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# Existence of Solutions for Fractional Differential Inclusions with Initial Value Condition and Non-Instantaneous Impulses

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**Abstract.** In this paper, we mainly consider the existence of solutions for a kind of $\psi$-Hilfer fractional differential inclusions involving non-instantaneous impulses. Utilizing another nonlinear alternative of Leray-Schauder type, we present a new constructive result for the addressed system with the help of generalized Gronwall inequality and Lagrange mean-value theorem, and some achievements in the literature can be generalized and improved. As an application, a typical example is delineated to demonstrate the effectiveness of our theoretical results.

## 1. Introduction

Impulsive differential equations are considered to the dynamics of many processes subject to abrupt changes, such as shocks, harvesting, natural disasters and so on. For some of these applications, we refer the reader to monographs [2–4]. There is a growing tendency nowadays that many experts show their great enthusiasm for a class of differential equations with a new type impulses (called non-instantaneous impulses), impulsive action starts at an arbitrary fixed point and remains active on a finite time interval, which is much different from classical instantaneous impulsive that changes is relatively short compared to the overall duration of the whole process. For an extensive collection of such results, we recommend readers to the related literatures, such as monograph [1] and papers [5–8, 14–18, 20].

The study about the existence of solutions for differential equations with non-instantaneous impulses is one of the most interesting and valuable topics. Detailedly, in paper [14], the authors utilized Krasnoselskii's fixed point theorem to study the integral boundary value problems for two classes of nonlinear non-instantaneous impulsive ordinary differential equations, and got a series of existence results. By the Darbo fixed point theorem and the Hausdorff measure of non-compactness, Suganya et al. [15] analyzed the existence results for an impulsive fractional neutral integro-differential equation with state-dependent delay and non-instantaneous impulses in Banach spaces. Bai et al. [16] acknowledged the existence of at least two distinct nonzero bounded weak solutions by using variational methods and critical point theory. Agarwal et al. in [6] studied the following initial value problem for the nonlinear scalar non-instantaneous

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impulsive differential equation

$$\begin{cases} x' = f_k(t,x), & t \in (s_k,t_{k+1}] \cap [t_0,T],\ k = 0,1,\cdots,p, \\ x(t) = \phi_k\left(t,x(t),x(t_k-0)\right), & t \in (t_k,s_k] \cap [t_0,T],\ k = 1,2,\cdots,p, \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

where $x,\ x_0 \in \mathbb{R},\ f_k : [s_k,t_{k+1}] \cap [t_0,T] \times \mathbb{R} \to \mathbb{R},\ k = 0,1,\cdots,p,\ \phi_k : [t_k,s_k] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ k = 1,2,\cdots,p.$ The impulses started abruptly at some points and their actions continued on given finite intervals, and authors used the monotone iterative technique combined with the method of lower and upper solutions to find approximately solution of the given problem. Regarding the techniques, it is worth pointing out that, recently, without iterative techniques, some authors obtained qualitative results for solutions of partial differential equations with discontinuous coefficients. For instance, in [9] and [10], the authors used explicit representation formulas for the higher order derivatives of solutions to parabolic equations and divergence form elliptic equations, respectively.

We know that differential inclusion is a generalization of an ordinary differential equation. Therefore, all problems considered for differential equations, that is, existence of solutions, dependence on initial conditions, etc, can be presented in the theory of differential inclusions. As a consequence, differential inclusions have been the subject of an intensive study of many researchers. Impulsive differential inclusions have also been researched by many experts, and we can see the monograph [4] and papers [21, 22, 24, 25, 27]. Also, fractional differential equations have been studied by many researchers. See [11] for an account of the recent developments and [12] for recent applications to fractional type advection-diffusion equation.

What will happen if the nonlinear differential equation (1.1) extends into fractional differential inclusions system? We are particularly interested in fractional differential inclusions with non-instantaneous impulses.

Deeply inspired by [6], in this paper, we first assume two increasing finite sequences of points $\{t_i\}_{i=1}^{p+1}$ and $\{s_i\}_{i=0}^{p}$ are given such that $s_0 = 0 < t_i \le s_i < t_{i+1},\ i = 1,2,\cdots,p$, and points $t_0,\ T \in \mathbb{R}_+$ are given such that $s_0 = 0 < t_0 < t_1,\ t_p < T \le t_{p+1}$, $p$ is a natural number.

Consider the following initial value problem for the nonlinear non-instantaneous impulsive $\psi$-Hilfer fractional differential inclusions

$$\begin{cases} {}^{H}D^{\alpha,\beta;\psi}_{t_0^+} x(t) \in A(t)x(t) + G\left(t,x(t)\right), & t \in J_1 = (s_k,t_{k+1}] \cap [t_0,T],\ k = 0,1,\cdots,p, \\ x(t) = \frac{\phi_k(t,x(t),x(t_k-0))}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in J_2 = (t_k,s_k] \cap [t_0,T],\ k = 1,2,\cdots,p, \\ x(t_0) = x_0, \end{cases} \tag{1.2}$$

where $J = J_1 \cup J_2$, $A(t) : D \subset X \to X$ is a bounded operator, $G : J_1 \times X \to P(X)$ is a multi-valued mapping, $P(X)$ is the family of all nonempty subsets of a real separable Banach space $X$, $x_0 \in X$, $\phi_k : J_2 \times X \times X \to X$, $(k = 1,2,\cdots,p)$. ${}^{H}D^{\alpha,\beta;\psi}_{0^+}$ is the $\psi$-Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $0 < \beta \le 1$, with respect to function $\psi$ (see [28]), $\gamma = \alpha + \beta(1-\alpha)$. The intervals $(t_k,s_k]$, $k = 1,2,\cdots$ are called intervals of non-instantaneous impulses and the functions $\phi_k(t,x,y)$, $k = 1,2,\cdots$ are called non-instantaneous impulsive functions.

**Remark 1.1.** *([5]) If $t_k = s_k$, $k = 1,2,\cdots,p$, then the system (1.2) reduces to an initial value problem for impulsive differential equations. In this case at any point of instantaneous impulse $t_k$ the amount of jump of the solution $x(t)$ is given by $I_k = \frac{\phi_k(t,x(t),x(t_k-0))}{\Gamma(\gamma)\Gamma(2-\gamma)}$.*

Compared with some recent results in the literatures, such as [14–16] and some others, the chief contributions of this study contain at least the following three:

(1) Non-instantaneous impulse is introduced into the $\psi$-Hilfer fractional differential inclusions system, and this is a significant breakthrough in researching the fractional differential inclusions system.

(2) The model we are concerned with is more generalized, if $\alpha \to 1$, $\beta = 1$, and there exists a function $f \in G(t,x)$, according to the theory mentioned in Section 2 and Section 3, the system (1.2) will degrade into system (1.1). Thus, the comprehensive model is originally discussed in the present paper.

(3) In papers [14–16], the authors respectively utilized Krasnoselskii's fixed point theorem, Darbo fixed point theorem and variational methods and critical point theory. Different from the approaches mentioned above, a ground-breaking method based on nonlinear alternative of Leray-Schauder type is exploited to discuss the initial value problem for $\psi$-Hilfer fractional differential inclusions with non-instantaneous impulses in this paper, and the results established are essentially new.

The following article is organized as follows: In Section 2, some lemmas and definitions will be given, and we will recall some known results for our considerations. Section 3 is devoted to researching the existence of solutions for system (1.2). To explain the results clearly, we finally provide an example in Section 4.

## 2. Preliminaries

In this section, we will introduce some basic definitions and lemmas which are useful throughout this paper.

**Definition 2.1.** *([29])(One parameter Mittag-Leffler function) The Mittag-Leffler function is given by the series*

$$E_\mu(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+1)},$$

*where $\mu\in\mathbb{C}$, $\Re(\mu)>0$ and $\Gamma(z)$ is a gamma function given by*

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt,$$

$\Re(z)>0$. *In particular, if $\mu=1$, we have $E_1(z)=\sum\limits_{j=0}^{\infty}\frac{z^j}{\Gamma(j+1)}=\sum\limits_{j=0}^{\infty}\frac{z^j}{j!}=e^z$.*

**Lemma 2.2.** *([29])(generalized Gronwall inequality) Let $u,v$ be two integrable functions and $g$ a continuous function, with domain $[a,b]$. Let $\psi\in C^1[a,b]$ an increasing function such that $\psi'(t)\neq 0$, $\forall\ t\in[a,b]$. Assume that*
*(1) $u$ and $v$ are nonnegative;*
*(2) $g$ is nonnegative and nondecreasing.*
*If*

$$u(t)\le v(t)+g(t)\int_a^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}u(s)ds,$$

*then*

$$u(t)\le v(t)+\int_a^t\left[\sum_{k=1}^{\infty}\frac{\left(g(t)\Gamma(\alpha)\right)^k}{\Gamma(\alpha k)}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha k-1}v(s)\right]ds,\ \forall\ t\in[a,b].$$

*Moreover, if $v(t)$ is a nondecreasing function on $[a,b]$, then*

$$u(t)\le v(t)E_\alpha\left(g(t)\Gamma(\alpha)\left[\psi(t)-\psi(s)\right]^{\alpha}\right),$$

*where $E_\alpha(\cdot)$ is the Mittag-Leffler function defined by Definition 2.1.*

**Definition 2.3.** *([28],[29]) Let $I=(a,b)$ $(-\infty\le a<b\le+\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha>0$. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b)$, having a continuous derivative $\psi'(t)$ on $(a,b)$. The left and right-sided fractional integrals of a function $\Phi$ with respect to another function $\psi$ on $[a,b]$ are defined by*

$$I_{a^+}^{\alpha;\psi}\Phi(t):=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(\xi)\left(\psi(t)-\psi(\xi)\right)^{\alpha-1}\Phi(\xi)d\xi$$

*for each $t\in I$.*

**Definition 2.4.** *([28]) Let $n-1<\alpha<n$ with $n\in\mathbb{N}^+$, $I=[a,b]$ be the interval such that $-\infty\le a<b\le+\infty$ and $\psi\in C^n([a,b],\mathbb{R})$ be a function such that $\psi$ is increasing and $\psi'(t)\neq 0$, for all $t\in I$. The left-sided $\psi$-Hilfer functional derivative ${}^HD_{0^+}^{\alpha,\beta;\psi}(\cdot)$ of order $\alpha$ and type $0\le\beta\le 1$ is defined by*

$$ {}^HD_{a^+}^{\alpha,\beta;\psi}f(t)=I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi}f(t). $$

*The right-sided $\psi$-Hilfer functional derivative is defined in an analogous form [28].*

**Lemma 2.5.** *([28]) If $f\in\mathcal{PC}^1_{\gamma;\psi}[a,b]$, $0<\alpha<1$, $0\le\beta\le1$, $\gamma=\alpha+\beta(1-\alpha)$, then*

$$ I_{a^+}^{\alpha;\psi}\,{}^HD_{a^+}^{\alpha,\beta;\psi}f(t)=f(t)-\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}I_{a^+}^{(1-\beta)(1-\alpha);\psi}f(a). $$

In this paper, let $X$ be a real separable Banach space with norm $\|\cdot\|$, let $Z\subset X$. Conveniently, we denote $P(X)=\{Z\subset X\mid Z\neq\emptyset\}$, $P_{cl}(X)=\{Z\subset P(X)\mid Z \text{ is closed}\}$, $P_b(X)=\{Z\subset P(X)\mid Z \text{ is bounded}\}$, $P_{cv}(X)=\{Z\subset P(X)\mid Z \text{ is}$
convex$\}$, $P_{cp}(X)=\{Z\subset P(X)\mid Z \text{ is compact}\}$, $P_{cv,cp}(X)=P_{cv}(X)\cap P_{cp}(X)$, $P_{b,cl}(X)=P_b(X)\cap P_{cl}(X)$ and so forth.

Let $C(J,X)$ denote the Banach space of all continuous functions from $J$ into $X$ with the norm $\|x\|_\infty=\max\{\|x(t)\|,$
$t\in J\}$.

We let $L^1(J,X)=\{x:J\to X\mid \|x\|:J\to[0,+\infty) \text{ be Lebesgue integrable}\}$, then $L^1(J,X)$ is a Banach space with the norm $\|x\|_{L^1}=\int_J\|x(t)\|dt$.

Assume that $PC(J,X)=\{x:J\to X\mid x(t) \text{ be continuous except abrupt moments}\}$, then $PC(J,X)$ is a Banach space with the norm $\|x\|_{PC}=\sup\{\|x(t)\|,t\in J\}$.

**Definition 2.6.** *([23]) The map $G$ is called upper semicontinuous (u.s.c.) on $x$ if for each $x_0\in X$, the set $G(x_0)$ is a nonempty, closed subset of $X$, and if for each open set $N$ of $X$ containing $G(x_0)$, there exists an open neighborhood $M$ of $x_0$ such that $G(M)\subseteq N$.*

**Definition 2.7.** *([13]) The multi-valued map $G:J_1\times X\to P(X)$ is said to be $L^1$-Carathéodory if*

*(i) $t\to G(t,x)$ is measurable for all $x\in X$, and*

*(ii) $x\to G(t,x)$ is u.s.c. on $X$ for almost each $t\in J_1$, in addition,*

*(iii) for each $q>0$, there exists $h_q\in L^1(J_1,R^+)$ such that*

$$ \|G(t,x)\|\le h_q(t),\text{for all } \|x\|\le q \text{ and almost each } t\in J_1. $$

For $x\in X$ and $Y$, $Z\in P_{b,cl}(X)$, let $D(x,Y)=\inf\{\|x-y\|\mid y\in Y\}$, $\rho(Y,Z)=\sup\limits_{y\in Y}D(y,Z)$ and Hausdorff metric $H:P_{b,cl}(X)\times P_{b,cl}(X)\to[0,+\infty)$ by $H(A,B)=\max\{\rho(A,B),\rho(B,A)\}$.

**Definition 2.8.** *([13]) Let $F:X\to P_{b,cl}(X)$ be multi-valued map. If there exists a constant $0<k<1$ such that $H(F(x),F(y))\le k\|x-y\|$ holds for each $x$, $y\in X$, then we say $F$ is a multi-valued contraction. The constant $k$ is called a contraction constant.*

**Definition 2.9.** *([13]) A multi-valued map $F$ is said to be completely continuous if $F(U)$ is relatively compact for every bounded subset $U\subseteq X$.*

**Definition 2.10.** *([13]) Let $S$ be a bounded subset of $X$.The Hausdorff measure of noncompactness of $S$ is define by*

$$ \alpha(s)=\inf\{\varepsilon>0\mid S \text{ has a finite cover by closed balls of radius } <\varepsilon\}. $$

**Definition 2.11.** *([13]) A multi-valued map $F : X \to P(X)$ is said to be a condensing map with respect to $\alpha$ (abbreviated, $\alpha$-condensing) if for every bounded set $D \subset X$, $\alpha(D) > 0$, $\alpha(F(D)) < \alpha(D)$.*

**Remark 2.12.** *([13])([26]) If multi-valued map $F$ is contractive, $G$ is completely continuous, then $F + G$ is a $\alpha$-condensing.*

**Lemma 2.13.** *([13]) Let $X$ be a Banach space, $G : J \times X \to P_{cv,cp}(X)$ be a $L^1$-Carathéodory mulitivalued map with*

$$S_{G,x} = \left\{ g \in L^1(J, X) \mid g(t) \in G(t, x) \text{ for a.e. } t \in J \right\} \neq \emptyset,$$

*and let $\Theta$ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$, then the operator*

$$\Theta \circ S_G : C(J, X) \to P_{cv,cp}(C(J, X)), u \mapsto (\Gamma \circ S_G)(u) := \Gamma(S_{G,u})$$

*is a closed graph operator in $C(J, X) \times C(J, X)$.*

The consideration of this paper is based on another nonlinear alternative of Leray-Schauder type for multi-valued maps due to O' Regan [30], we state it as follows.

**Lemma 2.14.** *Let $E$ be a Banach space with $U$ an open, convex subset of $E$ and $u_0 \in U$. Suppose*

*(a) $F : \overline{U} \to CD(E)$ has closed graph, and*

*(b) $F : \overline{U} \to CD(E)$ is a condensing map with $F(\overline{U})$ a subset of bounded set in $E$ hold. Then either*
*(i) $F$ has a fixed point in $\overline{U}$, or*
*(ii) there exists $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u) + (1 - \lambda)u_0$, where $CD(E)$ denotes the family of nonempty, closed and acyclic (see [19]) subsets of $E$.*

**Lemma 2.15.** *One can give a brief description of the solution $x(t; t_0, x_0)$ of initial value problem for system (1.2) as following*

$$x(t; t_0, x_0) = \begin{cases} X_k(t), & t \in (s_{k-1}, t_k] \cap [t_0, T],\ k = 1, 2, \cdots, \\ \frac{\phi_k(t, x(t; t_0, x_0), X_k(t_k - 0))}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in (t_k, s_k] \cap [t_0, T],\ k = 1, 2, \cdots. \end{cases} \tag{2.1}$$

**Proof** Assume that there exists $g \in S_{G,x}$ such that

$$^{H}D^{\alpha,\beta;\psi}_{t_0^+} x(t) = A(t)x(t) + g(t). \tag{2.2}$$

For any $t \in [t_0, t_1] \cap [t_0, T]$, applying the fractional integral operator $I^{\alpha;\psi}_{t_0{}^+}$ on both sides of the equation (2.2), and using Lemma 2.5, we have

$$\begin{aligned} x(t) =& \frac{(\psi(t) - \psi(t_0))^{\gamma-1}}{\Gamma(\gamma)} \frac{1}{\Gamma(1-\gamma)} \int_{t_0}^{t} \psi'(\xi) (\psi(t) - \psi(\xi))^{-\gamma} x(t_0) d\xi + I^{\alpha;\psi}_{t_0^+} (A(t)x(t) + g(t)) \\ =& \frac{x_0}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi) (\psi(t) - \psi(\xi))^{\alpha-1} (A(\xi)x(\xi) + g(\xi)) d\xi. \end{aligned} \tag{2.3}$$

On the interval $(t_1, s_1] \cap [t_0, T]$, the solution $x(t; t_0, x_0)$ satisfies the equation

$$\begin{aligned} x(t; t_0, x_0) =& \frac{\phi_1(t, x(t; t_0, x_0), X_1(t_1 - 0))}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ =& \frac{\phi_1(t, x(t; t_0, x_0), x(t_1 - 0; t_0, x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)}. \end{aligned} \tag{2.4}$$

For each $t \in (s_1, t_2] \cap [t_0, T]$, by the same approach, we have

$$x(t) = \frac{\phi_1(s_1, x(s_1; t_0, x_0), x(t_1 - 0, t_0, x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)} \int_{s_1}^{t} \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} (A(\xi)x(\xi) + g(\xi)) d\xi. \tag{2.5}$$

The functions on the rest intervals can be done in the same manner. Hence, the solution $x(t)$ of system (1.2) satisfies the following system of integral equations

$$x(t; t_0, x_0) = \begin{cases} \frac{x_0}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} (A(\xi)x(\xi) + g(\xi)) d\xi, \\ \qquad t \in [t_0, t_1] \cap [t_0, T], \\ \frac{\phi_k(t, x(t; t_0, x_0), x(t_k - 0; t_0, x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)}, \qquad t \in (t_k, s_k] \cap [t_0, T], k = 1, 2, \cdots, p, \\ \frac{\phi_k(s_k, x(s_k; t_0, x_0), x(t_k - 0; t_0, x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} (A(\xi)x(\xi) + g(\xi)) d\xi, \\ \qquad t \in [s_k, t_{k+1}] \cap [t_0, T], k = 1, 2, \cdots, p. \end{cases} \tag{2.6}$$

The proof of this Lemma is completed.

## 3. Main result

In this section, we will state and prove our main theorem. Before the main work, we impose the following assumptions.

$(H_0)$ For any $x \in J$, there exists a constant $M_A > 0$ such that $\|A(t) \cdot x(t)\| \le M_A \|x(t)\|$.

$(H_1)$ There exist $\theta_k, L_{1k}, L_{2k} > 0$ such that, for each $t \in J$, $x, y \in X$, $\|\phi_k(t, x, y)\| \le \theta_k$, $\|\phi_k(t, x_1, y_1) - \phi_k(t, x_2, y_2)\| \le L_{1k}\|x_1 - x_2\| + L_{2k}\|y_1 - y_2\|$, $k = 1, 2, \cdots, p$, and

$$0 < \frac{L_{1k} + L_{2k}}{\Gamma(\gamma)\Gamma(2-\gamma)} < 1$$

holds.

$(H_2)$ $G : J \times X \to P_{b,cv,cp}(X)$, $(t, x) \mapsto G(t, x)$ is measurable with respect to $t$, for each $x \in X$; u.s.c. with respect to $x$, for each $t \in J$; and for each fixed $x \in X$, the set

$$S_{G,x} = \left\{ g \in L^1(J, X) \mid g(t) \in G(t, x(t)) \text{ for a.e. } t \in J \right\} \ne \emptyset.$$

$(H_3)$ There exists a continuous nondecreasing function $\Phi : [0, +\infty) \to [0, +\infty)$, a function $p \in L^1(J, [0, +\infty))$, and a bounded constant $K > 0$, such that

$$\|G(t, x)\| = \sup\{\|g\| \mid g \in G(t, x)\} \le p(t)\Phi(\|x\|), \ \forall\, x \in X, \ a.e. \ t \in J,$$

and

$$\frac{K}{1 + \int_{t_0}^{T} \sum_{n=1}^{\infty} \frac{(M_A)^n}{\Gamma(n\alpha)} \psi'(\xi)(\psi(T) - \psi(\xi))^{n\alpha-1} d\xi} > M, \tag{3.1}$$

where

$$M = \max\{M_1, M_2\},$$

and

$$M_1 = \frac{\Phi(K)}{\Gamma(\alpha)} \int_{t_0}^{T} \psi'(\xi)(\psi(T) - \psi(\xi))^{\alpha-1} p(\xi) d\xi + \frac{\|x_0\|_{PC}}{\Gamma(\gamma)\Gamma(2-\gamma)},$$

and

$$M_2 = \frac{\Phi(K)}{\Gamma(\alpha)} \int_{t_0}^{T} \psi'(\xi)(\psi(T) - \psi(\xi))^{\alpha-1} p(\xi) d\xi + \frac{\theta_k}{\Gamma(\gamma)\Gamma(2-\gamma)},$$

where $k = 1, 2, \cdots, p$.

**Theorem 3.1.** *Suppose that the conditions* $(H_0)$-$(H_3)$ *are satisfied, then the system* (1.2) *has at least one solution on* $J$.

**Proof** One can transform system (1.2) into a fixed point problem. By Lemma 2.15, we consider the multi-valued map $N : PC(J,X) \to P(PC(J,X))$ defined by:

$$N(x) = \left\{ h \in PC(J,X) \mid h(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}(A(\xi)x(\xi)+g(\xi))d\xi \\ \quad + \frac{x_0}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in [t_0,t_1] \cap [t_0,T], \\ \frac{\phi_k(t,x(t;t_0,x_0),x(t_k-0;t_0,x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in (t_k,s_k] \cap [t_0,T], \\ \frac{1}{\Gamma(\alpha)} \int_{s_k}^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}(A(\xi)x(\xi)+g(\xi))d\xi \\ \quad + \frac{\phi_k(s_k,x(s_k;t_0,x_0),x(t_k-0;t_0,x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in [s_k,t_{k+1}] \cap [t_0,T], \end{cases} \right\}. \tag{3.2}$$

where $k = 1,2,\cdots,p$.

We can easily know that the fixed points of $N$ are the solutions of (1.2). Hence, our key work is to demonstrate that $N$ has a fixed point. The proof will be given in following five steps.

Step 1. Choose $u_0 = 0 \in U$ and an open, convex set $U$ in Banach space $PC(J,X)$. The most important and difficult step is to find the proper set $U$.

Let $x \in \lambda N(x)$, for some $\lambda \in (0,1)$. As for $t \in [t_0,t_1] \cap [t_0,T]$, then there exists $g \in S_{G,x}$ such that

$$x(t) = \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}(A(\xi)x(\xi)+g(\xi))d\xi + \frac{x_0\lambda}{\Gamma(\gamma)\Gamma(2-\gamma)}. \tag{3.3}$$

This implies by $(H_0)$, we have

$$\begin{aligned} \|x(t)\| \le& \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}(\|A(\xi)\cdot x(\xi)\| + \|g(\xi)\|)d\xi + \frac{\|x_0\|}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ \le& \frac{M_A}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\|x(\xi)\|d\xi + \frac{\|x_0\|}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\|g(\xi)\|d\xi. \end{aligned} \tag{3.4}$$

Let

$$a(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\|g(\xi)\|d\xi + \frac{\|x_0\|}{\Gamma(\gamma)\Gamma(2-\gamma)}, \quad 0 < \alpha \le 1, \tag{3.5}$$

then

$$\begin{aligned} a'(t) =& \frac{1}{\Gamma(\alpha)} \lim_{\xi \to t^-} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\|g(\xi)\| + \frac{\alpha-1}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-2}\psi'(t)\|g(\xi)\|d\xi \\ \ge& 0. \end{aligned} \tag{3.6}$$

Therefore, $a(t)$ is a non-decreasing function, and combined with conditions $(H_3)$, we have

$$\begin{aligned} a(t) \le& a(t_1) \le \frac{1}{\Gamma(\alpha)} \int_{t_0}^T \psi'(\xi)(\psi(T)-\psi(\xi))^{\alpha-1}\|g(\xi)\|d\xi + \frac{\|x_0\|}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ \le& \frac{\Phi(\|x\|)}{\Gamma(\alpha)} \int_{t_0}^T \psi'(\xi)(\psi(T)-\psi(\xi))^{\alpha-1}p(\xi)d\xi + \frac{\|x_0\|}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ \le& M_1. \end{aligned} \tag{3.7}$$

By (3.4), (3.5), (3.7) and generalized Gronwall inequality, we have

$$\begin{aligned}\|x(t)\| \leq & \frac{M_A}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\|x(\xi)\|d\xi+M_1\\ \leq & M_1+\int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{(M_A)^n}{\Gamma(n\alpha)}\psi'(\xi)(\psi(t)-\psi(\xi))^{n\alpha-1}M_1 d\xi\\ = & M_1\left(1+\int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{(M_A)^n}{\Gamma(n\alpha)}\psi'(\xi)(\psi(t)-\psi(\xi))^{n\alpha-1}d\xi\right)\\ \leq & M_1\left(1+\int_{t_0}^{T}\sum_{n=1}^{\infty}\frac{(M_A)^n}{\Gamma(n\alpha)}\psi'(\xi)(\psi(T)-\psi(\xi))^{n\alpha-1}d\xi\right).\end{aligned} \tag{3.8}$$

For any $t\in[s_k,t_{k+1}]\cap[t_0,T], k=1,2,\cdots,p$. By $(H_0)$ and $(H_1)$, we have

$$\begin{aligned}\|x(t)\| \leq & \frac{M_A}{\Gamma(\alpha)}\int_{s_k}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\|x(\xi)\|d\xi+\frac{\theta_k}{\Gamma(\gamma)\Gamma(2-\gamma)}\\ & +\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\|g(\xi)\|d\xi.\end{aligned} \tag{3.9}$$

Let

$$b(t)=\frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\|g(\xi)\|d\xi+\frac{\theta_k}{\Gamma(\gamma)\Gamma(2-\gamma)}, \tag{3.10}$$

and $b(t)$ is a non-decreasing function. We have

$$\begin{aligned}b(t) \leq & b(T)\leq\frac{\Phi(\|x\|)}{\Gamma(\alpha)}\int_{t_0}^{T}\psi'(\xi)(\psi(T)-\psi(\xi))^{\alpha-1}p(\xi)d\xi+\frac{\theta_k}{\Gamma(\gamma)\Gamma(2-\gamma)}\\ \leq & M_2\,.\end{aligned} \tag{3.11}$$

By (3.9)-(3.11) and generalized Gronwall inequality, we have

$$\|x(t)\|\leq M_2\left(1+\int_{t_0}^{T}\sum_{n=1}^{\infty}\frac{(M_A)^n}{\Gamma(n\alpha)}\psi'(\xi)(\psi(T)-\psi(\xi))^{n\alpha-1}d\xi\right). \tag{3.12}$$

For $t\in(t_k,s_k]\cap[t_0,T]$, $k=1,2,\cdots,p$, we have

$$\|x(t)\|\leq\frac{\theta_k}{\Gamma(\gamma)\Gamma(2-\gamma)}<M_2. \tag{3.13}$$

Let

$$M=\max\{M_1,M_2\}, \tag{3.14}$$

then for all $t\in J$, one can obtain

$$\|x\|_{PC}\leq M\left(1+\int_{t_0}^{T}\sum_{n=1}^{\infty}\frac{(M_A)^n}{\Gamma(n\alpha)}\psi'(\xi)(\psi(T)-\psi(\xi))^{n\alpha-1}d\xi\right). \tag{3.15}$$

Consequently,

$$\frac{\|x\|_{PC}}{1+\int_{t_0}^{T}\sum\limits_{n=1}^{\infty}\frac{(M_A)^n}{\Gamma(n\alpha)}\psi'(\xi)(\psi(T)-\psi(\xi))^{n\alpha-1}d\xi}\leq M\,. \tag{3.16}$$

By $(H_3)$, we let $U = \{x \in PC(J,X) \mid \|x\|_{PC} < K\}$, where $K$ satisfies $\|x\|_{PC} \neq K$. Therefore, there is no $x \in \partial U$ such that $x = \lambda N(x)$, for some $\lambda \in (0,1)$.

Step 2. $N(x)$ is convex, for all $x \in PC(J,X)$.

Indeed, if $h_1, h_2 \in N(x)$, then there exist $g_1, g_2 \in S_{G,x}$ such that for each $t \in J$, we have

$$h_i(t) = \begin{cases} \frac{x_0}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\,(A(\xi)x(\xi)+g_i(\xi))d\xi, \\ \qquad\qquad t \in [t_0,t_1]\cap[t_0,T], \\ \frac{\phi_k(t,x(t;t_0,x_0),x(t_k-0;t_0,x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)}, \qquad t \in (t_k,s_k]\cap[t_0,T], k=1,2,\cdots,p, \\ \frac{\phi_k(s_k,x(s_k;t_0,x_0),x(t_k-0;t_0,x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}\times \\ (A(\xi)x(\xi)+g_i(\xi))d\xi, \qquad t \in [s_k,t_{k+1}]\cap[t_0,T], k=1,2,\cdots,p, \end{cases} \tag{3.17}$$

where $i = 1,2$.

Let $0 \le \lambda \le 1$, then for each $t \in J$, we have

$$[\lambda h_1 + (1-\lambda)h_2](t) = \begin{cases} \frac{x_0}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}(A(\xi)x(\xi)+\lambda g_1(\xi)+ \\ (1-\lambda)g_2(\xi))d\xi, \qquad t \in [t_0,t_1]\cap[t_0,T], \\ \frac{\phi_k(t,x(t;t_0,x_0),x(t_k-0;t_0,x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)}, \qquad t \in (t_k,s_k]\cap[t_0,T], k=1,2,\cdots,p, \\ \frac{\phi_k(s_k,x(s_k;t_0,x_0),x(t_k-0;t_0,x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}(A(\xi)x(\xi)+ \\ \lambda g_1(\xi)+(1-\lambda)g_2(\xi))d\xi, \qquad t \in [s_k,t_{k+1}]\cap[t_0,T], k=1,2,\cdots,p, \end{cases} \tag{3.18}$$

By $(H_2)$, $S_{G,x}$ has convex values, then $\lambda h_1 + (1-\lambda)h_2 \in N(x)$, so $N(\overline{U})$ is convex, which implies that $N(\overline{U})$ is a acyclic subset of $P(PC(J,X))$.

Step 3. $N(U)$ is bounded.

That is to say, there exists a positive constant $l$ satisfying $\|N(x)\| = \{\|h\|_{PC} \mid h \in N(x)\} \le l$ for each $h \in N(x)$, $x \in U$.

Similar to (3.15), for each $t \in J$, we have

$$\|h(t)\| \le M\left(1 + \int_{t_0}^{T}\sum_{n=1}^{\infty}\frac{(M_A)^n}{\Gamma(n\alpha)}\psi'(\xi)(\psi(T)-\psi(\xi))^{n\alpha-1}\,d\xi\right). \tag{3.19}$$

Then for each $h \in N(U)$, we have

$$\begin{aligned}\|h(t)\|_{PC} \le& M\left(1 + \int_{t_0}^{T}\sum_{n=1}^{\infty}\frac{(M_A)^n}{\Gamma(n\alpha)}\psi'(\xi)(\psi(T)-\psi(\xi))^{n\alpha-1}\,d\xi\right) \\ :=& l.\end{aligned} \tag{3.20}$$

Step 4. $N : \overline{U} \to P(PC(J,X))$ has a closed graph. Consider the operator $\Theta : L^1(J,X) \to C(J,X)$, $\Lambda = t_0$ or $s_k$, $k = 1,2,\cdots,p$,

$$g \mapsto \Theta(g)(t) = \frac{1}{\Gamma(\alpha)}\int_{\Lambda}^{t}\psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}g(\xi)d\xi.$$

We can see that the operator $\Theta$ is linear and continuous. By Lemma 2.13, we can see $\Theta_i \circ S_G$ is closed graph operator.

Step 5. $N : \overline{U} \to P(PC(J,X))$ is a condensing mapping.

For $t \in [t_0,t_1]\cap[t_0,T]$, we define two maps. Let $N_1 : \overline{U} \to PC(J,X)$ be defined by

$$N_1(x) = \frac{x_0}{\Gamma(\gamma)\Gamma(2-\gamma)}, \tag{3.21}$$

and $N_2 : \overline{U} \to PC(J, X)$ be defined by

$$N_2(x) = \left\{ h_2 \in PC(J, X) \mid h_2(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} (A(\xi)x(\xi) + g(\xi))d\xi \right\}, \tag{3.22}$$

where $g \in S_{G,x}$, then $N(x) = N_1(x) + N_2(x)$ on $\overline{U}$. From Remark 2.12, we need to prove that $N_1$ is contractive and $N_2$ is completely continuous.

The key process is to demonstrate $N_2 : \overline{U} \to P(PC(J, X))$ is completely continuous. We will accomplish it with following steps $S_1$-$S_3$:

$S_1$. Similar to (3.4)-(3.8), and by $(H_3)$, we have

$$\|h_2\| \le M_3 \left( 1 + \int_{t_0}^{T} \sum_{n=1}^{\infty} \frac{(M_A)^n}{\Gamma(n\alpha)} \psi'(\xi)(\psi(T) - \psi(\xi))^{n\alpha-1} d\xi \right) < K, \tag{3.23}$$

where $M_3 = \frac{\Phi(K)}{\Gamma(\alpha)} \int_{t_0}^{T} \psi'(\xi)(\psi(T) - \psi(\xi))^{\alpha-1} p(\xi)d\xi < M$, $M$ is defined as (3.14). Thus, $N_2(\overline{U}) \subset P(PC(J, X))$. In view of nonnegativeness and continuity of $A$, $x$, $g$, we have $N_2(\overline{U}) \to P(PC(J, X))$ is continuous.

$S_2$. Let $\Omega \subset \overline{U}$ be bounded, it is easy to get

$$\|N_2(x)\| < +\infty. \tag{3.24}$$

Hence, $N_2(\Omega)$ is uniformly bounded.

$S_3$. We show that $N_2$ sends $\overline{U}$ into equicontinuous sets of $P(PC(J, X))$. Let $\tau_1, \tau_2 \in J_1$, $\tau_1 < \tau_2$, then for $x \in \overline{U}$, we have

$$\begin{aligned} \|h_2(\tau_2) - h_2(\tau_1)\| \le & \frac{1}{\Gamma(\alpha)} \int_{t_0}^{\tau_1} \psi'(\xi) \left[ (\psi(\tau_2) - \psi(\xi))^{\alpha-1} - (\psi(\tau_1) - \psi(\xi))^{\alpha-1} \right] \times \\ & (\|A(\xi)x(\xi)\| + \|g(\xi)\|)d\xi + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \psi'(\xi)(\psi(\tau_2) - \psi(\xi))^{\alpha-1} (\|A(\xi)x(\xi)\| + \|g(\xi)\|)d\xi, \end{aligned} \tag{3.25}$$

and by the Lagrange mean-value theorem, we have

$$\begin{aligned} \|h_2(\tau_2) - h_2(\tau_1)\| \le & \frac{(\alpha-1)(\tau_2 - \tau_1)\psi'(\eta)}{\Gamma(\alpha)} \int_{t_0}^{\tau_1} \psi'(\xi)(\psi(\eta) - \psi(\xi))^{\alpha-2} (\|A(\xi)x(\xi)\| + \|g(\xi)\|)d\xi \\ & + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} (\|A(\xi)x(\xi)\| + \|g(\xi)\|)d\xi, \end{aligned} \tag{3.26}$$

where $\tau_1 < \eta < \tau_2$. As $\tau_1 \to \tau_2$ , the right-hand side of the above inequality tends to zero. Therefore, $N_2(\Omega)$ is equicontinuous. By Arzela-Ascoli theorem, we have $N_2 : \overline{U} \to P(PC(J, X))$ is completely continuous. Coupled with the $N_1$ is contractive, we conclude that $N = N_1 + N_2$ is a condensing mapping.

For $t \in [s_k, t_{k+1}] \cap [t_0, T], k = 1, 2, \cdots, p$, we define two maps. Let $N_3 : \overline{U} \to PC(J, X)$ be defined by

$$N_3(x) = \frac{\phi_k(s_k, x(s_k; t_0, x_0), x(t_k - 0; t_0, x_0))}{\Gamma(\gamma)\Gamma(2-\gamma)}, \tag{3.27}$$

and $N_4 : \overline{U} \to PC(J, X)$ be defined by

$$N_4(x) = \left\{ h_4 \in PC(J, X) \mid h_4(t) = \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} (A(\xi)x(\xi) + g(\xi))d\xi \right\}, \tag{3.28}$$

where $g \in S_{G,x}$, then $N = N_3 + N_4$ on $\overline{U}$. We prove that $N_3$ is a contraction on $PC(J, X)$ as following:

Let $x$, $y \in \overline{U}$, by $(H_1)$, we have

$$\begin{aligned} \|N_3(x) - N_3(y)\| \le & \frac{L_{1k} + L_{2k}}{\Gamma(\gamma)\Gamma(2-\gamma)} \cdot \|x - y\| \\ < & \|x - y\|, \end{aligned} \tag{3.29}$$

we see that $N_3$ is a contractive mapping.

Same works as proving that $N_2$ is completely continuous, $N_4 : \overline{U} \to P(PC(J,X))$ is also completely continuous. we demonstrate that $N = N_3 + N_4$ is a condensing mapping.

For $t \in (t_k, s_k] \cap [t_0, T]$, $k = 1, 2, \cdots, p$, we also get $N$ is a condensing map.

All of the conditions of Lemma 2.14 are satisfied. Hence, $N$ has a fixed point $x \in \overline{U}$, which is a solution of (1.2).

## 4. Illustrative Example

In this section, we will present an example to illustrate our main results.

**Example 4.1.** *Suppose that* $\psi(t) = 0.1\ln(2+t)$, $\alpha = 0.3$, $\beta \to 1$, *then* $\gamma = \alpha + \beta(1-\alpha) \to 1$, *and* $\Gamma(\alpha) = 2.99$, $\Gamma(\beta) = \Gamma(\gamma) = 1$. *Let* $[t_0, T] = [1, 10]$, *and* $A(t)x(t) = e^{-t}x(t)$, *then* $M_A = \frac{1}{e}$. *Assume that* $G(t, x(t)) = [0, \|x(t)\|]$, $\phi_k(t, x(t), x(t_k - 0)) = a_k + \frac{1}{10}\sin x(t) + \frac{1}{5}\cos x(t_k - 0)$, $a_k$ *are constants,* $k = 1, 2, \cdots, p$, *then* $L_{1k} = \frac{1}{10}$, $L_{2k} = \frac{1}{5}$, *and we choose* $g = |\sin t| \cdot \|x(t)\|$, $p(t) = 1$, $\Phi(\|x\|) = \|x\|$. *If* $\|x_0\|_{PC} = 1$, *and* $\max\{\theta_k : k = 1, 2, \cdots, p\} = 1$, *then we have*

$$M = \frac{\Phi(K)}{\Gamma(\alpha)} \int_{t_0}^{T} \psi'(\xi)(\psi(T) - \psi(\xi))^{\alpha-1} p(\xi) d\xi + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)} = 0.62K + 1. \tag{5.1}$$

*By Mathematica software, we know*

$$1 + \int_{t_0}^{T} \sum_{n=1}^{\infty} \frac{(M_A)^n}{\Gamma(n\alpha)} \psi'(\xi)(\psi(T) - \psi(\xi))^{n\alpha-1} d\xi \approx 1.28.$$

*Then inequality* (3.1) *can be simplified as*

$$\frac{K}{1.28} > M. \tag{5.2}$$

*then we can get*

$$K > 6.10. \tag{5.3}$$

*By Theorem 3.1, we know the differential inclusions system has at least one solution* $x$, $\|x\| \le K$, *and* $K$ *satisfies inequality 5.3.*

## 5. Conclusions

In this paper, we are concerned with a kind of $\psi$-Hilfer fractional order differential inclusions. The addressed system with the fractional order has non-instantaneous impulsive effects, which are quite different from the related references discussed in the literatures [22, 24, 25, 27]. The fractional-order nonlinear differential system studied in the present paper is more generalized and more practical. With the help of generalized Gronwall inequality and Lagrange mean-value theorem, we use another nonlinear alternative of Leray-Schauder type and employ a novel argument, and the easily verifiable sufficient conditions have been provided to determine the existence of solutions for the system (1.2). Finally, a typical example has been presented at the end of this paper to illustrate the effectiveness and feasibility of the proposed criterion. Consequently, this paper shows theoretically and numerically that some related references known in the literature can be enriched and complemented.

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