



## Some Approximation Properties of New Families of Positive Linear Operators

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**Abstract.** In the present article, we propose the new class positive linear operators, which discrete type depending on a real parameters. These operators are similar to Jain operators but its approximation properties are different then Jain operators. Theorems of degree of approximation, direct results, Voronovskaya Asymptotic formula and statistical convergence are discussed.

### 1. Introduction

During the last decade two types of generalizations of the classical Poisson, binomial and negative binomial distribution, useful in biology, ecology and medicine have been introduced by considering the two basic forms of Lagrange series

$$\phi(z) = \phi(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[ \frac{d^{k-1}}{dz^{k-1}} (f(z))^k \phi'(z) \right]_{z=0} \left( \frac{z}{f(z)} \right)^k; \quad (1)$$

$$\phi(z) \left[ 1 - \frac{z}{f(z)} \frac{df(z)}{dz} \right]^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{d^k}{dz^k} (f(z))^k \phi(z) \right]_{z=0} \left( \frac{z}{f(z)} \right)^k \quad (2)$$

and expanding suitable function  $\phi(z)$  into powers of  $\frac{z}{f(z)}$  for suitably chosen function  $f(z)$ .

By putting  $\phi(z) = e^{\alpha z}$  and  $f(z) = e^{\beta z}$  in formulas (1) and (2), we achieved Generalized Poisson Distribution (GPD) studied by Consul and Jain [1, 2] and Linear Function Poisson Distribution(LFPD) was introduced and studied by Jain [3].

Since 1912, Bernstein Polynomial and its various generalization have been studied by Bernstein [4], Szász [5], Meyer-Konig and Zeller [6], Cheney and Sharma [7], Stancu [8]. Bernstein polynomials are based on binomial and negative binomial distributions. In 1941, Szász and Mirakyan [9] have introduced operator using the Poisson distribution. We mention that rate of convergence developed by Rempulska and Walczak [10], asymptotic expansion introduced by Abel *et al.* [11]. In 1976, May [12] showed that the Baskakov

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operators can reduce to the Szász-Mirakyan operators.

The Lagrange formula (1) was used to establish the Jain operators [13], which are as follows

$$J_n^{[\beta]}(f, x) = \sum_{k=0}^{\infty} \omega_{\beta}(k, nx) f\left(\frac{k}{n}\right), \tag{3}$$

where  $\omega_{\beta}(k, nx) = nx(nx + k\beta)^{k-1} \frac{e^{-(nx+k\beta)}}{k!}$ ;  $0 \leq \beta < 1$  and  $f \in C[0, \infty)$ .

The operators (3) are generalization of well known the Szász-Mirakyan operators. In 2013, Agratini [14] discussed the relation between the local smoothness of function and local approximation. Also, the degree of approximation and the statistical convergence of the sequence (3) was studied in [14]. We mention that a Kantorovich-type extension of the Jain operators was given in [15]. Additionally, the Durrmeyer type generalization of the Jain operators was established in [16–19]. The Jain operators was also developed in two variables in [20]. The Jain type variant of Lupas operators [21] was studied by Patel and Mishra in [22]. Due to their properties, the operators  $J_n^{[\beta]}$  and  $J_n^{[0]}$  have been intensively studied by many mathematicians. Thus, in our opinion, the class  $P_n^{[\beta]}$  defined in (7) should deeper investigate.

In this manuscript, we use Lagrange formula (2) to establish new sequence of positive linear operators. The approximation properties establish in this manuscript are different then the Jain operators (3). Local approximation properties, the rate of convergence, weighted approximation, asymptotic formula and statistical convergence are investigated for the sequence of the operators (7).

For  $0 < \alpha < \infty$  and  $|\beta| < 1$ , proceed by setting  $\phi(z) = e^{\alpha z}$  and  $f(z) = e^{\beta z}$ , in Lagrange formula (2), we get

$$e^{\alpha z} [1 - \beta z]^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + k\beta)^k \left(\frac{z}{e^{\beta z}}\right)^k.$$

Therefore, we shall have

$$e^{\alpha z} = (1 - \beta z) \sum_{k=0}^{\infty} (\alpha + \beta k)^k \frac{u^k}{k!}; \quad u = ze^{-\beta z}, \tag{4}$$

where  $z$  and  $u$  are sufficiently small such that  $|\beta u| < a^{-1}$  and  $|\beta z| < 1$ .

By taking  $z = 1$ , we have

$$1 = (1 - \beta) \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + \beta k)^k e^{-(\alpha + \beta k)}. \tag{5}$$

Define

$$p_{\beta}(k, \alpha) = (1 - \beta) \frac{1}{k!} (\alpha + \beta k)^k e^{-(\alpha + \beta k)}. \tag{6}$$

Now, from equality (5), we can write

$$\sum_{k=0}^{\infty} p_{\beta}(k, \alpha) = 1,$$

for  $0 < \alpha < \infty$  and  $|\beta| < 1$ .

## 2. Construction of the Operators

We may now define the operator as

$$P_n^{[\beta]}(f, x) = \sum_{k=0}^{\infty} p_{\beta}(k, nx) f\left(\frac{k}{n}\right), \tag{7}$$

where  $0 \leq \beta < 1$  and  $\mathbf{p}_\beta(k, nx)$  is as defined in (6).

**Lemma 2.1.** Let  $0 < \alpha < \infty$ ,  $|\beta| < 1$  and  $r \in \mathbb{N}$ ,

$$S(r, \alpha, \beta) = \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + \beta k)^{k+r} e^{-(\alpha+\beta k)}$$

and

$$(1 - \beta)S(0, \alpha, \beta) = 1.$$

Then

$$S(r, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k (\alpha + k\beta) S(r - 1, \alpha + k\beta, \beta).$$

**Proof:** Notice that

$$\begin{aligned} S(r, \alpha, \beta) &= \alpha \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + \beta k)^{k+r-1} e^{-(\alpha+\beta k)} + \beta \sum_{k=0}^{\infty} \frac{1}{k!} k (\alpha + \beta k)^{k+r-1} e^{-(\alpha+\beta k)} \\ &= \alpha S(r - 1, \alpha, \beta) + \beta \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + \beta + \beta k)^{k+r} e^{-(\alpha+\beta+\beta k)} \\ &= \alpha S(r - 1, \alpha, \beta) + \beta S(r, \alpha + \beta, \beta). \end{aligned} \tag{8}$$

By a repeated use of (8), the proof of the lemma is archived.

Now when  $|\beta| < 1$ , we have

$$S(1, \alpha, \beta) = \sum_{k=0}^{\infty} \frac{\beta^k (\alpha + k\beta)}{1 - \beta} = \frac{\alpha}{(1 - \beta)^2} + \frac{\beta^2}{(1 - \beta)^3}; \tag{9}$$

$$S(2, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k (\alpha + k\beta) \left( \frac{\alpha + k\beta}{(1 - \beta)^2} + \frac{\beta^2}{(1 - \beta)^3} \right) = \frac{\alpha^2}{(1 - \beta)^3} + \frac{3\alpha\beta^2}{(1 - \beta)^4} + \frac{\beta^3(1 + 2\beta)}{(1 - \beta)^5}; \tag{10}$$

$$S(3, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k (\alpha + k\beta) S(2, \alpha + k\beta, \beta) = \frac{\alpha^3}{(1 - \beta)^4} + \frac{6\alpha^2\beta^2}{(1 - \beta)^5} + \frac{\alpha\beta^3(4 + 11\beta)}{(1 - \beta)^6} + \frac{\beta^4 + 8\beta^5 + 6\beta^6}{(1 - \beta)^7}; \tag{11}$$

$$\begin{aligned} S(4, \alpha, \beta) &= \sum_{k=0}^{\infty} \beta^k (\alpha + k\beta) S(3, \alpha + k\beta, \beta) \\ &= \frac{\alpha^4}{(1 - \beta)^5} + \frac{10\alpha^3\beta^2}{(1 - \beta)^6} + \frac{5\alpha^2(2\beta^3 + 7\beta^4)}{(1 - \beta)^7} \\ &\quad + \frac{5\alpha(\beta^4 + 10\beta^5 + 10\beta^6)}{(1 - \beta)^8} + \frac{\beta^5 + 22\beta^6 + 58\beta^7 + 24\beta^8}{(1 - \beta)^9}. \end{aligned} \tag{12}$$

### 3. Estimation of Moments

We should note that, the first moment of the operators (3) gives  $J_n^{[\beta]}(t, x) = ax$ , for some real constant  $a$ , but for the operators (7), we have  $P_n^{[\beta]}(t, x) = cx + b$  for some real constants  $b$  and  $c$ . Due to this operators (3) and (7) have different approximation properties. We required following results to prove main results.

**Lemma 3.1.** The operators  $P_n^{[\beta]}$   $n \geq 1$ , defined by (7) satisfy the following relations

- $P_n^{[\beta]}(1, x) = 1;$

$$\begin{aligned}
 2. P_n^{[\beta]}(t, x) &= \frac{x}{(1-\beta)} + \frac{\beta}{n(1-\beta)^2}; \\
 3. P_n^{[\beta]}(t^2, x) &= \frac{x^2}{(1-\beta)^2} + \frac{x(1+2\beta)}{n(1-\beta)^3} + \frac{\beta(1+2\beta)}{n^2(1-\beta)^4}; \\
 4. P_n^{[\beta]}(t^3, x) &= \frac{x^3}{(1-\beta)^3} + \frac{3x^2(1+\beta)}{n(1-\beta)^4} + \frac{x(1+8\beta+6\beta^2)}{n^2(1-\beta)^5} + \frac{\beta(1+8\beta+6\beta^2)}{n^3(1-\beta)^6}; \\
 5. P_n^{[\beta]}(t^4, x) &= \frac{x^4}{(1-\beta)^4} + \frac{2x^3(3+2\beta)}{n(1-\beta)^5} + \frac{x^2(7+26\beta+12\beta^2)}{n^2(1-\beta)^6} \\
 &\quad + \frac{x(1+22\beta+58\beta^2+24\beta^3)}{n^3(1-\beta)^7} + \frac{\beta(1+22\beta+58\beta^2+24\beta^3)}{n^4(1-\beta)^8}.
 \end{aligned}$$

**Proof:** By the relation (5), it clear that  $P_n^{[\beta]}(1, x) = 1$ .  
 By the simple computation, we get

$$\begin{aligned}
 P_n^{[\beta]}(t, x) &= (1-\beta) \sum_{k=0}^{\infty} (nx+k\beta)^k \frac{e^{-(nx+k\beta)}}{(k)!} \frac{k}{n} \\
 &= \frac{(1-\beta)}{n} S(1, nx+\beta, \beta) = \frac{x}{(1-\beta)} + \frac{\beta}{n(1-\beta)} + \frac{\beta^2}{n(1-\beta)^2} = \frac{x}{(1-\beta)} + \frac{\beta}{n(1-\beta)^2}; \\
 P_n^{[\beta]}(t^2, x) &= \frac{(1-\beta)}{n^2} \{S(2, nx+2\beta, \beta) + S(1, nx+\beta, \beta)\} \\
 &= \frac{(1-\beta)}{n^2} \left\{ \frac{(nx+2\beta)^2}{(1-\beta)^3} + \frac{3(nx+2\beta)\beta^2}{(1-\beta)^4} + \frac{\beta^3(1+2\beta)}{(1-\beta)^5} + \frac{(nx+\beta)}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3} \right\} \\
 &= \frac{x^2}{(1-\beta)^2} + \frac{x(1+2\beta)}{n(1-\beta)^3} + \frac{\beta(1+2\beta)}{n^2(1-\beta)^4}; \\
 P_n^{[\beta]}(t^3, x) &= \frac{(1-\beta)}{n^3} \{S(3, nx+3\beta, \beta) + 3S(2, nx+2\beta, \beta) + S(1, nx+\beta, \beta)\} \\
 &= \frac{(1-\beta)}{n^3} \left\{ \frac{(nx+3\beta)^3}{(1-\beta)^4} + \frac{6(nx+3\beta)^2\beta^2}{(1-\beta)^5} + \frac{(nx+3\beta)(4\beta^3+11\beta^4)}{(1-\beta)^6} + \frac{\beta^4+8\beta^5+6\beta^6}{(1-\beta)^7} \right. \\
 &\quad \left. + 3 \left( \frac{(nx+2\beta)^2}{(1-\beta)^3} + \frac{3(nx+2\beta)\beta^2}{(1-\beta)^4} + \frac{\beta^3(1+2\beta)}{(1-\beta)^5} \right) + \frac{(nx+\beta)}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3} \right\} \\
 &= \frac{x^3}{(1-\beta)^3} + \frac{3x^2(1+\beta)}{n(1-\beta)^4} + \frac{x(1+8\beta+6\beta^2)}{n^2(1-\beta)^5} + \frac{\beta(1+8\beta+6\beta^2)}{n^3(1-\beta)^6}; \\
 P_n^{[\beta]}(t^4, x) &= \frac{(1-\beta)}{n^4} \{S(4, nx+4\beta, \beta) + 6S(3, nx+3\beta, \beta) + 7S(2, nx+2\beta, \beta) + S(1, nx+\beta, \beta)\} \\
 &= \frac{x^4}{(1-\beta)^4} + \frac{2x^3(3+2\beta)}{n(1-\beta)^5} + \frac{x^2(7+26\beta+12\beta^2)}{n^2(1-\beta)^6} \\
 &\quad + \frac{x(1+22\beta+58\beta^2+24\beta^3)}{n^3(1-\beta)^7} + \frac{\beta(1+22\beta+58\beta^2+24\beta^3)}{n^4(1-\beta)^8}.
 \end{aligned}$$

The proof of Lemma 3.1 is complete.

We also introduce the  $s$ -th order central moment of the operator  $P_n^{[\beta]}$ , that is  $P_n^{[\beta]}(\varphi_x^s, x)$ , where  $\varphi_x(t) = t-x$ ,  $(x, t) \in \mathbf{R}^+ \times \mathbf{R}^+$ . On the basis of above lemma and by linearity of operators (7), by a straightforward calculation, we obtain

**Lemma 3.2.** Let the operator  $P_n^{[\beta]}$  be defined by relation as (7) and let  $\varphi_x = t - x$  be given by

1.  $P_n^{[\beta]}(\varphi_x, x) = \frac{x\beta}{(1-\beta)} + \frac{\beta}{n(1-\beta)^2};$
2.  $P_n^{[\beta]}(\varphi_x^2, x) = \frac{x^2\beta^2}{(1-\beta)^2} + \frac{x(1+2\beta^2)}{n(1-\beta)^3} + \frac{\beta(1+2\beta)}{n^2(1-\beta)^4};$
3.  $P_n^{[\beta]}(\varphi_x^3, x) = \frac{x^3\beta^3}{(1-\beta)^3} + \frac{3x^2\beta(1+\beta^2)}{n(1-\beta)^4} + \frac{x(1+5\beta+3\beta^2+6\beta^3)}{n^2(1-\beta)^5} + \frac{\beta(1+8\beta+6\beta^2)}{n^3(1-\beta)^6};$
4.  $P_n^{[\beta]}(\varphi_x^4, x) = \frac{x^4\beta^4}{(1-\beta)^4} + \frac{2x^3\beta^2(3+2\beta^2)}{n(1-\beta)^5} + \frac{x^2(3+4\beta+20\beta^2+6\beta^3+12\beta^4)}{n^2(1-\beta)^6}$   
 $+ \frac{x(1+18\beta+30\beta^2+32\beta^3+24\beta^4)}{n^3(1-\beta)^7} + \frac{\beta(1+22\beta+58\beta^2+24\beta^3)}{n^4(1-\beta)^8}.$

**Lemma 3.3.** Let  $n > 1$  be a given number. For every  $0 < \beta < 1$ , one has

$$P_n^{[\beta]}(\varphi_x^2, x) \leq \frac{3+n}{n(1-\beta)^4} \left( \phi(x) + \frac{1}{n} \right),$$

where  $\phi^2(x) = x(1+x)$ ,  $x \in [0, \infty)$ .

**Proof:** Since,  $\max\{x, x^2\} \leq x + x^2$ ,  $0 < \beta < 1$  and  $(1-\beta)^{-3} \leq (1-\beta)^{-4}$ , we have

$$\begin{aligned} P_n^{[\beta]}(\varphi_x^2, x) &= (x^2 + x) \left( \frac{\beta^2}{(1-\beta)^2} + \frac{(1+2\beta^2)}{n(1-\beta)^3} \right) + \frac{\beta(1+2\beta)}{n^2(1-\beta)^4}; \\ &\leq \phi^2(x) \left( \frac{1+(2+n)\beta^2-n\beta^3}{n(1-\beta)^3} \right) + \frac{\beta(1+2\beta)}{n^2(1-\beta)^4}; \\ &\leq \phi^2(x) \left( \frac{1+(2+n)\beta^2}{n(1-\beta)^3} \right) + \frac{\beta(1+2\beta)}{n^2(1-\beta)^4}; \\ &\leq \frac{1+(2+n)\beta^2}{n(1-\beta)^4} \left( \phi^2(x) + \frac{1}{n} \right), \end{aligned}$$

which is required.

#### 4. Approximation Properties

The convergence property of the operators (7) is proved in the following theorem:

**Theorem 4.1.** Let  $f$  be a continuous function on  $[0, \infty)$  and  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$ , then the sequence  $P_n^{[\beta_n]}$  converges uniformly to  $f$  on  $[a, b]$ , where  $0 \leq a < b < \infty$ .

**Proof:** Since  $P_n^{[\beta_n]}$  is a positive linear operator for  $0 \leq \beta_n < 1$ , it is sufficient, by Korovkin’s result [23], to verify the uniform convergence for test functions  $f(t) = 1, t$  and  $t^2$ .

It is clear that  $P_n^{[\beta_n]}(1, x) = 1$ .

Going to  $f(t) = t$ ,

$$\lim_{n \rightarrow \infty} P_n^{[\beta_n]}(t, x) = \lim_{n \rightarrow \infty} \left[ \frac{x}{(1-\beta_n)} + \frac{\beta_n}{n(1-\beta_n)^2} \right] = x, \text{ as } \beta_n \rightarrow 0.$$

Proceeding to the function  $f(t) = t^2$ , it can easily be shown that

$$\lim_{n \rightarrow \infty} P_n^{[\beta_n]}(t^2, x) = \lim_{n \rightarrow \infty} \left[ \frac{x^2}{(1-\beta_n)^2} + \frac{x(1+2\beta_n)}{n(1-\beta_n)^3} + \frac{\beta_n(1+2\beta_n)}{n^2(1-\beta_n)^4} \right] = x^2, \text{ as } \beta_n \rightarrow 0.$$

The proof of theorem 4.1 is complete.

4.1. Local Approximation

Let  $C_B[0, \infty)$  be denote the set of all bounded continuous real-valued functions on  $[0, \infty)$ . The space is endowed with sup-norm  $\| \cdot \|$ , where  $\|f\| = \sup_{x \in [0, \infty)} |f(x)|$ ,  $f \in C_B[0, \infty)$ . In connection with the estimation of the degree of approximation, the so called moduli of smoothness play important role. Further, let us consider the following  $K$ -functional:

$$K_2(f, \delta) = \inf_{g \in W^2} (\|f - g\| + \delta \|g''\|),$$

where  $\delta > 0$  and  $W^2 = \{g \in C_B[0, \infty) : g', g'' \in [0, \infty)\}$ . By [24, 14, p. 177, Theorem 2.4] there exists an absolute constant  $C > 0$  such that

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \tag{13}$$

where

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x + 2h) - 2f(x + h) + f(x)|$$

is the second order modulus of smoothness of  $f \in C_B[0, \infty)$ . By

$$\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x + h) - f(x)|$$

we denote the usual modulus of continuity of  $f \in C_B[0, \infty)$ . In what follows we shall use the notations  $\phi(x) = \sqrt{x(1+x)}$  and  $\delta^2(x) = \phi^2(x) + \frac{1}{n}$ , where  $x \in [0, \infty)$  and  $n \geq 1$ .

Now, we establish local approximation theorems in connection with the operators  $P_n^{[\beta]}$ .

**Theorem 4.2.** Let  $P_n^{[\beta]}$ ,  $n \in \mathbb{N}$ , be given by (7). For every  $f \in C_B[0, \infty)$ , one has

$$|P_n^{[\beta]}(f, x) - f(x)| \leq C \omega_2 \left( f, \frac{1}{2} \sqrt{\frac{1 + (2+n)\beta^2}{n(1-\beta)^4} \delta^2(x) + \left(\frac{\beta(1+nx(1-\beta))}{n(1-\beta)^2}\right)^2} \right) + \omega \left( f, \frac{\beta(1+nx(1-\beta))}{n(1-\beta)^2} \right). \tag{14}$$

**Proof:** Let us introduce the auxiliary operators  $\widetilde{P}_n^{[\beta]}$  defined by

$$\widetilde{P}_n^{[\beta]}(f, x) = P_n^{[\beta]}(f, x) - f \left( \frac{\beta + nx(1-\beta)}{n(1-\beta)^2} \right) + f(x), \tag{15}$$

for  $x \in [0, \infty)$ . The operators  $\widetilde{P}_n^{[\beta]}$  are linear. By Lemma 3.2, we have

$$\widetilde{P}_n^{[\beta]}(t-x, x) = 0. \tag{16}$$

Let  $g \in W^2$ . From Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad t \in [0, \infty). \tag{17}$$

Applying the linear operator  $\widetilde{P}_n^{[\beta]}$  and taking in view (15) and (16), we can write

$$\begin{aligned}
 |\widetilde{P}_n^{[\beta]}(g, x) - g(x)| &= |\widetilde{P}_n^{[\beta]}(g - g(x), x)| = \left| g'(x)\widetilde{P}_n^{[\beta]}(t - x, x) + \widetilde{P}_n^{[\beta]} \left( \int_x^t (t - u)g''(u)du, x \right) \right| \\
 &\leq \left| P_n^{[\beta]} \left( \int_x^t (t - u)g''(u)du, x \right) - \int_x^{\frac{\beta + nx(1 - \beta)}{n(1 - \beta)^2}} \left( \frac{\beta + nx(1 - \beta)}{n(1 - \beta)^2} - u \right) g''(u)du \right| \\
 &\leq P_n^{[\beta]} \left( \left| \int_x^t (t - u)g''(u)du \right|, x \right) + \left| \int_x^{\frac{\beta + nx(1 - \beta)}{n(1 - \beta)^2}} \left( \frac{\beta + nx(1 - \beta)}{n(1 - \beta)^2} - u \right) |g''(u)|du \right| \\
 &\leq P_n^{[\beta]} \left( (t - x)^2 \frac{\|g''\|}{2}, x \right) + \|g''\| \left| \int_x^{\frac{\beta + nx(1 - \beta)}{n(1 - \beta)^2}} \left( \frac{\beta + nx(1 - \beta)}{n(1 - \beta)^2} - u \right) du \right| \\
 &\leq \frac{\|g''\|}{2} P_n^{[\beta]}(\varphi_x^2, x) + \frac{\|g''\|}{2} \left( \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right)^2 \\
 &\leq \|g''\| \left( \frac{1 + (2 + n)\beta^2}{n(1 - \beta)^4} \delta^2(x) + \left( \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right)^2 \right). \tag{18}
 \end{aligned}$$

Let  $f \in C_B[0, \infty)$ , Further on, taking in view that

$$|P_n^{[\beta]}(f - g, x)| \leq \|f - g\|, \quad |\widetilde{P}_n^{[\beta]}(f - g, x)| \leq 3\|f - g\| \tag{19}$$

and by definition of modulus of continuity, we have

$$\left| f \left( \frac{\beta + nx(1 - \beta)}{n(1 - \beta)^2} \right) - f(x) \right| \leq \omega \left( f, \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right). \tag{20}$$

Now, (18), (19) and (20) imply

$$\begin{aligned}
 |P_n^{[\beta]}(f, x) - f(x)| &\leq |\widetilde{P}_n^{[\beta]}(f - g, x) - (f - g)(x)| + |\widetilde{P}_n^{[\beta]}(g, x) - g(x)| + \left| f \left( \frac{\beta + nx(1 - \beta)}{n(1 - \beta)^2} \right) - f(x) \right| \\
 &\leq 4 \left( \|f - g\| + \frac{\|g''\|}{4} \left( \frac{1 + (2 + n)\beta^2}{n(1 - \beta)^4} \delta^2(x) + \left( \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right)^2 \right) \right) \\
 &\quad + \omega \left( f, \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right).
 \end{aligned}$$

Hence, taking infimum on the right hand side over all  $g \in W^2$ , we get

$$|P_n^{[\beta]}(f, x) - f(x)| \leq 4K_2 \left( f, \frac{1}{4} \left( \frac{1 + (2 + n)\beta^2}{n(1 - \beta)^4} \delta^2(x) + \left( \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right)^2 \right) \right) + \omega \left( f, \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right).$$

In view of (13), we get

$$|P_n^{[\beta]}(f, x) - f(x)| \leq C\omega_2 \left( f, \frac{1}{2} \sqrt{\frac{1 + (2 + n)\beta^2}{n(1 - \beta)^4} \delta^2(x) + \left( \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right)^2} \right) + \omega \left( f, \frac{\beta(1 + nx(1 - \beta))}{n(1 - \beta)^2} \right).$$

This completes the proof of the theorem.

We recall that a continuous function  $f$  defined on  $J$  is locally  $Lip \alpha$  on  $E$  ( $0 < \alpha \leq 1$ ), if it satisfies the condition

$$|f(x) - f(y)| \leq M_f |x - y|^\alpha, \quad (x, y) \in J \times E, \tag{21}$$

where  $M_f$  is a constant depending only on  $f$ .

**Theorem 4.3.** Let  $P_n^{[\beta]}$ ,  $n \in \mathbb{N}$ , be given by (7),  $0 < \alpha \leq 1$  and  $E$  be any subset of  $[0, \infty)$ . If  $f$  is locally Lip  $\alpha$  on  $E$ , then we have

$$|P_n^{[\beta]}(f, x) - f(x)| \leq M_f \left( \left( \frac{1 + (2 + n)\beta^2}{n(1 - \beta)^4} \delta^2(x) \right)^{\alpha/2} + 2d^\alpha(x, E) \right), \quad x \geq 0,$$

where  $d(x, E)$  is the distance between  $x$  and  $E$  defined as

$$d(x, E) = \inf\{|x - y| : y \in E\}.$$

**Proof:** By using the continuity of  $f$ , it is obvious that (21) holds for any  $x \geq 0$  and  $y \in \bar{E}$ ,  $\bar{E}$  being the closure in  $\mathbb{R}$  of the set  $E$ . Let  $(x, x_0) \in [0, \infty) \times \bar{E}$  be such that  $|x - x_0| = d(x, E)$ .

On the other hand, we can write  $|f(t) - f(x_0)| \leq |f(t) - f(x)| + |f(x) - f(x_0)|$  and applying the linear positive operators  $P_n^{[\beta]}$ , we have

$$\begin{aligned} |P_n^{[\beta]}(f, x) - f(x)| &\leq |P_n^{[\beta]}(|f(t) - f(x_0)|, x) - f(x)| + |f(x_0) - f(x)| \\ &\leq |P_n^{[\beta]}(M_f|t - x_0|^\alpha, x)| + M_f|x_0 - x|^\alpha. \end{aligned}$$

Note that  $P_n^{[\beta]}$  is positive, so it is monotone.

In the inequality  $(A + B)^\alpha \leq A^\alpha + B^\alpha$  ( $A \geq 0, B \geq 0, 0 < \alpha \leq 1$ ), we put  $A = |t - x|$ ,  $B = |x - x_0|$  and using Holder's inequality, we get

$$\begin{aligned} |P_n^{[\beta]}(f, x) - f(x)| &\leq M_f P_n^{[\beta]}(|(t - x) + (x - x_0)|^\alpha, x) + M_f|x_0 - x|^\alpha \\ &\leq M_f \left( P_n^{[\beta]}(|t - x|^\alpha, x) + |x - x_0|^\alpha \right) + M_f|x_0 - x|^\alpha \\ &\leq M_f \left( \left( P_n^{[\beta]}((t - x)^2, x) \right)^{\alpha/2} + 2|x - x_0|^\alpha \right) \\ &\leq M_f \left( \left( \frac{1 + (2 + n)\beta^2}{n(1 - \beta)^4} \left( \phi(x) + \frac{1}{n} \right) \right)^{\alpha/2} + 2|x - x_0|^\alpha \right), \end{aligned}$$

which is required results.

#### 4.2. Rate of convergence

Let  $B_{x^2}[0; \infty)$  be the set of all functions  $f$  defined on  $[0, \infty)$  satisfying the condition  $|f(x)| \leq M_f(1 + x^2)$ , where  $M_f$  is a constant depending only on  $f$ . By  $C_{x^2}[0, \infty)$ , we denote the subspace of all continuous functions belonging to  $B_{x^2}[0, \infty)$ . Also, let  $C_{x^2}^*[0, \infty)$  be the subspace of all functions  $f \in C_{x^2}[0, \infty)$ , for which

$\lim_{x \rightarrow \infty} \frac{f(x)}{1 + x^2}$  is finite. The norm on  $C_{x^2}[0, \infty)$  is  $\|f\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1 + x^2}$ . For any positive  $a$ , by

$$\omega_a(f, \delta) = \sup_{|t-x| \leq \delta} \sup_{x, t \in [0, a]} |f(t) - f(x)|,$$

we denote the usual modulus of continuity of  $f$  on the closed interval  $[0, a]$ . We know that for a function  $f \in C_{x^2}[0, \infty)$ , the modulus of continuity  $\omega_a(f, \delta)$  tends to zero. Now, we give a rate of convergence theorem for the operator  $P_n^{[\beta]}$ :

**Theorem 4.4.** Let  $f \in C_{x^2}[0, \infty)$  and let the operator  $P_n^{[\beta]}$  be defined as in (7), where  $0 \leq \beta < 1$  and  $\omega_a(f, \delta)$  be its modulus of continuity on the finite interval  $[0, a] \subset [0, \infty)$ , where  $a > 0$ . Then for every  $n \geq 1$ ,

$$\|P_n^{[\beta]}(f, \cdot) - f\| \leq \frac{(1 + (2 + n)\beta^2)K}{n(1 - \beta)^4} + 2\omega_{a+1} \left( f, \sqrt{\frac{(1 + (2 + n)\beta^2)K}{n(1 - \beta)^4}} \right),$$

where  $K = 6M_f(1 + a^2)(1 + a + a^2)$ .



**Proof:** For  $x \in [0, a]$  and  $t > a + 1$ , since  $t - x > 1$ , we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(2 + x^2 + t^2) \\ &\leq M_f(2 + 3x^2 + 2(t - x)^2) \\ &\leq 6M_f(1 + a^2)(t - x)^2. \end{aligned} \tag{22}$$

For  $x \in [0, a]$  and  $t \leq a + 1$ , we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f, |t - x|) + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{a+1}(f, \delta), \tag{23}$$

with  $\delta > 0$ . From (22) and (23), we have

$$|f(t) - f(x)| \leq 6M_f(1 + a^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{a+1}(f, \delta), \tag{24}$$

for  $x \in [0, a]$  and  $t \geq 0$ . Thus

$$|P_n^{[\beta]}(f, x) - f(x)| \leq 6M_f(1 + a^2)P_n^{[\beta]}((t - x)^2, x) + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta} P_n^{[\beta]}((t - x)^2, x)\right)^{\frac{1}{2}}. \tag{25}$$

Hence, by Schwarz’s inequality and Lemma 3.3, for  $0 < \beta < 1$  and  $x \in [0, a]$

$$\begin{aligned} |P_n^{[\beta]}(f, x) - f(x)| &\leq \frac{6M_f(1 + a^2)(1 + (2 + n)\beta^2)}{n(1 - \beta)^4} \left(\phi(x) + \frac{1}{n}\right) \\ &\quad + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta} \sqrt{\frac{(1 + (2 + n)\beta^2)}{n(1 - \beta)^4} \left(\phi(x) + \frac{1}{n}\right)}\right) \\ &\leq \frac{(1 + (2 + n)\beta^2)K}{n(1 - \beta)^4} + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta} \sqrt{\frac{(1 + (2 + n)\beta^2)K}{n(1 - \beta)^4}}\right). \end{aligned} \tag{26}$$

By taking  $\delta = \sqrt{\frac{(1 + (2 + n)\beta^2)K}{n(1 - \beta)^4}}$ , we get the assertion of theorem.

### 4.3. Weighted approximation

Now, we shall discuss the weighted approximation theorem, where the approximation formula holds true on the interval  $[0, \infty)$ .

**Theorem 4.5.** Let the operator  $P_n^{[\beta_n]}$  be defined as in (7), where  $(\beta_n)_{n \geq 1}$ ,  $0 \leq \beta_n < 1$ , satisfies  $\lim_n \beta_n = 0$ . For each  $f \in C_{x^2}^*[0, \infty)$ , we have

$$\lim_{n \rightarrow \infty} \|P_n^{[\beta_n]}(f, \cdot) - f\|_{x^2} = 0.$$

**Proof:** Using the theorem in [25], we see that it is sufficient to verify the following three conditions

$$\lim_{n \rightarrow \infty} \|P_n^{[\beta_n]}(t^v, \cdot) - x^v\|_{x^2} = 0, \text{ for } v = 0, 1, 2, \tag{27}$$

for every  $x \in [0, \infty)$ .

Since  $P_n^{[\beta_n]}(1, x) = 1$ , the first condition of (27) is fulfilled for  $v = 0$ . By Lemma 3.1 we have for  $n \geq 1$

$$\begin{aligned} \|P_n^{[\beta_n]}(t, x) - x\|_{x^2} &= \sup_{x \in [0, \infty)} \frac{|P_n^{[\beta_n]}(t, x) - x|}{1 + x^2} \\ &\leq \frac{\beta}{(1 - \beta)} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + \frac{\beta}{n(1 - \beta)^2} \leq \frac{\beta}{(1 - \beta)} + \frac{\beta}{n(1 - \beta)^2} \end{aligned}$$

and the second condition of (27) holds for  $v = 1$  as  $n \rightarrow \infty$  with  $\beta_n \rightarrow 0$ . Similarly, we can write for  $n \geq 1$

$$\begin{aligned} \|P_n^{[\beta_n]}(t^2, x) - x^2\|_{x^2} &\leq \frac{(2 - \beta_n)\beta_n}{(1 - \beta_n)^2} \sup_{x \in [0, \infty)} \left( \frac{x^2}{1 + x^2} \right) + \frac{(1 + 2\beta_n)}{n(1 - \beta_n)^3} \sup_{x \in [0, \infty)} \left( \frac{x^2}{1 + x^2} \right) + \frac{\beta_n(1 + 2\beta_n)}{n^2(1 - \beta_n)^4} \\ &\leq \frac{(2 - \beta_n)\beta_n}{(1 - \beta_n)^2} + \frac{(1 + 2\beta_n)}{n(1 - \beta_n)^3} + \frac{\beta_n(1 + 2\beta_n)}{n^2(1 - \beta_n)^4}, \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \|P_n^{[\beta_n]}(t^2, x) - x^2\|_{x^2} = 0 \text{ with } \beta_n \rightarrow \infty.$$

Thus the proof is completed.

#### 4.4. Asymptotic Formula

In order to present asymptotic formula, we need the following lemma.

**Lemma 4.6.** Let  $P_n^{[\beta]}$  be defined as (7). In addition,  $0 < \beta < 1$ , then

$$P_n^{[\beta]}((t - x)^4, x) \leq \frac{267(x + x^2 + x^3 + x^4)}{n^4(1 - \beta)^8}.$$

**Proof:** Since  $\max\{x, x^2, x^3, x^4\} \leq x + x^2 + x^3 + x^4$ ,  $(1 - \beta)^2 \leq 1$  and  $(1 - \beta)^{-i} \leq (1 - \beta)^{-(i+1)}$ , for  $i \in \mathbb{N}$ ,

$$\begin{aligned} P_n^{[\beta]}((t - x)^4, x) &\leq \frac{x^4}{(1 - \beta)^4} + \frac{10x^3}{n(1 - \beta)^5} + \frac{45x^2}{n^2(1 - \beta)^6} + \frac{105x}{n^3(1 - \beta)^7} + \frac{105}{n^4(1 - \beta)^8} \\ &\leq \frac{267(x + x^2 + x^3 + x^4)}{n^4(1 - \beta)^8}, \end{aligned}$$

we obtain claim inequality.

Notice that,  $\lim_{n \rightarrow \infty} n^2 P_n^{[\beta_n]}((t - x)^4, x) = 3x^2$  with  $\beta_n \rightarrow 0$ .

**Theorem 4.7.** Let  $f, f', f'' \in C[0, \infty)$  and let the operator  $P_n^{[\beta_n]}$  be defined as in (7), where  $(\beta_n)_{n \geq 1}$ ,  $0 \leq \beta_n < 1$ , satisfies  $\lim_n \beta_n = 0$ . then

$$\lim_{n \rightarrow \infty} n \left( P_n^{[\beta_n]}(f, x) - f(x) \right) = \frac{x}{2} f''(x), \quad \forall x > 0.$$

**Proof:** Let  $f, f', f'' \in C[0, \infty)$  and  $x \in [0, \infty)$  be fixed. By the Taylor formula, we have

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + r(t; x)(t - x)^2, \tag{28}$$

where  $r(\cdot; x)$  is the Peano form of the remainder and  $\lim_{t \rightarrow x} r(t; x) = 0$ .

We apply  $P_n^{[\beta_n]}$  to equation (28), we get

$$\begin{aligned} P_n^{[\beta_n]}(f, x) - f(x) &= f'(x)P_n^{[\beta_n]}((t - x), x) + \frac{1}{2} f''(x)P_n^{[\beta_n]}((t - x)^2, x) + P_n^{[\beta_n]}(r(t; x)(t - x)^2, x) \\ &= f'(x) \left[ \frac{x\beta_n}{(1 - \beta_n)} + \frac{\beta_n}{n(1 - \beta_n)^2} \right] + P_n^{[\beta_n]}(r(t; x)(t - x)^2, x) \\ &\quad + \frac{f''(x)}{2} \left[ \frac{x^2\beta_n^2}{(1 - \beta_n)^2} + \frac{x(1 + 2\beta_n^2)}{n(1 - \beta_n)^3} + \frac{\beta_n(1 + 2\beta_n)}{n^2(1 - \beta_n)^4} \right]. \end{aligned}$$

In the second term  $P_n^{[\beta_n]}(r(t;x)(t-x)^2, x)$  applying the Cauchy-Schwartz inequality, we have

$$0 \leq |nP_n^{[\beta_n]}(r(t;x)(t-x)^2, x)| \leq \sqrt{n^2 P_n^{[\beta_n]}((t-x)^4, x)} \sqrt{P_n^{[\beta_n]}(r(t;x)^2, x)}. \quad (29)$$

Observe that  $r^2(x, x) = 0$  and  $r^2(\cdot, x) \in C_{x^2}^*[0, \infty)$ . Then, it follows that

$$\lim_{t \rightarrow x} nP_n^{[\beta_n]}(r(t, x)^2, x) = r^2(x, x) = 0 \quad (30)$$

uniformly with respect to  $x \in [0, A]$  for any  $A > 0$ .

On the basis of (29), (30) and Lemma 4.6, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left( P_n^{[\beta_n]}(f, x) - f(x) \right) &= f'(x) \lim_{n \rightarrow \infty} n \left[ \frac{x\beta_n}{(1-\beta_n)} + \frac{\beta_n}{n(1-\beta_n)^2} \right] \\ &\quad + \frac{f''(x)}{2} \lim_{n \rightarrow \infty} n \left[ \frac{x^2\beta_n^2}{(1-\beta_n)^2} + \frac{x(1+2\beta_n^2)}{n(1-\beta_n)^3} + \frac{\beta_n(1+2\beta_n)}{n^2(1-\beta_n)^4} \right] \\ &= \frac{x}{2} f''(x) \text{ with } \beta_n \rightarrow 0, \end{aligned}$$

which completes the proof.

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