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Higher Schwarzian Derivative and Dirichlet Morrey Space

Shuan Tang[a], Guangmin Hu[b,*], Qingtian Shi[c], Jianjun Jin[d]

[a] *School of Mathematics Sciences, Guizhou Normal University, Guiyang, 550001, P.R.China*
[b] *College of Science, Jinling Institute of Technology, Nanjing, 211169, P.R.China*
[c] *School of Mathematics, Quanzhou Normal University, Fujian, 362000, P.R.China*
[d] *School of Mathematics Sciences, Hefei University of Technology, Xuancheng Campus, Xuancheng, 242000, P.R.China*

Abstract. We treat the logarithmic derivative model and Schwarzian derivative model of the Dirichlet-Morrey Teichmüller space. It is shown that the higher Bers maps, induced by the higher Schwarzian differential operators, are holomorphic in Dirichlet-Morrey Teichmüller space. It is also shown that the logarithmic derivative model of this Teichmüller space is connected.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disc in the extended complex plane $\widehat{\mathbb{C}}$. Let $\mathbb{D}^*$ be the exterior of $\overline{\mathbb{D}}$ and $S^1 = \partial\mathbb{D}$ be the boundary of $\mathbb{D}$. Denote by $M(\mathbb{D}^*)$ the open unit ball of the Banach space $L^\infty(\mathbb{D}^*)$ of all Beltrami differentials $\mu(z)$ on $\mathbb{D}^*$. It is well konwn that for each $\mu(z) \in M(\mathbb{D}^*)$, there exists a unique quasiconformal mapping $f^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ whose complex dilatation is equal to μ in $\mathbb{D}^*$ and is zero in $\mathbb{D}$, normalized by

$$f^\mu(0) = (f^\mu)'(0) - 1 = (f^\mu)''(0) = 0, \tag{1}$$

(see [1] [3]). Two Beltrami coefficients μ_1 and μ_2 in $M(\mathbb{D}^*)$ are said to be Teichmüller equivalent, denoted by $\mu_1 \sim \mu_2$, if $f^{\mu_1}(\mathbb{D}) = f^{\mu_2}(\mathbb{D})$. The universal Teichmüller space T is defined as $T = M(\mathbb{D}^*)/\sim$, where $[\mu]$ is the Teichmüller equivalent class containing $\mu \in M(\mathbb{D}^*)$.

The Schwarzian derivative S_f of a conformal mapping f in $\mathbb{D}$ is defined by

$$S_f = \left(N_f\right)' - \frac{1}{2}\left(N_f\right)^2, \qquad \text{where } N_f = (\log f')'.$$

Denote by $B_n(\mathbb{D})$ the Banach space of all holomorphic functions φ in $\mathbb{D}$ with the following finite norm

$$\|\varphi\|_n = \sup_{z\in\mathbb{D}} |\varphi(z)|(1-|z|^2)^n, \quad n = 1, 2, \cdots. \tag{2}$$

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*Corresponding author: Guangmin Hu
Email addresses: tsaflyhigher@163.com (Shuan Tang), 18810692738@163.com (Guangmin Hu), shiqingtian2013@gmail.com (Qingtian Shi), jinjjhb@163.com (Jianjun Jin)

It is well known that the Bers projection

$$\beta_3 : M(\mathbb{D}^*) \to B_2(\mathbb{D}), \quad \beta_3(\mu) = S_{f^\mu|_{\mathbb{D}}},$$

is a holomorphic split submersion from $M(\mathbb{D}^*)$ onto its image, which descends down to the Bers embedding $\mathbb{B} : T \to B_2(\mathbb{D})$. Via the Bers embedding, T carries a natural complex Banach manifold structure so that $\Phi : M(\mathbb{D}^*) \to T$ is a holomorphic split submersion which seeds μ to the equivalent class $[\mu]$ (see [17], [18]).

It is of interest to embed the universal Teichmüller space onto an open subset of some complex Banach space of holomorphic functions in $\mathbb{D}$ in terms of some general differential operators. Krushkal considered in [16] some nonlinear differential operators of higher order of the form

$$P_n(f) = F\left(\frac{f''(z)}{f'(z)}, \frac{f'''(z)}{f'(z)}, \cdots, \frac{f^{(n)}(z)}{f'(z)}, f''(z), \cdots, f^{(n)}(z)\right), \quad z \in \mathbb{D},$$

where F is an analytic function of its arguments($n \geq 2$). It was proved [16] that the map $P_n : M(\mathbb{D}^*) \to B_{n-1}(\mathbb{D})$, which is defined by the correspondence of $\mu \in M(\mathbb{D}^*)$ to $P_n(f^\mu) \in B_{n-1}(\mathbb{D})$, $n \geq 3$, is holomorphic.

Schippers considered in [20] some other nonlinear differential operators. Let $n \geq 3$, define $\sigma_3(f) = S_f$ and

$$\sigma_{n+1}(f)(z) = \sigma_n'(f)(z) - (n-1)N_f(z)\sigma_n(f)(z). \tag{3}$$

For more general differential operators, we refer the reader to [2], [13], [15], [23] and references therein.

Buss [7] proved that the higher Bers map $\beta_n : M(\mathbb{D}^*) \to B_{n-1}(\mathbb{D})$, which is defined by the correspondence of $\mu \in M(\mathbb{D}^*)$ to $\sigma_n(f^\mu) \in B_{n-1}(\mathbb{D})$, $n \geq 3$, is holomorphic.

Theorem 1.1. *[7] Let $n \geq 3$. The higher Bers map $\beta_n : M(\mathbb{D}^*) \to B_{n-1}(\mathbb{D})$ is holomorphic. The differential $D_0\beta_n$ at the origin is given by the following correspondence*

$$\nu \mapsto \frac{(-1)^n n!}{\pi} \iint_{\mathbb{D}^*} \frac{\nu(w)}{(z-w)^{n+1}} dudv,$$

which induces a bounded surjective operator from $L^\infty(\mathbb{D}^*)$ onto $B_{n-1}(\mathbb{D})$.

It should be pointed out that the case $n = 3$ is the classical result of Bers [6]. The higher Bers maps on Weil-Petersson and BMO Teichmüller space were also investigated recently by the authors (see [25] [26]). In this paper, we will treat the higher Bers maps on the Dirichlet-Morrey Teichmüller space.

Let

$$S_{\mathbb{D}}(I) = \{r\zeta \in \mathbb{D} : 1 - |I| \leq r < 1, \zeta \in I\}$$

denote the Carleson square in $\mathbb{D}$ and

$$S_{\mathbb{D}^*}(I) = \{r\zeta \in \mathbb{D}^* : 1 \leq r < 1 + |I|, \zeta \in I\}$$

denote the Carleson square in $\mathbb{D}^*$, where I be an open sub-arc of S^1. For $0 < q < \infty$, a non-negative Borel measure μ on $\mathbb{D}$ is called q-Carleson measure if

$$\|\mu\|^2_{\mathbb{D},q} := \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S_{\mathbb{D}}(I))}{|I|^q} < \infty.$$

Replacing $S_{\mathbb{D}}(I)$ by $S_{\mathbb{D}^*}(I)$, we can define q-Carleson measure on $\mathbb{D}^*$ similarly. Clearly, μ is the classical Carleson measure for the case $q = 1$. Denote by $CM_q(\mathbb{D})$ the set of all q-Carleson measures on $\mathbb{D}$ and $CM_q(\mathbb{D}^*)$ the set of all q-Carleson measures on $\mathbb{D}^*$. It is well known that a non-negative Borel measure μ belongs to $CM_q(\mathbb{D})$ if and only if

$$\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)^p}{|1-\bar{w}z|^{p+q}} d\mu(z) < \infty, \tag{4}$$

where $p \in (0, \infty)$ (see [30]).

The Bloch space $\mathfrak{B}$ consists of all analytic functions f in $\mathbb{D}$ so that

$$\|f\|_{\mathfrak{B}} := \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty.$$

The little Bloch space $\mathfrak{B}_0$, a closed subspace of $\mathfrak{B}$, consists of functions $f \in \mathfrak{B}$ such that

$$\lim_{|z|\to 1^-} |f'(z)|(1-|z|^2) = 0.$$

Let $0 \le p < \infty$, the weighted Dirichlet space $\mathcal{D}^p(\mathbb{D})$ is the set of all analytic functions f in $\mathbb{D}$ for which

$$\|f\|^2_{\mathcal{D}^p} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^p dm(z) < \infty,$$

where $dm(z)$ denotes the normalized Lebesgue area measure.

For $0 < \lambda, p \le 1$, the Dirichlet-Morrey space $\mathcal{D}^p_\lambda(\mathbb{D})$, introduced recently in [12], consists of those analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{D}^p_\lambda(\mathbb{D})} = \sup_{a\in\mathbb{D}} \left((1-|a|^2)^{\frac{p(1-\lambda)}{2}} \|f\circ\varphi_a - f(a)\|_{\mathcal{D}^p}\right) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}, a \in \mathbb{D}$. Some basic properties of Dirichlet-Morrey space were characterized in [12].

The authors obtained in [24] the following result.

Theorem 1.2. *[24] Suppose that f is a bounded univalent function in $\mathbb{D}$ and $\log f' \in \mathfrak{B}_0$, $0 < \lambda < 1$ and $0 < p \le 1$. Then the following statements are equivalent:*

(1) $\log f' \in \mathcal{D}^p_\lambda(\mathbb{D})$;

(2) $|S_f(z)|^2(1-|z|^2)^{p+2}dm(z) \in CM_{p\lambda}(\mathbb{D})$;

(3) *f can be extended to a quasiconformal mapping in the extended plane $\widehat{\mathbb{C}}$ such that its complex dilatation μ satisfies $\frac{|\mu(z)|^2}{(|z|^2-1)^{2-p}}dm(z) \in CM_{p\lambda}(\mathbb{D}^*)$.*

Denote by $\mathcal{L}(\mathbb{D}^*)$ the Banach space of all essentially bounded measurable functions μ on $\mathbb{D}^*$ each of which induces a $p\lambda$-Carleson measure $\eta_\mu = \frac{|\mu(z)|^2}{(|z|^2-1)^{2-p}}dm(z)$. The norm on $\mathcal{L}(\mathbb{D}^*)$ is defined as

$$\|\mu\|_{\mathcal{L}} = \|\mu\|_\infty + \|\eta_\mu\|_{\mathbb{D}^*,p\lambda} < \infty.$$

Let $\mathfrak{M}(\mathbb{D}^*) = M(\mathbb{D}^*) \cap \mathcal{L}(\mathbb{D}^*)$. Dirichlet-Morrey Teichmüller space T_{DM} is defined as $\mathfrak{M}(\mathbb{D}^*)/\sim$, where $\sim$ denotes the Teichmüller equivalent relation defined as above.

We use $\mathcal{N}_{p\lambda,n}(\mathbb{D})$ $(n \ge 3)$ to denote the space of all analytic functions f in $\mathbb{D}$ with the norm

$$\|f\|^2_{\mathcal{N}_{p\lambda,n}} = \sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{2n-4+p}\left(\frac{(1-|a|^2)}{|1-\bar{a}z|^2}\right)^p dm(z) < \infty.$$

In this paper, we shall prove the following

Theorem 1.3. *Let $n \ge 3$. The higher Bers map $\beta_n : \mathfrak{M}(\mathbb{D}^*) \to \mathcal{N}_{p\lambda,n}(\mathbb{D})$ is well defined and holomorphic. The differential $D_0\beta_n$ at the origin is given by the following correspondence*

$$\mu \mapsto \frac{(-1)^n n!}{\pi} \int_{\mathbb{D}^*} \frac{\mu(w)}{(z-w)^{n+1}} dudv.$$

We also consider the pre-logarithmic derivative model of the Dirichlet-Morrey Teichmüller space. Let us first recall some notions and definitions.

Let S_Q be the class of all univalent analytic functions f in $\mathbb{D}$, which can be extended to a quasiconformal mapping in $\widehat{\mathbb{C}}$, normalized by $f(0) = f'(0) - 1 = 0$. Then the universal Teichmüller space can be described as $T(1) = \{\log f' : f$ belongs to $S_Q\}$. It is well known that $T(1)$ is a disconnected subset of Bloch space $\mathfrak{B}$, and $T_b = \{\log f' \in T(1) : f(\mathbb{D})$ is bounded$\}$, $T_\theta = \{\log f' \in T(1) : f(e^{i\theta}) = \infty\}$, $\theta \in [0, 2\pi)$ are connected components of $T(1)$ (see [32]).

In recent years, the pre-logarithmic derivative model of the universal Teichmüller space and its subspaces have been much investigated (See [4] [5] [8] [9] [10] [14] [21] [22] [11] [27] [28] [29] [32]).

We consider the pre-logarithmic derivative model $T^0_{DM}(1)$ of Dirichlet-Morrey Teichmüller space, which is defined as

$$T^0_{DM}(1) = \{\log f' : f \in S_Q \text{ and } \log f' \in \mathfrak{B}_0 \cap \mathcal{D}^p_\lambda(\mathbb{D})\}.$$

We endow the space $\mathfrak{B}_0 \cap \mathcal{D}^p_\lambda(\mathbb{D})$ the following norm

$$\|\psi\|_{\mathfrak{B},\mathcal{D}^p_\lambda} = \|\psi\|_{\mathfrak{B}} + \|\psi\|_{\mathcal{D}^p_\lambda(\mathbb{D})}.$$

Let $T^0_{DM,b}(1) = \{\ \log f' \in T^0_{DM}(1) : f(\mathbb{D})$ is bounded $\}$. We obtain the following

Theorem 1.4. *$T^0_{DM,b}(1)$ is connected in $\mathfrak{B}_0 \cap \mathcal{D}^p_\lambda(\mathbb{D})$.*

Throughout this paper, we use the notation $a \lesssim b$ to denote that there is a constant $C > 0$ such that $a \leq Cb$, and the notation $a \approx b$ to indicate that $a \lesssim b \lesssim a$.

2. Proof of Theorem 1.3

We shall prove Theorem 1.3 in this section. Some lemmas are needed. The following result gives some higher derivative characterizations of $\mathcal{D}^p_\lambda(\mathbb{D})$ (see [12]).

Lemma 2.1. [12] *Let f be an analytic function on $\mathbb{D}$ and $0 < p, \lambda \leq 1$. Then $d\mu(z) = |f(z)|^2(1-|z|^2)^p dm(z)$ is a $p\lambda$-Carleson measure if and only if $d\nu(z) = |f'(z)|^2(1-|z|^2)^{p+2}dm(z)$ is a $p\lambda$-Carleson measure. Furthermore,*

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}|f(z)|^2(1-|z|^2)^p\Big(\frac{(1-|a|^2)}{|1-\bar{a}z|^2}\Big)^p dm(z) \approx$$

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)^{p+2}\Big(\frac{(1-|a|^2)}{|1-\bar{a}z|^2}\Big)^p dm(z).$$

We also need the following result (see [31]).

Lemma 2.2. [31] *Suppose that $k > -1$, $r, t > 0$, and $r + t - k > 2$. If $t < k + 2 < r$, then there exists a universal constant $C > 0$ such that for all $z, \zeta \in \mathbb{D}$,*

$$\int_{\mathbb{D}}\frac{(1-|w|^2)^k}{|1-\overline{w}z|^r|1-\overline{w}\zeta|^t}dm(w) \leq C\frac{(1-|z|^2)^{2+k-r}}{|1-\overline{\zeta}z|^t},$$

where $w = u + iv$.

We now show that the higher Bers map is well defined on Dirichlet-Morrey Teichmüller space.

Lemma 2.3. *Let $n \geq 3$. If $\mu \in \mathfrak{M}(\mathbb{D}^*)$, then $\sigma_n(f^\mu) \in \mathcal{N}_{p\lambda,n}(\mathbb{D})$.*

Proof. We will prove this Lemma by using mathematical induction. It follows from Corollary 2.5 in [24] that if $\mu \in \mathfrak{M}(\mathbb{D}^*)$, then $\sigma_3(f^\mu)(z) \in \mathcal{N}_{p\lambda,3}$. Now suppose that $\sigma_n(f^\mu) \in \mathcal{N}_{p\lambda,n}, n \geq 3$, we shall prove that $\sigma_{n+1}(f^\mu) \in \mathcal{N}_{p\lambda,n+1}$.

Indeed, by Lemma 2.1, we have

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p|\sigma_n'(f^\mu)(z)|^2(1-|z|^2)^{2n-2+p}dm(z)<\infty. \tag{5}$$

Observing that

$$\sigma_{n+1}(f^\mu)(z)=\sigma_n'(f^\mu)(z)-(n-1)N_{f^\mu}(z)\sigma_n(f^\mu)(z), n\geq 3,$$

we deduce that

$$|\sigma_{n+1(f^\mu)}(z)|\leq|\sigma_n'(f^\mu)(z)|+|(n-1)N_{f^\mu}\sigma_n(f^\mu)(z)|. \tag{6}$$

Noting that f^μ is a univalent analytic function in $\mathbb{D}$, we conclude from [19] that

$$\sup_{z\in\mathbb{D}}|N_{f^\mu}(z)|(1-|z|^2)\leq 6. \tag{7}$$

Consequently, combing (5), (6) with (7) gives

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p|\sigma_{n+1}(f^\mu)(z)|^2(1-|z|^2)^{2n-2+p}dm(z)\\
&\lesssim\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p|\sigma_n'(f^\mu)(z)|^2(1-|z|^2)^{2n-2+p}dm(z)\\
&+\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p|\sigma_n(f^\mu)(z)|^2(1-|z|^2)^{2n-4+p}dxdy\\
&<\infty.
\end{aligned}$$

This implies that $\sigma_{n+1}(f^\mu) \in \mathcal{N}_{p\lambda,n+1}$. The proof of Lemma 2.3 is completed. □

The following result shows that the Bers map $\beta_3 : \mathfrak{M}(\mathbb{D}^*) \to \mathcal{N}_{p\lambda,3}(\mathbb{D})$ is Lipschitz continuous.

Lemma 2.4. *Let $0 < p, \lambda \leq 1$. For any $\mu, \nu \in \mathfrak{M}(\mathbb{D}^*)$, the following inequality holds.*

$$\|\beta_3(\mu)-\beta_3(\nu)\|_{\mathcal{N}_{p\lambda,3}}\lesssim\|\mu-\nu\|_{\mathcal{L}}.$$

Proof. In [4], it is proved that for any two elements $\mu, \nu \in M(\mathbb{D}^*)$,

$$|\beta_3(\mu)-\beta_3(\nu)|^2(1-|z|^2)^2\lesssim\int_{\mathbb{D}^*}\frac{|\mu(\zeta)-\nu(\zeta)|^2+\|\mu-\nu\|_\infty^2|\mu(\zeta)|^2}{|\zeta-z|^4}dm(\zeta).$$

Therefore,

$$\begin{aligned}
\|\beta_3(\mu)-\beta_3(\nu)\|_{\mathcal{N}_{p\lambda,3}}^2&=\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p|\beta_3(\mu)-\beta_3(\nu)|^2(1-|z|^2)^{2+p}dm(z)\\
&\lesssim\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\int_{\mathbb{D}^*}\frac{|\mu(\zeta)-\nu(\zeta)|^2}{|\zeta-z|^4}dm(\zeta)\right)(1-|z|^2)^p\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p dm(z)\\
&+\|\mu-\nu\|_\infty^2\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\int_{\mathbb{D}^*}\frac{|\mu(\zeta)|^2}{|\zeta-z|^4}dm(\zeta)\right)(1-|z|^2)^p\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p dm(z).
\end{aligned}$$

Consequently, by a change of variable $\zeta = \frac{1}{\tau}$, we get

$$\|\beta_3(\mu) - \beta_3(\nu)\|^2_{\mathcal{N}_{p\lambda,3}} \lesssim \sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} \frac{|\mu(\frac{1}{\bar{\tau}}) - \nu(\frac{1}{\bar{\tau}})|^2}{(1-|\tau|^2)^{2-p}} \Big(\frac{1-|a|^2}{|1-\bar{a}\tau|^2}\Big)^p dudv$$
$$\times \int_{\mathbb{D}} \frac{(1-|z|^2)^p(1-|\tau|^2)^{2-p}|1-\bar{a}\tau|^{2p}}{|1-\overline{\tau}z|^4|1-\bar{a}z|^{2p}} dxdy$$
$$+ \|\mu-\nu\|^2_\infty \sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} \frac{|\mu(\frac{1}{\bar{\tau}})|^2}{(1-|\tau|^2)^{2-p}} \Big(\frac{1-|a|^2}{|1-\bar{a}\tau|^2}\Big)^p dudv$$
$$\times \int_{\mathbb{D}} \frac{(1-|z|^2)^p(1-|\tau|^2)^{2-p}|1-\bar{a}\tau|^{2p}}{|1-\overline{\tau}z|^4|1-\bar{a}z|^{2p}} dxdy \tag{8}$$

In [24], we have proved that if the complex dilatation μ satisfies

$$\frac{|\ \mu(z)\ |^2}{(|z|^2-1)^{2-p}} dm(z) \in CM_{p\lambda}(\mathbb{D}^*),$$

then

$$\frac{|\ \mu(\frac{1}{\bar{z}})\ |^2}{(1-|z|^2)^{2-p}} dm(z) \in CM_{p\lambda}(\mathbb{D}). \tag{9}$$

Therefore, combing (4), (8) with (9) and using Lemma 2.2 yields

$$\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathcal{N}_{K,3}} \lesssim \|\mu-\nu\|_{\mathcal{L}}.$$

This completes the proof of Lemma 2.4. □

It should be pointed out that the case $p = 1, \lambda = 1$ has been proved in [22].

We are now in a position to prove Theomren 1.3.

Proof. We first show that the higher Bers map $\beta_n : \mathfrak{M}(\mathbb{D}^*) \to \mathcal{N}_{p\lambda,n}(\mathbb{D})$ is continuous. For simplicity of notations, for any $\mu, \nu \in \mathfrak{M}(\mathbb{D}^*)$, we use f to denote the quasiconformal mapping whose complex dilatation is equal to μ in $\mathbb{D}^*$ and is zero in $\mathbb{D}$, and g to denote the quasiconformal mapping whose complex dilatation is equal to ν in $\mathbb{D}^*$ and is zero in $\mathbb{D}$, both normalized

$$f(0) = f'(0) - 1 = f''(0) = 0 \quad \text{and} \quad g(0) = g'(0) - 1 = g''(0) = 0.$$

By the definition of the higher Schwarzian derivative, we have

$$\|\sigma_{n+1}(f) - \sigma_{n+1}(g)\|_{\mathcal{N}_{p\lambda,n+1}} \le \|\sigma'_n(f) - \sigma'_n(g)\|_{\mathcal{N}_{p\lambda,n+1}}$$
$$+ (n-1)\|N_f\sigma_n(f) - N_g\sigma_n(g)\|_{\mathcal{N}_{p\lambda,n+1}}. \tag{10}$$

It follows from Lemma 2.1 that

$$\|\sigma'_n(f) - \sigma'_n(g)\|_{\mathcal{N}_{p\lambda,n+1}} \approx \|\sigma_n(f) - \sigma_n(g)\|_{\mathcal{N}_{p\lambda,n}}. \tag{11}$$

Note that

$$|N_f\sigma_n(f) - N_g\sigma_n(g)| \le |N_f||\sigma_n(f) - \sigma_n(g)| + |\sigma_n(g)||N_f - N_g|. \tag{12}$$

We conclude from (11) and (12) that

$$\|N_f\sigma_n(f) - N_g\sigma_n(g)\|_{\mathcal{N}_{p\lambda,n+1}} \le \|N_f\|_{\mathfrak{B}}\|\sigma_n(f) - \sigma_n(g)\|_{\mathcal{N}_{p\lambda,n}}$$
$$+ \|\sigma_n(g)\|_{\mathcal{N}_{p\lambda,n}}\|N_f - N_g\|_{\mathfrak{B}}. \tag{13}$$

By Theorem 3.1 in Chapter II in [17], there is a constant $C > 0$ such that

$$\|N_f - N_g\|_{\mathfrak{B}} \le C\|\mu - \nu\|_\infty. \tag{14}$$

Consequently, combing (7), (10), (13) with (14) yields

$$\|\sigma_{n+1}(f) - \sigma_{n+1}(g)\|_{\mathcal{N}_{p\lambda,n+1}} \lesssim \|\sigma_n(f) - \sigma_n(g)\|_{\mathcal{N}_{p\lambda,n}} + \|\mu - \nu\|_\infty.$$

Repeating this process $n-3$ times gives

$$\|\sigma_{n+1}(f) - \sigma_{n+1}(g)\|_{\mathcal{N}_{p\lambda,n+1}} \lesssim \|\sigma_3(f) - \sigma_3(g)\|_{\mathcal{N}_{p\lambda,3}} + \|\mu - \nu\|_\infty.$$

By Lemma 2.4, we get

$$\|\sigma_{n+1}(f) - \sigma_{n+1}(g)\|_{\mathcal{N}_{p\lambda,n+1}} \lesssim \|\mu - \nu\|_{\mathcal{L}}.$$

This implies that the higher Bers map $\beta_n : \mathfrak{M}(\mathbb{D}^*) \to \mathcal{N}_{p\lambda,n}(\mathbb{D})$ is continuous.

We now turn to show that the higher Bers map $\beta_n : \mathfrak{M}(\mathbb{D}^*) \to \mathcal{N}_{p\lambda,n}(\mathbb{D})$ is holomorphic. Since we have proved that β_n is continuous, it is sufficient to show that for any $\mu \in \mathfrak{M}(\mathbb{D}^*)$ and $\nu \in \mathcal{L}(\mathbb{D}^*)$, $\beta_n(\mu + t\nu)$ is holomorphic in a small neighborhood of $t = 0$ in the complex plane. Since $\mu \in \mathfrak{M}(\mathbb{D}^*)$, there exists a positive constant ϵ such that for any t with $|t| < 2\epsilon$,

$$\|\mu + t\nu\|_\infty < 1 \quad \text{and} \quad \|\mu + t\nu\|_{\mathcal{L}} < \infty.$$

For simplicity of notations, we use $\psi(t)$ to denote $\beta_n(\mu + t\nu)$. For fixed $z \in \mathbb{D}$, the function $\psi(t)$ is holomorphic in $|t| < 2\epsilon$. For $|t| < \epsilon$, $|t_0| < \epsilon$, it follows from Cauchy formula that

$$\begin{aligned}\left|\frac{\psi(t)(z) - \psi(t_0)(z)}{t - t_0} - \frac{d}{dt}|_{t=t_0}\psi(t)(z)\right| &= \frac{|t - t_0|}{2\pi}\left|\int_{|s|=2\epsilon} \frac{\psi(s)(z)}{(s-t)(s-t_0)^2}ds\right| \\ &\le \frac{|t - t_0|}{2\pi\epsilon^3}\int_{|s|=2\epsilon} |\psi(s)(z)||ds|. \end{aligned} \tag{15}$$

Using Fubini theorem yields

$$\begin{aligned}&(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p\left|\frac{\psi(t)(z) - \psi(t_0)(z)}{t - t_0} - \frac{d}{dt}|_{t=t_0}\psi(t)(z)\right|^2(1-|z|^2)^{2n-4+p}dxdy \\ &\le (1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p\frac{|t-t_0|^2}{4\pi^2\epsilon^6}\int_{\mathbb{D}}\left(\int_{|s|=2\epsilon}|\psi(s)(z)||ds|\right)^2(1-|z|^2)^{2n-4+p}dxdy \\ &\lesssim |t-t_0|^2(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\int_{|s|=2\epsilon}|\psi(s)(z)|^2|ds|(1-|z|^2)^{2n-4+p}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p dxdy \\ &= |t-t_0|^2\int_{|s|=2\epsilon}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p|\psi(s)(z)|^2(1-|z|^2)^{2n-4+p}dxdy|ds| \\ &\lesssim |t-t_0|^2.\end{aligned}$$

This implies that the limit

$$\lim_{t\to t_0}\frac{\psi(t) - \psi(t_0)}{t - t_0} = \frac{d}{dt}|_{t=t_0}\psi(t)$$

exists in $\mathcal{N}_{p\lambda,n}(\mathbb{D})$. Thus, we conclude that $\beta_n : \mathfrak{M}(\mathbb{D}^*) \to \mathcal{N}_{p\lambda,n}(\mathbb{D})$ is holomorphic.

Furthermore, Buss proved in Theorem 3.4 in [7] that

$$\frac{d}{dt}|_{t=0}\psi(t)(z) = \frac{(-1)^n n!}{\pi}\int_{\mathbb{D}^*}\frac{\mu(w)}{(z-w)^{n+1}}dudv.$$

The proof follows. □

3. The connectivity of $T^0_{DM,b}(1)$

In this section, we shall prove Theorem 1.4. Let $r > 1$ and $\Delta_r = \{z : |z| < r\}$. A Beltrami differential $\mu(z) \in M(\mathbb{D}^*)$ is called a vanishing Beltrami differential if for any $\epsilon > 0$, there exists $r > 1$ such that $\|\mu|_{\Delta_r}\|_\infty < \epsilon$. Denote by $M^0(\mathbb{D}^*)$ the collection of all vanishing Beltrami differentials.

Let $\mathfrak{M}^0(\mathbb{D}^* = \mathfrak{M}(\mathbb{D}^*) \cap M^0(\mathbb{D}^*)$. For each $\mu(z) \in \mathfrak{M}^0(\mathbb{D}^*$, there exists a unique quasiconformal mapping $f^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined as in the introduction such that $f^\mu(\mathbb{D})$ is bounded. Define the pre-Bers projection mapping L_b on $\mathfrak{M}^0(\mathbb{D}^*)$ by setting $L_b(\mu) = \log(f^\mu)'$. To prove Theorem 1.4, we need the following result which has its own interest.

Proposition 3.1. *The pre-Bers projection mapping $L_b : \mathfrak{M}^0(\mathbb{D}^*) \to \mathcal{D}^p_\lambda(\mathbb{D})$ is well defined and holomorphic.*

Proof. For any $\mu \in \mathfrak{M}^0(\mathbb{D}^*) \subset M^0(\mathbb{D}^*)$, it follows from [5] that $\log(f^\mu)' \in \mathfrak{B}_0$. It also follows from Theorem 1.2 that $\log(f^\mu)' \in \mathcal{D}^p_\lambda(\mathbb{D})$. Therefore, the pre-Bers projection mapping $L_b : \mathfrak{M}^0(\mathbb{D}^*) \to \mathcal{D}^p_\lambda(\mathbb{D})$ is well defined. To prove that $L_b : \mathfrak{M}^0(\mathbb{D}^*) \to \mathcal{D}^p_\lambda(\mathbb{D})$ is holomorphic, we first show that it is continuous. For $\mu, \nu \in \mathfrak{M}^0(\mathbb{D}^*)$, it follows from Theorem 3.1 in Chapter II in [17] that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{(f^\mu)''}{(f^\mu)'} - \frac{(f^\nu)''}{(f^\nu)'}\right| \lesssim \|\mu - \nu\|_\infty.$$

By Lemma 2.4, we have

$$\|\beta_3(\mu) - \beta_3(\nu)\|_{\mathcal{N}_{p\lambda,3}} \lesssim \|\mu - \nu\|_{\mathcal{L}}.$$

Therefore, from Lemma 2.1, we get

$$\begin{aligned}
\|\log(f^\mu)' - \log(f^\nu)'\|^2_{\mathcal{D}^p_\lambda(\mathbb{D})} &\approx \sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p \\
&\quad\times\left|\frac{(f^\mu)''}{(f^\mu)'} - \frac{(f^\nu)''}{(f^\nu)'}\right|^2(1-|z|^2)^p dxdy \\
&\approx \sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p\left|\left(\frac{(f^\mu)''}{(f^\mu)'}\right)' - \left(\frac{(f^\nu)''}{(f^\nu)'}\right)'\right|^2(1-|z|^2)^{p+2}dxdy \\
&\lesssim \sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p\left|S_{f^\mu} - S_{f^\nu}\right|^2(1-|z|^2)^{p+2}dxdy \\
&+ \sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p\left|\left(\frac{(f^\mu)''}{(f^\mu)'}\right)^2 - \left(\frac{(f^\nu)''}{(f^\nu)'}\right)^2\right|^2(1-|z|^2)^{p+2}dxdy \\
&\lesssim \|\beta_3(\mu) - \beta_3(\nu)\|^2_{\mathcal{N}_{p\lambda,3}} + \sup_{z\in\mathbb{D}}\left\{(1-|z|^2)^2\left|\frac{(f^\mu)''}{(f^\mu)'} - \frac{(f^\nu)''}{(f^\nu)'}\right|^2\right\} \\
&\times\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p\left|\frac{(f^\mu)''}{(f^\mu)'} + \frac{(f^\nu)''}{(f^\nu)'}\right|^2(1-|z|^2)^p dxdy \\
&\lesssim \|\mu-\nu\|^2_{\mathcal{L}} + \|\mu-\nu\|^2_\infty(\|\log(f^\mu)'\|^2_{\mathcal{D}^p_\lambda(\mathbb{D})} + \|\log(f^\nu)'\|^2_{\mathcal{D}^p_\lambda(\mathbb{D})}) \\
&\lesssim \|\mu-\nu\|^2_{\mathcal{L}}.
\end{aligned}$$

This implies that $L_b : \mathfrak{M}^0(\mathbb{D}^*) \to \mathcal{D}^p_\lambda(\mathbb{D})$ is continuous.

Similar to the proof of Theorem 1.3, it remains to show that for any $\mu \in \mathfrak{M}^0(\mathbb{D}^*)$ and $\nu \in \mathcal{L}(\mathbb{D}^*)$, $L_b(\mu + t\nu)$ is holomorphic in a small neighborhood of $t = 0$ in the complex plane. Chose a positive constant ϵ such that for any t with $|t| < 2\epsilon$, $\|\mu + t\nu\|_\infty < 1$ and $\|\mu + t\nu\|_{\mathcal{L}} < \infty$. We abbreviate the function $L_b(\mu + t\nu)$ by $\phi(t)$. For fixed $z \in \mathbb{D}$, the function $\phi(t)$ is holomorphic in $|t| < 2\epsilon$ (see [18]) and (15) still holds for $\phi(t)$.

Thus, by Fubini theorem, we deduce that

$$
\begin{aligned}
&(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}\left|\frac{\phi(t)(z)-\phi(t_0)(z)}{t-t_0}-\frac{d}{dt}|_{t=t_0}\phi(t)(z)\right|^2(1-|z|^2)^p\Big(\frac{(1-|a|^2)}{|1-\bar{a}z|^2}\Big)^p dxdy\\
&\le (1-|a|^2)^{p(1-\lambda)}\frac{|t-t_0|^2}{4\pi^2\epsilon^6}\int_{\mathbb{D}}\Big(\int_{|s|=2\epsilon}|\phi(s)(z)||ds|\Big)^2(1-|z|^2)^p\Big(\frac{(1-|a|^2)}{|1-\bar{a}z|^2}\Big)^p dxdy\\
&\lesssim (1-|a|^2)^{p(1-\lambda)}|t-t_0|^2\int_{\mathbb{D}}\int_{|s|=2\epsilon}|\phi(s)(z)|^2|ds|(1-|z|^2)^p\Big(\frac{(1-|a|^2)}{|1-\bar{a}z|^2}\Big)^p dxdy\\
&=|t-t_0|^2\int_{|s|=2\epsilon}(1-|a|^2)^{p(1-\lambda)}\int_{\mathbb{D}}|\phi(s)(z)|^2(1-|z|^2)^p\Big(\frac{(1-|a|^2)}{|1-\bar{a}z|^2}\Big)^p dxdy|ds|\\
&\lesssim |t-t_0|^2.
\end{aligned}
$$

Therefore, we deduce that the limit

$$\lim_{t\to t_0}\frac{\phi(t)-\phi(t_0)}{t-t_0}=\frac{d}{dt}|_{t=t_0}\phi(t)$$

exists in $\mathcal{D}^p_\lambda(\mathbb{D})$. This implies that $L_b:\mathfrak{M}^0(\mathbb{D}^*)\to\mathcal{D}^p_\lambda(\mathbb{D})$ is holomorphic and the proof follows. □

We now start our proof of Theorem 1.4.

Proof. Let $\log f'\in T^0_{DM}(1)$. By Theorem 1.2, f can be extended to a quasiconformal mapping to the whole plane such that its complex dilatation μ satisfies $\frac{|\mu(z)|^2}{(|z|^2-1)}dxdy\in CM_{p\lambda}(\mathbb{D}^*)$. Let f^t be the quasiconformal mapping in $\widehat{\mathbb{C}}$ with $f^{-1}(\infty)=(f^t)^{-1}(\infty)$ and $\bar\partial f^t=t\mu\partial f^t$. We now prove the path $t\longmapsto\log(f^t)', 0\le t\le 1$ is continuous in $\mathfrak{B}_0\cap\mathcal{D}^p_\lambda(\mathbb{D})$.

For f^{t_1}, f^{t_2}, we conclude from Proposition 3.1 that

$$\|\log(f^{t_1})'-\log(f^{t_2})'\|_{\mathcal{D}^p_\lambda(\mathbb{D})}\lesssim|t_1-t_2|\cdot\|\mu\|_{\mathcal{L}}.$$

On the other hand, by (14) (see Theorem 3.1 in Chapter II in [17]), we get

$$\|\log(f^{t_1})'-\log(f^{t_2})'\|_{\mathfrak{B}}\lesssim|t_1-t_2|\cdot\|\mu\|_\infty.$$

Thus, we deduce that

$$\|\log(f^{t_1})'-\log(f^{t_2})'\|_{\mathfrak{B},\mathcal{D}^p_\lambda}\lesssim|t_1-t_2|\cdot\|\mu\|_{\mathcal{L}}.$$

This means that the path $t\longmapsto\log(f^t)', 0\le t\le 1$ is continuous in $\mathfrak{B}_0\cap\mathcal{D}^p_\lambda(\mathbb{D})$.

Therefore, each $\log f'\in T^0_{DM}(1)$ can be connected by a continuous path in $\mathfrak{B}_0\cap\mathcal{D}^p_\lambda(\mathbb{D})$ to a Möbius transformation γ with $\log\gamma'\in T^0_{DM}(1)$. Observe that $\gamma(\mathbb{D})$ is bounded, it follows that the path $\rho\longmapsto\log\gamma'_\rho$ connects the point $\log\gamma'$ to the point 0 in $T^0_{DM}(1)$, where $\gamma_\rho=\gamma(\rho z)$. This implies that $T^0_{DM,b}(1)=\{\log f'\in T^0_{DM}(1): f(\mathbb{D})\text{ is bounded}\}$ is connected in $\mathfrak{B}_0\cap\mathcal{D}^p_\lambda(\mathbb{D})$. □

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