(s,\bar{z}(s))\right| ds\right\} \\ &\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)} \sup_{i \geq 1} \left\{\int_0^T \beta_i \left|h_i(s,y(s)) - h_i(s,\bar{z}(s))\right| ds\right\} \\ &< \frac{2T^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{\epsilon\Gamma(\alpha)}{2T^{\alpha}} .T < \epsilon.\end{aligned}$$

Thus S is continuous on $B_1 \subset C(I, c^{\beta})$. Since t is arbitrarily fixed therefore S is continuous on B_1 for all $t \in I$.

We have for arbitrarily fixed $t \in I$,

$$\begin{aligned}
&\mu_{c^\beta}(SB_1)\\
&= \lim_{i\to\infty}\left[\sup_{y(t)\in B_1}\sup_{k\geq i}\left\{\left|\beta_k\int_0^T G(t,s)h_k(s,y(s))ds - \lim_{m\to\infty}\left(\beta_m\int_0^T G(t,s)h_m(s,y(s))ds\right)\right|\right\}\right]\\
&= \lim_{i\to\infty}\left[\sup_{y(t)\in B_1}\sup_{k\geq i}\left\{\left|\int_0^T G(t,s)\left(\beta_k\hat{b}_k(s)y_k(s) - \lim_{m\to\infty}\beta_m\hat{b}_m(s)y_m(s)\right)ds\right|\right\}\right]\\
&\leq \lim_{i\to\infty}\left[\sup_{y(t)\in B_1}\sup_{k\geq i}\left\{\int_0^T |G(t,s)|\left|\beta_k\hat{b}_k(s)y_k(s) - \lim_{m\to\infty}\beta_m\hat{b}_m(s)y_m(s)\right|ds\right\}\right]\\
&\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)} \leq \lim_{i\to\infty}\left[\sup_{y(t)\in B_1}\sup_{k\geq i}\left\{\int_0^T \left|\beta_k\hat{b}_k(s)y_k(s) - \lim_{m\to\infty}\beta_m\hat{b}_m(s)y_m(s)\right|ds\right\}\right]\\
&\leq \frac{2T^{\alpha-1}}{\Gamma(\alpha)}\lim_{i\to\infty}\left[\sup_{y(t)\in B_1}\sup_{k\geq i}\left\{\int_0^T \left(\left|\hat{b}_k(s)\right|\left|\beta_k(s)y_k(s) - \lim_{m\to\infty}\beta_m y_m(s)\right| + \left|\lim_{m\to\infty}\beta_m y_m(s)\left(\hat{b}_k(s)-\hat{b}_m(s)\right)\right|\right)ds\right\}\right]\\
&\leq \frac{2T^\alpha\hat{B}}{\Gamma(\alpha)}\mu_{c^\beta}(B_1)
\end{aligned}$$

i.e. $\mu_{c^\beta}(SB_1) \leq \frac{2\hat{B}T^\alpha}{\Gamma(\alpha)}\mu_{c^\beta}(B_1) \implies \sup_{t\in I}\mu_{c^\beta}(SB_1) \leq \frac{2\hat{B}T^\alpha}{\Gamma(\alpha)}\sup_{t\in I}\mu_{c^\beta}(B_1)$.
Hence $\mu_{C(I,c^\beta)}(SB_1) \leq \frac{2\hat{B}T^\alpha}{\Gamma(\alpha)}\mu_{C(I,c^\beta)}(B_1) < \epsilon \implies \mu_{C(I,c^\beta)}(B_1) < \frac{\epsilon\Gamma(\alpha)}{2\hat{B}T^\alpha}$.
Taking $\delta = \frac{\epsilon}{2\hat{B}T^\alpha}\left(\Gamma(\alpha) - 2\hat{B}T^\alpha\right)$ we get $\epsilon \leq \mu_{C(I,c^\beta)}(B_1) < \epsilon+\delta$. Therefore S is a Meir–Keeler condensing operator defined on the set $B_1 \subset C(I,c^\beta)$. Since t is arbitrarily fixed , thus for all $t \in I$, S satisfies all the conditions of Theorem 3 which implies S has a fixed point in B_1. Thus the systems (3) has a solution in $C(I,c^\beta)$. □

6. Examples

Let us consider the following systems of differential equations

$$D^{\frac{3}{2}}y_i(t) = -\frac{t}{i^2} - \left(1 + \sum_{j=i}^{\infty}\frac{1}{4j^2}\right)y_i(t) \tag{7}$$

with $y_i(0) = y_i(\frac{1}{4}) = 0$, where $h_i(t,y(t)) = \frac{t}{i^2} + \left(1 + \sum_{j=i}^{\infty}\frac{1}{4j^2}\right)y_i(t), \forall\, i \in \mathbb{N},\ t \in \left(0,\frac{1}{4}\right)$.
Let $\beta_i = \frac{1}{i^2}$ for all $i \in \mathbb{N}$.
Here $T = \frac{1}{4},\ \alpha = \frac{3}{2}$.
Also $a_i(t) = \frac{t}{i^2},\ b_i(t) = 1 + \sum_{j=i}^{\infty}\frac{1}{4j^2}$ are real continuous functions on $I = \left[0,\frac{1}{4}\right]$ and $\hat{B} = 1 + \frac{\pi^2}{24}$. We observe that $(\beta_i a_i(t)) = \left(\frac{t}{i^4}\right)$ converges uniformly to zero on I and the sequence $\{b_i(t)\}$ is convergent on I.
Also $\frac{2\hat{B}T^\alpha}{\Gamma(\alpha)} = 2.\left(1+\frac{\pi^2}{24}\right).\frac{1}{8}.\frac{1}{\Gamma(\frac{3}{2})} \approx \frac{1.41}{3.55} < 1$.
If $y(t) \in C(I,c^\beta)$ then for any $t \in [0,\frac{1}{4}]$ we have

$$\lim_{i\to\infty}\beta_i h_i(t,y(t)) = \lim_{i\to\infty}\left(\frac{t}{i^4} + \frac{y_i(t)}{i^2}\left(1 + \sum_{j=i}^{\infty}\frac{1}{4j^2}\right)\right)$$

is unique and finite science $y(t) \in C(I,c^\beta)$ i.e. $(h_i(t,y(t))) \in c^\beta$.

Let $t \in I$ and $z(t) \in C(I, c^\beta)$ be arbitrary, where $z(t) = (z_i(t))_{i=1}^{\infty}$. For $\epsilon > 0$, we have

$$\begin{aligned} \| (hy)(t) - (hz)(t) \|_{c^\beta} &= \sup_{i\geq 1} \left\{ \beta_i \left| h_i(t, y(t)) - h_i(t, z(t)) \right| \right\} \\ &\leq \sup_{i\geq 1} \left\{ \frac{1}{i^2} \left| y_i(t) - z_i(t) \right| \left(1 + \sum_{j=i}^{\infty} \frac{1}{4j^2} \right) \right\} \\ &\leq \left(1 + \frac{\pi^2}{24} \right) \| y(t) - z(t) \|_{c^\beta} < \epsilon \end{aligned}$$

whenever $\| y(t) - z(t) \|_{c_0^\beta} < \delta = \frac{\epsilon}{1+\frac{\pi^2}{24}}$, which implies the equicontinuity of $((hy)(t))_{t\in I}$ on c^β.
Thus by Theorem 5.1, the systems (7) has unique solution in $C(I, c^\beta)$.

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