# Global Stability in a Diffusive Beddington-Deangelis and Tanner Predator-Prey Model

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**Abstract.** Our goal is to study a diffusive Beddington-DeAngelis and Tanner predator-prey system with no-flux boundary condition. It is proved that the unique constant equilibrium is globally asymptotically stable under a new simpler parameter condition.

## 1. Introduction

In this article, we consider the following Beddington-DeAngelis and Tanner reaction-diffusion system of predator-prey model:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = d_1 \Delta u + au - u^2 - \dfrac{uv}{u + v + m}, & x \in \Omega, t > 0, \\[2mm]
\dfrac{\partial v}{\partial t} = d_2 \Delta v + bv - \dfrac{v^2}{\gamma u}, & x \in \Omega, t > 0, \\[2mm]
\dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\[2mm]
u(x,0) = u_0(x) > 0, v(x,0) = v_0(x) \geq (\not\equiv)0, & x \in \Omega.
\end{cases}
\tag{1}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N = 1, 2, 3)$ with smooth boundary $\partial\Omega$, $0 < T \leq +\infty$, initial condition $u_0(x)$ and $v_0(x)$ are continuous functions on $\overline{\Omega}$ and compatible on $\partial\Omega$, constants $d_1, d_2, a, b, m, \gamma > 0$, and $\nu$ is the outward directional derivative normal to $\partial\Omega$.

There are the Beddington-DeAngelis and Tanner type functional response contained in the first and second equation of model (1), respectively, where $u(x,t)$ and $v(x,t)$ represent the population density of the predator and the prey at time $t$ with diffusion rates $d_1$ and $d_2$, respectively. We suppose that the two diffusion rate are positive and equal, but not necessary constants. $a$ denote the death rate of the predator $u$. The constant $r$ is called the intrinsic growth rate of the prey $v$. The constants $\delta$ is the conversion rate of the predator. The term $\beta u$ measures the mutual interference between predators.

The Beddington-DeAngelis type functional response term $\frac{\delta uv}{\beta u + v + \alpha}$ was proposed by Beddington [2] and DeAngelis [3]. They introduced the following predator-prey model with this functional response term

$$
\begin{cases}
\dot{x} = rx - \theta x^2 - \dfrac{\gamma xy}{a + bx + cy}, \\
\dot{y} = -dy + \dfrac{\delta xy}{a + bx + cy}.
\end{cases}
\tag{2}
$$

Huang et al. [4, 5] introduced a class of virus dynamics model with intracellular delay and nonlinear Beddington-DeAngelis infection rate. Liu and Kong [6] considered the dynamics of a predator-prey system with Beddington-DeAngelis functional response and delays.

Besides the Beddington-DeAngelis and Tanner type functional responses term mentioned above, there exist several other famous functional responses, such as well-known Holling type (I, II, III, IV), Hassel-Verley type and Monod-Haldane type functional responses and so on. Some researchers investigated and raised some well-known open questions for structured predator-prey models with different types of functional responses. Particularly, Peng and Wang [7] studied the steady states of a diffusive Holling-Tanner type predator-prey system

$$
\begin{cases}
u_t - d_1 \Delta u = au - u^2 - \dfrac{uv}{u + m}, & x \in \Omega, t \in (0, \infty), \\
v_t - d_2 \Delta v = rv - \dfrac{v^2}{\gamma u}, & x \in \Omega, t \in (0, \infty), \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, \infty), \\
u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \overline{\Omega},
\end{cases}
\tag{3}
$$

They proved the existence and non-existence of positive non-constant steady solutions for (3), and argued that (3) possesses no positive non-constant steady solution under a certain condition. In the another paper [8], the authors studied the stability of diffusive predator-prey model of Holling-Tanner type (3) by the construction of a standard linearization procedure and a Lyapunov function. Chen and Shi [26] focused attention on the steady states of (3). They applied the defined iteration and comparison principle sequences to prove the global asymptotic stability. Their scientific research achievement improves the earlier one proposed by Wang and Peng [8] which used Lyapunov method. We also note here that the (non-spatial) kinetic equation of system (3) was first introduced by May [11] and Tanner [10], see also [12, 13] and references therein.

Recently, Qi and Zhu [14] studied the global stability of a reaction-diffusion system of predator-prey model (3). Indeed, they established improved global asymptotic stability of the unique positive equilibrium solution in [14]. Besides the papers mentioned above, one can see [15–23] for more detailed information and biological significances of the studied system.

In the present paper by incorporating the ratio-dependent Beddington-DeAngelis functional response and diffusion term into system (3), motivated by the previous works [1], we will study the global stability of the positive equilibrium solution. Therefore, we argue that it is interesting, beneficial and significant to study the global asymptotic stability of (1) since it possesses biological implications and extends the former researches.

## 2. The main results

It is effortless to verify that model (1) possesses a unique positive equilibrium $(u_*, v_*)$, where

$$
u_* = \frac{a(1 + b\gamma) - m - b\gamma + \sqrt{(a(1 + b\gamma) - m - b\gamma)^2 + 4am(1 + b\gamma)}}{2(1 + b\gamma)},
$$

$$
v_* = b\gamma u_*.
$$

Our proof is based on the upper and lower solution method in [24, 25]. Our main theorem is stated as:

**Theorem 2.1.** *Suppose that the parameters* $m$, $a$, $b$, $\gamma$, $d_1$, $d_2$ *are all positive. Then for system (1), the positive equilibrium* $(u_*, v_*)$ *is globally asymptotically stable, that is, for any initial values* $u_0(x) > 0$, $v_0(x) \geq (\not\equiv)0$,

$$\lim_{t\to\infty}(u(t,x), v(t,x)) = (u_*, v_*),$$

*uniformly for* $x \in \overline{\Omega}$, *if*

$$m > b\gamma. \tag{4}$$

*Proof.* It is well known that if $c > 0$, and $w(x, t) > 0$ satisfies the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta w + w(c - w) & x \in \Omega, t > 0, \\ \dfrac{\partial w(x,t)}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) \geq (\not\equiv)0, & x \in \Omega, \end{cases}$$

then $w(x, t) \to c$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$.

Since (4) holds, we can choose an $\epsilon_0$ satisfying

$$0 < \epsilon_0 < \frac{b\gamma(m - b\gamma)a}{b\gamma(b\gamma + 1) + mb\gamma + m}. \tag{5}$$

Since $u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} = d_1\Delta u + au - u^2 - \frac{uv}{u + v + m} \leq d_1\Delta u + au - u^2,$$

and the Neumann boundary condition, then from comparison principle of parabolic equations, there exists $t_1$ such that for any $t > t_1$, $u(x, t) \leq \bar{c}_1$, where $\bar{c}_1 = a + \epsilon_0$. This in turn implies

$$\frac{\partial v}{\partial t} = d_2\Delta v + bv - \frac{v^2}{\gamma u} \leq d_2\Delta v + v\left(b - \frac{v}{\gamma(a + \epsilon_0)}\right)$$

for $t > t_1$. Hence there exists $t_2 > t_1$ such that for any $t > t_2$, $v(x, t) \leq \bar{c}_2$, where $\bar{c}_2 = b\gamma(a + \epsilon_0) + \epsilon_0$. Again this implies

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1\Delta u + au - u^2 - \frac{uv}{u + v + m} \\ &\geq d_1\Delta u + au - u^2 - \frac{b\gamma(a + \epsilon_0) + \epsilon_0}{m}u, \end{aligned}$$

for $t > t_2$. Since $m > b\gamma$, then for $\epsilon_0$ chosen as in (5),

$$a - \frac{b\gamma(a + \epsilon_0) + \epsilon_0}{m} > 0,$$

and

$$a - \frac{b\gamma(a + \epsilon_0) + \epsilon_0}{m} - \epsilon_0 > 0.$$

Hence there exists $t_3 > t_2$ such that for any $t > t_3$, $u(x, t) \geq \underline{c}_1 > 0$, where

$$\underline{c}_1 = a - \frac{b\gamma(a + \epsilon_0) + \epsilon_0}{m} - \epsilon_0.$$

Finally we apply the lower bound of $u$ considered two lines above to the equation of $v$, and we have

$$\frac{\partial v}{\partial t} = d_2 \Delta v + bv - \frac{v^2}{\gamma u} \geq d_2 \Delta v + v \left( b - \frac{v}{\gamma \underline{c}_1} \right)$$

for $t > t_3$. Since for the $\epsilon_0$ chosen above in (5),

$$b\gamma \left( a - \frac{b\gamma(a + \epsilon_0) + \epsilon_0}{m} - \epsilon_0 \right) - \epsilon_0 > 0,$$

then there exists $t_4 > t_3$ such that for any $t > t_4$, $v(x,t) \geq \underline{c}_2 > 0$, where

$$\underline{c}_2 = b\gamma \left( a - \frac{b\gamma(a + \epsilon_0) + \epsilon_0}{m} - \epsilon_0 \right) - \epsilon_0.$$

Therefore for $t > t_4$ we obtain that

$$\underline{c}_1 \leq u(x,t) \leq \bar{c}_1, \quad \underline{c}_2 \leq u(x,t) \leq \bar{c}_2,$$

and $\underline{c}_1, \bar{c}_1, \underline{c}_2, \bar{c}_2$ satisfy

$$0 \geq a - \bar{c}_1 - \frac{\underline{c}_2}{m + \bar{c}_1 + \bar{c}_2},$$

$$0 \geq b - \frac{\bar{c}_2}{\gamma \bar{c}_1},$$

$$0 \leq a - \underline{c}_1 - \frac{\bar{c}_2}{m + \underline{c}_1 + \underline{c}_2}, \tag{6}$$

$$0 \geq b - \frac{\underline{c}_2}{\gamma \underline{c}_1}.$$

The inequalities (6) show that $(\underline{c}_1, \underline{c}_2)$ and $(\bar{c}_1, \bar{c}_2)$ are a pair of coupled lower and upper solutions of system (1) as in the definition in [24, 25](see also [26]), as the nonlinearities in (1) are mixed quasimonotone. It is clear that there exists $K > 0$ such that for any $(\underline{c}_1, \underline{c}_2) \leq (u_1, v_1), (u_2, v_2) \leq (\bar{c}_1, \bar{c}_2)$,

$$\left| au_1 - u_1^2 - \frac{u_1 v_1}{u_1 + v_1 + m} - au_2 + u_2^2 + \frac{u_2 v_2}{u_2 + v_2 + m} \right| \leq K(|u_1 - u_2| + |v_1 - v_2|),$$

$$\left| bv_1 - \frac{v_1^2}{\gamma u_1} - bv_2 + \frac{v_2^2}{\gamma u_2} \right| \leq K(|u_1 - u_2| + |v_1 - v_2|).$$

We define two iteration sequences $(\bar{c}_1^{(n)}, \bar{c}_2^{(n)})$ and $(\underline{c}_1^{(n)}, \underline{c}_2^{(n)})$ as follows: for $n \geq 1$,

$$\bar{c}_1^{(n)} = \bar{c}_1^{(n-1)} + \frac{1}{K} \left( a\bar{c}_1^{(n-1)} - (\bar{c}_1^{(n-1)})^2 - \frac{\bar{c}_1^{(n-1)} \underline{c}_2^{(n-1)}}{\bar{c}_1^{(n-1)} + \bar{c}_2^{(n-1)} + m} \right),$$

$$\bar{c}_2^{(n)} = \bar{c}_2^{(n-1)} + \frac{1}{K} \left( b\bar{c}_2^{(n-1)} - \frac{(\bar{c}_2^{(n-1)})^2}{\gamma \bar{c}_1^{(n-1)}} \right),$$

$$\underline{c}_1^{(n)} = \underline{c}_1^{(n-1)} + \frac{1}{K} \left( a\underline{c}_1^{(n-1)} - (\underline{c}_1^{(n-1)})^2 - \frac{\underline{c}_1^{(n-1)} \bar{c}_2^{(n-1)}}{\underline{c}_1^{(n-1)} + \underline{c}_2^{(n-1)} + m} \right),$$

$$\underline{c}_2^{(n)} = \underline{c}_2^{(n-1)} + \frac{1}{K} \left( b\underline{c}_2^{(n-1)} - \frac{(\underline{c}_2^{(n-1)})^2}{\gamma \underline{c}_1^{(n-1)}} \right).$$

where $(\bar{c}_1^0, \bar{c}_2^0) = (\bar{c}_1, \bar{c}_2)$ and $(\underline{c}_1^0, \underline{c}_2^0) = (\underline{c}_1, \underline{c}_2)$. Then for $n \geq 1$,

$$(\underline{c}_1, \underline{c}_2) \leq (\underline{c}_1^{(n)}, \underline{c}_2^{(n)}) \leq (\underline{c}_1^{(n+1)}, \underline{c}_2^{(n+1)}) \leq (\bar{c}_1^{(n+1)}, \bar{c}_2^{(n+1)}) \leq (\bar{c}_1^{(n)}, \bar{c}_2^{(n)}) \leq (\bar{c}_1, \bar{c}_2).$$

and there exists $(\tilde{c}_1, \tilde{c}_2)$ and $(\hat{c}_1, \hat{c}_2)$ such that

$$(\underline{c}_1, \underline{c}_2) \leq (\hat{c}_1, \hat{c}_2) \leq (\tilde{c}_1, \tilde{c}_2) \leq (\bar{c}_1, \bar{c}_2),$$

so $\lim_{n \to \infty} \bar{c}_1^n = \tilde{c}_1$, $\lim_{n \to \infty} \bar{c}_2^n = \tilde{c}_2$, $\lim_{n \to \infty} \underline{c}_1^n = \hat{c}_1$, $\lim_{n \to \infty} \underline{c}_2^n = \hat{c}_2$ and

$$0 = a - \tilde{c}_1 - \frac{\hat{c}_2}{m + \tilde{c}_1 + \tilde{c}_2}, \quad 0 = b - \frac{\tilde{c}_2}{\gamma \tilde{c}_1}, \quad 0 = a - \hat{c}_1 - \frac{\tilde{c}_2}{m + \hat{c}_1 + \hat{c}_2}, \quad 0 = b - \frac{\hat{c}_2}{\gamma \hat{c}_1}. \tag{7}$$

Simplifying (7) we obtain

$$(a - \tilde{c}_1)(m + (1 + b\gamma)\tilde{c}_1) = b\gamma \hat{c}_1, \quad (a - \hat{c}_1)(m + (1 + b\gamma)\hat{c}_1) = b\gamma \tilde{c}_1. \tag{8}$$

Subtracting the first equation of (8) from the second equation, we have

$$(\tilde{c}_1 - \hat{c}_1)(a(1 + b\gamma) - m + b\gamma - (1 + b\gamma)(\tilde{c}_1 + \hat{c}_1)) = 0. \tag{9}$$

If we suppose that $\tilde{c}_1 \neq \hat{c}_1$, then

$$\frac{a(1 + b\gamma) - m + b\gamma}{1 + b\gamma} = (\tilde{c}_1 + \hat{c}_1). \tag{10}$$

Substituting equation (10) into (8), we have

$$(a - \tilde{c}_1)(m + (1 + b\gamma)\tilde{c}_1) = b\gamma \left( \frac{a(1 + b\gamma) - m + b\gamma}{1 + b\gamma} - \tilde{c}_1 \right),$$
$$(a - \hat{c}_1)(m + (1 + b\gamma)\hat{c}_1) = b\gamma \left( \frac{a(1 + b\gamma) - m + b\gamma}{1 + b\gamma} - \hat{c}_1 \right). \tag{11}$$

Therefore, the following equation:

$$(a - x)(m + (1 + b\gamma)x) = b\gamma \left( \frac{a(1 + b\gamma) - m + b\gamma}{1 + b\gamma} - x \right) \tag{12}$$

possesses two positive roots $\tilde{c}_1$ and $\hat{c}_1$. Eq. (12) can be written as follows:

$$(1 + b\gamma)x^2 + (m - a(1 + b\gamma) - b\gamma)x + b\gamma \left( \frac{a(1 + b\gamma) - m + b\gamma}{1 + b\gamma} \right) - am = 0.$$

Since Eq. (12) cannot have two positive roots, then

$$\frac{b\gamma \left( \frac{a(1+b\gamma)-m+b\gamma}{1+b\gamma} \right) - am}{1 + b\gamma} < 0$$

$$\Longleftrightarrow b\gamma \left( \frac{a(1 + b\gamma) - m + b\gamma}{1 + b\gamma} \right) - am < 0$$

$$\Longleftrightarrow \frac{a(1 + b\gamma) - m + b\gamma}{1 + b\gamma} - \frac{am}{b\gamma} < 0$$

$$\Longleftrightarrow a(1 + b\gamma) - m + b\gamma - \frac{(1 + b\gamma)am}{b\gamma} < 0$$

$$\Longleftrightarrow a(1 + b\gamma)b\gamma - mb\gamma + b^2\gamma^2 - (1 + b\gamma)am < 0$$

$$\Longleftrightarrow (a + b\gamma + ab\gamma)b\gamma - (a + b\gamma + ab\gamma)m < 0$$

$$\Longleftrightarrow b\gamma < m.$$

Hence $\tilde{c}_1 = \hat{c}_1$, and consequently $\tilde{c}_2 = \hat{c}_2$. Then from the results in [24, 25], the solution $(u(x,t), v(x,t))$ of system (1) satisfies

$$\lim_{t\to\infty}(u(t,x), v(t,x)) = (u_*, v_*),$$

uniformly for $x \in \overline{\Omega}$. Hence from the above analysis, we can obtain that the constant equilibrium $(u_*, v_*)$ is globally asymptotically stable for system (1) if (4) holds. $\square$

For the diffusive Beddington-DeAngelis and Tanner system with same kinetic equations, there are one other version of nondimensionalized equations in [1]. Our main result can be used to this kind of equation with a conversion of the parameters.

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1-u) - \dfrac{uv}{a+u+v}, & x \in \Omega, t > 0, \\[2mm] \dfrac{\partial v}{\partial t} = d_2 \Delta v + v\left(\delta - \beta\dfrac{v}{u}\right), & x \in \Omega, t > 0, \\[2mm] \dfrac{\partial u(x,t)}{\partial v} = \dfrac{\partial v(x,t)}{\partial v} = 0, & x \in \partial\Omega, t > 0, \\[2mm] u(x,0) = u_0(x) > 0, v(x,0) = v_0(x) \geq (\not\equiv)0, & x \in \Omega. \end{cases} \tag{13}$$

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