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Applications of Measure of Non-Compactness and Modified Simulation Function for Solvability of Nonlinear Functional Integral Equations

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Abstract. In this work we introduce a modified version of simulation function and define a simulation type contraction mappings involving measure of non-compactness in the frame work of Banach space and derive some basic Darbo type fixed point results. Also, our theorem generalizes the Theorem 4 of [R. Arab, Some generalizations of Darbo fixed point theorem and its application, Miskolc Mathematical Notes, 18(2)(2017),595-610.] and extend some recent results. Further we show the applicability of obtained results to the theory of integral equations followed by two concrete examples.

1. Introduction

Integral equation create a very important and significant part of the mathematical analysis and has various applications into real world problems. Also, nonlinear functional-integral equations have been studied in the vehicular traffic, the biology, theory of optimal control and economics, etc., for example, see [1, 11, 15, 16, 18]. Recently, there have been several successful efforts to apply the concept of a measure of noncompactness in the study of the existence and behavior of solutions of nonlinear differential and integral equations [2–6, 8–10, 17, 19, 20, 22–24]. In our investigations, we apply the method associated with the technique of measures of noncompactness to generalize the Darbo fixed point theorem [14] and to extend some recent results of Arab [7]. Moreover, as an application, we study the existence of solutions of the nonlinear integral equation of the form

$$x(t) = g(t) + f_1\Big(t, x(t), \int_0^t u(t,s,x(s))ds\Big) + f_2\Big(t, x(t), \int_0^1 v(t,s,x(s))ds\Big), \tag{1}$$

where f_1, f_2, g, u and v satisfy certain conditions.
The rest of the paper is organized as follows. In Section 2, we present some definitions and preliminary

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results concerning the concept of measure of noncompactness. In Section 3, using the θ functions (set of all modified simulation functions), some generalizations of Darbo fixed point theorem and recent results due to Arab [7] are discussed. Finally in Section 4, using the obtained results in Section 3, we investigate the problem of existence of solutions for the nonlinear integral equation (1) followed by two suitable examples.

2. Preliminaries

In this section, we recall some notations, definitions and theorems to obtain all results of this work. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0,\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space. Let $\overline{B}(x,r)$ denote the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0,r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$ the closure and the convex closure of X, respectively. Moreover, let us denote $\mathfrak{M}_E$, the family of nonempty bounded subsets of E, and $\mathfrak{N}_E$, the subfamily of $\mathfrak{M}_E$ consisting of all relatively compact sets. We use the following definition of the measure of noncompactness given in [14].

Definition 2.1. *A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

(1^0) *The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$,*

(2^0) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,

(3^0) $\mu(\bar{X}) = \mu(X)$,

(4^0) $\mu(ConvX) = \mu(X)$,

(5^0) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$,

(6^0) *If (X_n) is a sequence of closed sets from m_E such that $X_{n+1} \subset X_n (n = 1, 2, \dots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.*

It follows from Definition 2.1 (6^0) that X_∞ is a member of the family $ker\mu$. Since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we can deduce that $\mu(X_\infty) = 0$. This implies that $X_\infty \in ker\mu$. Further facts concerning measures of noncompactness and their properties may be found in [12, 14].
Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorem.

Theorem 2.2. *[2, Schauder] Let C be a nonempty, bounded, closed, convex subset of a Banach space E. Then every compact, continuous map $T : C \longrightarrow C$ has at least one fixed point.*

In the following we state a fixed-point theorem of Darbo type proved by Banaś and Goebel [14].

Theorem 2.3. *Let C be a nonempty, closed, bounded, and convex subset of the Banach space E and $F : C \longrightarrow C$ be a continuous mapping. Assume that there exist a constant $k \in [0,1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset of C. Then F has a fixed-point in C.*

Remark 2.4. *[14] Under the assumptions of the above theorem it can be shown that the set FixF of fixed points of F belonging to Ω is an element of $ker\mu$.*

3. Fixed point theorem

The main result of the present paper is the following fixed point theorem which is a generalization of Darbo fixed point theorem (cf. Theorem 2.3) and extend Theorem 4 of [7].
The notion of a simulation function was introduced by Khojasteh et al. [21] as follows.

Definition 3.1. *[21] A simulation function is a mapping $\zeta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying the following conditions:*

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$, *for all* $t,s > 0$;

(ζ_3) *if* $\{t_n\}$, $\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

In this sequel, we modify the Definition 3.1 and introduce Θ, the class of functions $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying the following conditions:

(θ_1) $\theta(t,s) < s - t$, for all $t,s > 0$;

(θ_2) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = l > 0$ and $\lim_{n\to\infty} s_n = s > 0$, then

$$\limsup_{n\to\infty} \theta(t_n, s_n) < s - l.$$

Example 3.2. *Let* $\theta_i : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$, $i = 1,2,3,4,5$ *be defined by*

(i) $\theta_1(t,s) = \lambda s - t$ *for all* $t,s \in \mathbb{R}_+$ *and* $0 < \lambda < 1$.

(ii) $\theta_2(t,s) = s - \varphi(s) - t$ *for all* $t,s \in \mathbb{R}_+$, *where* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a lower semi-continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$.

(iii) $\theta_3(t,s) = \varphi(s) - \psi(t)$ *for all* $t,s \in \mathbb{R}_+$, *where* $\psi,\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *are two continuous functions such that* $\psi(t) = \varphi(t) = 0$ *if and only if* $t = 0$ *and* $\varphi(t) < t \le \psi(t)$ *for all* $t > 0$.

(iv) $\theta_4(t,s) = \varphi(s) - t$ *for all* $t,s \in \mathbb{R}_+$, *where* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a upper semi continuous function with* $\varphi(t) < t$ *for all* $t > 0$ *and* $\varphi(t) = 0$ *if and only if* $t = 0$.

(v) $\theta_5(t,s) = s\,\varphi(s) - t$ *for all* $t,s \ge 0$, *where* $\varphi : \mathbb{R}_+ \longrightarrow [0,1)$ *is a function with* $\limsup_{t\to r^+} \varphi(t) < 1$ *for all* $r > 0$.

Then $\theta_i \in \Theta$ *for* $i = 1,2,3,4,5$.

Our first result is as follows:

Theorem 3.3. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \longrightarrow C$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two continuous functions. Suppose that if for any $0 < a < b < \infty$ there exists $0 < \gamma(a,b) < 1$ such that for all $X \subseteq C$,*

$$\begin{aligned} &a \le \mu(X) + \varphi(\mu(X)) \le b \\ &\Longrightarrow \theta[\mu(TX) + \varphi(\mu(TX)), \gamma(a,b)(\mu(X) + \varphi(\mu(X)))] \ge 0, \end{aligned} \tag{2}$$

where μ is an arbitrary measure of noncompactness and $\theta \in \Theta$. Then T has at least one fixed point in C.

Proof. Let $C_0 = C$, we construct a sequence $\{C_n\}$ such that $C_{n+1} = Conv(TC_n)$, for $n \ge 0$. $TC_0 = TC \subseteq C = C_0, C_1 = Conv(TC_0) \subseteq C = C_0$, therefore by continuing this process, we have

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots.$$

If there exists a positive integer $N \in \mathbb{N}$ such that $\mu(C_N) + \varphi(\mu(C_N)) = 0$, i.e, $\mu(C_N) = 0$, then C_N is relatively compact. On the other hand, we have $T(C_N) \subseteq Conv(TC_N) = C_{N+1} \subseteq C_N$. Then Theorem 2.2 implies that T has a fixed point. So we assume that

$$0 < \mu(C_n) + \varphi(\mu(C_n)), \quad \forall n \ge 1. \tag{3}$$

Suppose that

$$\mu(C_{n_0}) + \varphi(\mu(C_{n_0})) < \mu(C_{n_0+1}) + \varphi(\mu(C_{n_0+1})) \tag{4}$$

for some $n_0 \in \mathbb{N}$. In addition, by (3) and (4), we have

$$0 < a := \mu(C_{n_0}) + \varphi(\mu(C_{n_0})) \leq \mu(C_{n_0}) + \varphi(\mu(C_{n_0})) < \mu(C_{n_0+1}) + \varphi(\mu(C_{n_0+1})) := b.$$

By using (2) and (θ_1) with $X = C_{n_0}$, there exists $0 < \gamma(a,b) < 1$ such that

$$\begin{aligned} 0 \leq &\theta[\mu(TC_{n_0}) + \varphi(\mu(TC_{n_0})), \gamma(a,b)(\mu(C_{n_0}) + \varphi(\mu(C_{n_0})))] \\ =&\theta[\mu(convTC_{n_0}) + \varphi(\mu(convTC_{n_0})), \gamma(a,b)(\mu(C_{n_0}) + \varphi(\mu(C_{n_0})))] \\ =&\theta[\mu(C_{n_0+1}) + \varphi(\mu(C_{n_0} + 1)), \gamma(a,b)(\mu(C_{n_0}) + \varphi(\mu(C_{n_0})))] \\ <&\gamma(a,b)(\mu(C_{n_0}) + \varphi(\mu(C_{n_0}))) - \mu(C_{n_0+1}) + \varphi(\mu(C_{n_0+1})), \end{aligned}$$

which implies that $\gamma(a,b) > 1$, a contradiction. This implies that

$$\mu(C_{n+1}) + \varphi(\mu(C_{n+1})) \leq \mu(C_n) + \varphi(\mu(C_n)),$$

for all $n \in \mathbb{N}$, that is, the sequence $\{\mu(C_n) + \varphi(\mu(C_n))\}$ is non-increasing and nonnegative, we infer that

$$\lim_{n\to\infty} \mu(C_n) + \varphi(\mu(C_n)) = r. \tag{5}$$

Now, we show that $r = 0$. Suppose to the contrary, that $r > 0$. Then

$$0 < a := r \leq \mu(C_n) + \varphi(\mu(C_n)) \leq \mu(C_0) + \varphi(\mu(C_0)) =: b \text{ for all } n \geq 0.$$

By using (2) with $X = C_{n_0}$, there exists $0 < \gamma(a,b) < 1$ such that

$$\begin{aligned} 0 \leq &\theta[\mu(TC_n) + \varphi(\mu(TC_n)), \gamma(a,b)(\mu(C_n) + \varphi(\mu(C_n)))] \\ =&\theta[\mu(ConvTC_n) + \varphi(\mu(ConvTC_n)), \gamma(a,b)(\mu(C_n) + \varphi(\mu(C_n)))] \\ =&\theta[\mu(C_{n+1}) + \varphi(\mu(C_{n+1})), \gamma(a,b)(\mu(C_n) + \varphi(\mu(C_n)))]. \end{aligned}$$

The above inequality and the condition (θ_2), with $t_n = \mu(C_{n+1}) + \varphi(\mu(C_{n+1}))$ and $s_n = \gamma(a,b)(\mu(C_n) + \varphi(\mu(C_n)))$, we have

$$\begin{aligned} 0 \leq &\limsup_{n\to\infty} \theta[\mu(C_{n+1}) + \varphi(\mu(C_{n+1})), \gamma(a,b)(\mu(C_n) + \varphi(\mu(C_n)))] \\ <&\gamma(a,b)r - r < 0, \end{aligned}$$

which is a contradiction. Then we conclude that $r = 0$ and from (5), since $\varphi \geq 0$, we get

$$\lim_{n\to\infty} \mu(C_n) = 0 \;\; \textit{and} \; \lim_{n\to\infty} \varphi(\mu(C_n)) = 0.$$

Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1, 2, \ldots$, it follows from (6^0) that

$$C_\infty = \bigcap_{n=1}^{\infty} C_n$$

is nonempty convex closed set, invariant under T and belongs to $Ker\mu$. Therefore Theorem 2.2 completes the proof. □

We show the unifying power of simulation functions by applying Theorem 3.3 to deduce different kinds of contractive conditions in the existing literature. Two immediate consequences of Theorem 3.3 are the following.

Theorem 3.4. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \to C$ be a continuous function. Suppose that if for any $0 < a < b < \infty$ there exists $0 < \gamma(a,b) < 1$ such that for all $X \subseteq C$,*

$$a \leq \mu(X) \leq b \Longrightarrow \theta[\mu(TX), \gamma(a,b)\mu(X)] \geq 0,$$

where μ is an arbitrary measure of noncompactness and $\theta \in \Theta$. Then T has at least one fixed point in C.

Theorem 3.5. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \longrightarrow C$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two continuous functions and $\theta \in \Theta$. Suppose that there exists a constant $\lambda \in (0,1)$ such that for all $X \subseteq C$,*

$$\theta[\mu(TX) + \varphi(\mu(TX)), \lambda\,(\mu(X) + \varphi(\mu(X)))] \geq 0,$$

where μ is an arbitrary measure of noncompactness. Then T has at least one fixed point in C.

An immediate consequence of Theorem 3.5 is the following.

Corollary 3.6. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $T : C \longrightarrow C$ be a continuous function. Suppose that there exist two continuous functions $\psi, \phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$ and a constant $0 < \lambda < 1$, such that*

$$\phi(\mu(TX)) \leq \psi(\lambda\,\mu(X)) \text{ for all } X \subseteq C,$$

where μ is an arbitrary measure of noncompactness. Then T has at least one fixed point in C.

Proof. The result follows from Theorem 3.5, by taking as function $\theta(t,s) = \psi(s) - \phi(t)$, for all $t, s \geq 0$ and $\varphi \equiv 0$. □

The following result is another consequence of Theorem 3.5.

Corollary 3.7. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \longrightarrow C$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two continuous functions. Suppose that there exists a constant $0 < \lambda < 1$ such that*

$$\mu(TX) + \varphi(\mu(TX)) \leq \lambda[\mu(X) + \varphi(\mu(X))] \text{ for all } X \subseteq C,$$

where μ is an arbitrary measure of noncompactness. Then T has at least one fixed point in C.

Remark 3.8. *Taking $\varphi \equiv 0$ in Corollary 3.7, we obtain the Darbo fixed point theorem.*

Now, the following fixed point theorem follows immediately from Theorem 3.3 is a generalization of [7].

Theorem 3.9. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \longrightarrow C$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be two continuous functions such that for any $0 < a < b < \infty$, there exists $0 < \gamma(a,b) < 1$ such that for all $X \subseteq C$,*

$$a \leq \mu(X) + \varphi(\mu(X)) \leq b \Longrightarrow \mu(TX) + \varphi(\mu(TX)) \leq \gamma(a,b)[\mu(X) + \varphi(\mu(X))],$$

where μ is an arbitrary measure of noncompactness. Then T has at least one fixed point in C.

Proof. The result follows from Theorem 3.3, by taking as function $\theta(t,s) = \lambda\,s - t$, for all $t, s \geq 0$ and $\gamma(a,b) = \lambda\,\gamma'(a,b)$ where $\lambda \in [0,1)$ and $0 < \gamma'(a,b) < 1$. □

4. Application

In this section, as an application of Theorem 3.4, we consider the integral equation (1) and prove the existence of solutions of that equation. In what follows we will work in the classical Banach space $C(I) = C[0,1]$ consisting of all real functions defined and continuous on the interval $I = [0,1]$. The space $C(I)$ is furnished by the standard norm

$$\|x\| = \max\{|x(t)| : t \in I\}.$$

Next, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in this Section. This measure was introduced and studied in [13].
Let X be a fixed nonempty and bounded subset of $C(I)$. For $x \in X$ and $\epsilon \geq 0$, denote by $\omega(x,\epsilon)$ the modulus of continuity of the function x on the interval $[0,1]$, i.e.

$$\omega(x,\epsilon) := \sup\{|x(t) - x(s)| : t,s \in [0,1], |t-s| \leq \epsilon\}.$$

Further, let us put

$$\omega(X,\epsilon) := \sup\{\omega(x,\epsilon) : x \in X\},\ \omega_0(X) := \lim_{\epsilon\to 0} \omega(X,\epsilon).$$

Define

$$i(x) := \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t,s \in I, t \leq s\},$$

and

$$i(X) := \sup\{i(x) : x \in X\}.$$

Observe that all functions belonging to X are nondecreasing on I if and only if $i(X) = 0$.
Now, let us define the function μ on the family $\mathfrak{M}_{C(I)}$ by the formula

$$\mu(X) := \omega_0(X) + i(X).$$

It can be shown [13] that the function μ is a measure of noncompactness in the space $C(I)$.
Now, equation (1) will be investigated under the assumptions:

(A_1) $g : I \longrightarrow \mathbb{R}_+$ is a continuous and nondecreasing function, let $b = \max\{|g(t)| : t \in I\}$.

(A_2) $u, v : I \times I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions such that $u, v : I \times I \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and for arbitrarily fixed $s \in I$ and $x \in \mathbb{R}_+$ the functions $t \longrightarrow u(t,s,x)$ and $t \longrightarrow v(t,s,x)$ are nondecreasing on I.

(A_3) There exists a nondecreasing function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the inequality

$$|u(t,s,x)|, |v(t,s,x)| \leq h(|x|),$$

holds for all $t,s \in I$ and $x \in \mathbb{R}$.

(A_4) $f_1, f_2 : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions such that $f_1, f_2 : I \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Moreover there exists constant $k \in [0,1)$ such that

$$|f_i(t,x,y) - f_i(t,z,w)| \leq \frac{k}{2}|x-z| + |y-w|.$$

(A_5) For arbitrarily $x, y \in \mathbb{R}_+$, $t \longrightarrow f_i(t,x,y)$ is nondecreasing on I, and for arbitrarily $t \in I$ and $x \in \mathbb{R}_+$, $y \longrightarrow f_1(t,x,y)$ is nondecreasing on $\mathbb{R}_+$.

(A_6) There exists $r_0 > 0$ with $b + kr_0 + 2h(r_0) + 2M < r_0$, where $M = \sup\{|f_1(t,0,0)|, |f_2(t,0,0)| : t \in I\}$.

Theorem 4.1. *Under assumptions $(A_1)-(A_6)$, the equation (1) has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing on I.*

Proof. Consider the operators F, G and T defined on the space $C(I)$ by the formulas

$$
\begin{aligned}
(Fx)(t) &= f_1\Big(t, x(t), \int_0^t u(t,s,x(s))ds\Big),\\
(Gx)(t) &= f_2\Big(t, x(t), \int_0^1 v(t,s,x(s))ds\Big),\\
(Tx)(t) &= g(t) + (Fx)(t) + (Gx)(t).
\end{aligned}
$$

By considering the conditions of theorem we infer that Tx is continuous on I for any function $x \in C(I)$, i.e. T transforms the space $C(I)$ into itself. Moreover, for each $t \in I$, we have

$$
\begin{aligned}
|(Fx)(t)| &\le \Big| f_1\Big(t, x(t), \int_0^t u(t,s,x(s))ds\Big) - f_1(t,0,0)\Big| + |f_1(t,0,0)|\\
&\le \frac{k}{2}|x(t)| + \Big|\int_0^t u(t,s,x(s))ds\Big| + |f_1(t,0,0)|\\
&\le \frac{k}{2}|x(t)| + \int_0^t h(|x(s)|)ds + |f_1(t,0,0)|\\
&\le \frac{k}{2}\|x\| + \int_0^t h(\|x\|)ds + M\\
&\le \frac{k}{2}\|x\| + h(\|x\|) + M.
\end{aligned}
\tag{6}
$$

Similarly one can show that

$$
|(Gx)(t)| \le \frac{k}{2}\|x\| + h(\|x\|) + M. \tag{7}
$$

Linking (6) and (7) we obtain

$$
|(Tx)(t)| \le |g(t)| + |(Fx)(t)| + |(Gx)(t)| \le b + k\|x\| + 2h(\|x\|) + 2M.
$$

Hence

$$
\|Tx\| \le b + k\|x\| + 2h(\|x\|) + 2M.
$$

Thus if $\|x\| \le r_0$ we obtain from assumption (A_6) the estimate

$$
\|Tx\| \le b + kr_0 + 2h(\|x\|) + 2M \le r_0.
$$

Consequently the operator T maps the ball $B_{r_0} \subset C(I)$ into itself. Next, we prove that the operator T is continuous on B_{r_0}. To do this, let $\{x_n\}$ be a sequence in B_{r_0} such that $x_n \to x$. We have to show that $Tx_n \to Tx$. In fact, for each $t \in I$, we have

$$
\begin{aligned}
&|(Fx_n)(t) - (Fx)(t)|\\
&= \Big| f_1\Big(t, x_n(t), \int_0^t u(t,s,x_n(s))ds\Big) - f_1\Big(t, x(t), \int_0^t u(t,s,x(s))ds\Big)\Big|\\
&\le \frac{k}{2}|x_n(t) - x(t)| + \Big|\int_0^t [u(t,s,x_n(s)) - u(t,s,x(s))]ds\Big|\\
&\le \frac{k}{2}\|x_n - x\| + \int_0^t U_{r_0}(\epsilon)ds\\
&\le \frac{k}{2}\|x_n - x\| + U_{r_0}(\epsilon),
\end{aligned}
$$

where we denoted

$$U_{r_0}(\epsilon) = \sup\{|u(t,s,x) - u(t,s,y)| : t,s \in I,\ x,y \in [0,r_0],\ |x-y| \le \epsilon\}.$$

Similarly we have

$$|(Gx_n)(t) - (Gx)(t)| \le \frac{k}{2}\|x_n - x\| + V_{r_0}(\epsilon),$$

where $V_{r_0}(\epsilon)$ is defined as

$$V_{r_0}(\epsilon) = \sup\{|v(t,s,x) - v(t,s,y)| : t,s \in I,\ x,y \in [0,r_0],\ |x-y| \le \epsilon\}.$$

As

$$\begin{aligned} |(Tx_n)(t) - (Tx)(t)| &\le |(Fx_n)(t) - (Fx)(t)| + |(Gx_n)(t) - (Gx)(t)| \\ &\le k\|x_n - x\| + U_{r_0}(\epsilon) + V_{r_0}(\epsilon). \end{aligned}$$

It follows that

$$\|Tx_n - Tx\| \le k\|x_n - x\| + U_{r_0}(\epsilon) + V_{r_0}(\epsilon).$$

This proves that T is continuous on B_{r_0} (obviously, $U_{r_0}(\epsilon) \to 0$ and $V_{r_0}(\epsilon) \to 0$ as $\epsilon \to 0$ which is a simple consequence of the uniform continuity of the functions u and v on the set $I \times I \times [0,r_0]$). Consider the operator T on the subset $B^+_{r_0}$ of the ball B_{r_0} defined in the following way:

$$B^+_{r_0} = \{x \in B_{r_0} : x(t) \ge 0,\ \text{for } t \in I\}.$$

Obviously the set $B^+_{r_0}$ is nonempty, bounded, closed and convex. In view of our assumptions (A_1) and (A_4), if $x(t) \ge 0$ then $(Tx)(t) \ge 0$ for all $t \in I$. Thus T transforms the set $B^+_{r_0}$ into itself. Moreover T is continuous on $B^+_{r_0}$. Let X be a nonempty subset of $B^+_{r_0}$. Fix $\epsilon > 0$ and $t_1, t_2 \in I$ with $|t_2 - t_1| \le \epsilon$. Without loss of generality assume that $t_2 \ge t_1$. Then we get

$$\begin{aligned} &|(Fx)(t_2) - (Fx)(t_1)| \\ \le&\Big|f_1\Big(t_2, x(t_2), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_2, x(t_1), \int_0^{t_2} u(t_2,s,x(s))ds\Big)\Big| \\ &+ \Big|f_1\Big(t_2, x(t_1), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_1, x(t_1), \int_0^{t_2} u(t_2,s,x(s))ds\Big)\Big| \\ &+ \Big|f_1\Big(t_1, x(t_1), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_1, x(t_1), \int_0^{t_2} u(t_1,s,x(s))ds\Big)\Big| \\ &+ \Big|f_1\Big(t_1, x(t_1), \int_0^{t_2} u(t_1,s,x(s))ds\Big) - f_1\Big(t_1, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big)\Big| \\ \le&\frac{k}{2}|x(t_2) - x(t_1)| + \omega(f_1,\epsilon) + \Big|\int_0^{t_2} [u(t_2,s,x(s)) - u(t_1,s,x(s))]ds\Big| \\ &+ \Big|\int_{t_1}^{t_2} u(t_1,s,x(s))ds\Big| \\ \le&\frac{k}{2}|x(t_2) - x(t_1)| + \omega(f_1,\epsilon) + \int_0^{t_2} \omega(u,\epsilon))ds + \int_{t_1}^{t_2} K_u ds. \end{aligned}$$

We obtain that

$$|(Fx)(t_2) - (Fx)(t_1)| \le \frac{k}{2}\omega(x,\epsilon) + \omega(f_1,\epsilon) + \omega(u,\epsilon)) + K_u\,\epsilon$$

where

$$\begin{aligned}\omega(u,\epsilon) &= \sup\{|u(t_2,s,x)-u(t_1,s,x)| : t_2,t_1,s\in I,\ |t_2-t_1|\le\epsilon,\ x\in[0,r_0]\},\\ K_u &= \sup\{|u(t,s,x)| : t,s\in I, x\in[0,r_0]\},\\ \omega(f_1,\epsilon) &= \sup\{|f_1(t,x,y)-f_1(s,x,y)| : t,s\in I,\ |t-s|\le\epsilon,\ x\in[0,r_0],\ y\in[0,K_u]\}.\end{aligned}$$

Also we have

$$\begin{aligned}&|(Gx)(t_2)-(Gx)(t_1)|\\ \le&\Big|f_2\Big(t_2,x(t_2),\int_0^1 v(t_2,s,x(s))ds\Big)-f_2\Big(t_2,x(t_1),\int_0^1 v(t_2,s,x(s))ds\Big)\Big|\\ &+\Big|f_2\Big(t_2,x(t_1),\int_0^1 v(t_2,s,x(s))ds\Big)-f_2\Big(t_1,x(t_1),\int_0^1 v(t_2,s,x(s))ds\Big)\Big|\\ &+\Big|f_2\Big(t_1,x(t_1),\int_0^1 v(t_2,s,x(s))ds\Big)-f_2\Big(t_1,x(t_1),\int_0^1 v(t_1,s,x(s))ds\Big)\Big|\\ \\ \le&\frac{k}{2}|x(t_2)-x(t_1)|+\omega(f_2,\epsilon)+\Big|\int_0^1[v(t_2,s,x(s))-v(t_1,s,x(s))]ds\Big|\\ \le&\frac{k}{2}\omega(x,\epsilon)+\omega(f_2,\epsilon)+\omega(v,\epsilon),\end{aligned}$$

where

$$\begin{aligned}\omega(v,\epsilon) &= \sup\{|v(t_2,s,x)-v(t_1,s,x)| : t_2,t_1,s\in I,\ |t_2-t_1|\le\epsilon,\ x\in[0,r_0]\},\\ K_v &= \sup\{|v(t,s,x)| : t,s\in I, x\in[0,r_0]\},\\ \omega(f_2,\epsilon) &= \sup\{|f_2(t,x,y)-f_2(s,x,y)| : t,s\in I,\ |t-s|\le\epsilon,\ x\in[0,r_0],\ y\in[0,K_v]\}.\end{aligned}$$

Hence

$$\begin{aligned}|(Tx)(t_2)-(Tx)(t_1)| \le&|g(t_2)-g(t_1)|+|(Fx)(t_2)-(Fx)(t_1)|+|(Gx)(t_2)-(Gx)(t_1)|\\ \le&\omega(g,\epsilon)+k\omega(x,\epsilon)+\omega(f_1,\epsilon)+\omega(u,\epsilon)+K_u\,\epsilon+\omega(f_2,\epsilon)+\omega(v,\epsilon).\end{aligned}$$

Thus taking the supremum on x, we obtain

$$\omega(TX,\epsilon)\le\omega(g,\epsilon)+k\omega(X,\epsilon)+\omega(f_1,\epsilon)+\omega(u,\epsilon)+K_u\,\epsilon+\omega(f_2,\epsilon)+\omega(v,\epsilon)$$

Now, in virtue of continuity of the function ψ and the uniform continuity of the functions g, f_1 and f_2 on I, $I\times[0,r_0]\times[0,K_u]$ and $I\times[0,r_0]\times[0,K_v]$, respectively, we have that $\omega(g,\epsilon)\longrightarrow 0$, $\omega(f_1,\epsilon)\longrightarrow 0$, $\omega(f_2,\epsilon)\longrightarrow 0$, $\omega(u,\epsilon)\longrightarrow 0$ and $\omega(v,\epsilon)\longrightarrow 0$ as $\epsilon\longrightarrow 0$. So let $\epsilon\longrightarrow 0$ to obtain

$$\omega_0(TX)\le k\omega_0(X). \tag{8}$$

Let $x\in X$ and $t_1,t_2\in I$ with $t_1<t_2$. Then

$$\begin{aligned}&|(Tx)(t_2)-(Tx)(t_1)|-[(Tx)(t_2)-(Tx)(t_1)]\\ &=|g(t_2)+(Fx)(t_2)+(Gx)(t_2)-g(t_1)-(Fx)(t_1)-(Gx)(t_1)|\\ &-[g(t_2)+(Fx)(t_2)+(Gx)(t_2)-g(t_1)-(Fx)(t_1)-(Gx)(t_1)]\\ &\le|g(t_2)-g(t_1)|-[g(t_2)-g(t_1)]+|(Fx)(t_2)-(Fx)(t_1)|-[(Fx)(t_2)-(Fx)(t_1)]\\ &+|(Gx)(t_2)-(Gx)(t_1)|-[(Gx)(t_2)-(Gx)(t_1)]\\ &\le k|x(t_2)-x(t_1)|-k[x(t_2)-x(t_1)].\end{aligned} \tag{9}$$

Indeed

$$
\begin{aligned}
&|(Fx)(t_2) - (Fx)(t_1)| - [(Fx)(t_2) - (Fx)(t_1)] \\
&\le \Big| f_1\Big(t_2, x(t_2), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_1, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big)\Big| \\
&- \Big[f_1\Big(t_2, x(t_2), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_1, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big)\Big] \\
&\le \Big| f_1\Big(t_2, x(t_2), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_2, x(t_1), \int_0^{t_2} u(t_2,s,x(s))ds\Big)\Big| \\
&+ \Big| f_1\Big(t_2, x(t_1), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_2, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big)\Big| \\
&+ \Big| f_1\Big(t_2, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big) - f_1\Big(t_1, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big)\Big| \\
&- \Big[f_1\Big(t_2, x(t_2), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_2, x(t_1), \int_0^{t_2} u(t_2,s,x(s))ds\Big)\Big] \\
&- \Big[f_1\Big(t_2, x(t_1), \int_0^{t_2} u(t_2,s,x(s))ds\Big) - f_1\Big(t_2, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big)\Big] \\
&- \Big[f_1\Big(t_2, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big) - f_1\Big(t_1, x(t_1), \int_0^{t_1} u(t_1,s,x(s))ds\Big)\Big] \\
&\le \frac{k}{2}|x(t_2) - x(t_1)| - \frac{k}{2}[x(t_2) - x(t_1)].
\end{aligned}
$$

Similarly we have

$$|(Gx)(t_2) - (Gx)(t_1)| \quad -[(Gx)(t_2) - (Gx)(t_1)] \le \tfrac{k}{2}|x(t_2) - x(t_1)| - \tfrac{k}{2}[x(t_2) - x(t_1)].$$

Hence we get

$$i(Tx) \le k\, i(x),$$

and consequently

$$i(TX) \le k\, i(X). \tag{10}$$

From (9) and (10) and the definition of the measure of noncompactness μ, we obtain

$$\mu(TX) = \omega_0(TX) + i(TX) \le k\omega_0(X) + k\, i(X) = k[\omega_0(X) + i(X)] = k\mu(X).$$

Now the result follows from Theorem 3.4 by taking as function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$

$$\theta(t,s) = k\,s - t, \forall\, t,s \in \mathbb{R}_+ \text{ and } \varphi \equiv 0.$$

This completes the proof. □

Now we provide two examples illustrating the result obtained.

Example 4.2. *Consider the following nonlinear functional-integral equation:*

$$
\begin{aligned}
x(t) = \frac{t}{t+1} + \frac{t^2}{4(1+t^4)}x(t) + \frac{t}{8}\int_0^t s\arctan(x^2(s))ds \\
+ \frac{t}{4}\int_0^1 \frac{1}{1+s}\arctan(x(s))ds.
\end{aligned}
\tag{11}
$$

Equation (11) is a special case of the integral equation (1), where

$$\begin{aligned} f_1(t,x,y) =& f_2(t,x,y) = \frac{t^2}{8(1+t^4)}x + y, \\ u(t,s,x) =& \frac{ts}{8}\arctan(x^2), \\ v(t,s,x) =& \frac{t}{4(1+s)}\arctan(x), \\ g(t) =& \frac{t}{t+1},\ h(x) = \frac{1}{4}x. \end{aligned}$$

Then it is easily seen that g satisfies the assumption (A_1) with $b = \frac{1}{2}$. Since $u(t,s,x) = \frac{ts}{8}\arctan(x^2)$ and $v(t,s,x) = \frac{t}{4(1+s)}\arctan(x)$, then for all $t,s \in I$ and $x \in \mathbb{R}$, we get

$$\begin{aligned} |u(t,s,x)| &= \left|\frac{ts}{8}\arctan(x^2)\right| \le \frac{1}{4}|x| = h(|x|), \\ |v(t,s,x)| &= \left|\frac{t}{4(1+s)}\arctan(x)\right| \le \frac{1}{4}|x| = h(|x|). \end{aligned}$$

In this example we have $f_1(t,x,y) = f_2(t,x,y) = \frac{t^2}{1+t^4}x + y$ and these functions satisfy assumption (A_5). On the other hand for all $t \in I$ and $x,y \in \mathbb{R}$, we get

$$\begin{aligned} |f_i(t,x,y) - f_i(t,z,w)| &= \left|\frac{t^2}{8(1+t^4)}x + y - \frac{t^2}{8(1+t^4)}z - w\right| \\ &\le \frac{t^2}{8(1+t^4)}|x-z| + |y-w| \\ &\le \frac{1}{16}|x-z| + |y-w|. \end{aligned}$$

So, $k = \frac{1}{8}$ and $M = 0$. Thus the existent inequalities in assumption (A_6) have the forms

$$\frac{1}{2} + \frac{1}{8}r_0 + 2 \times \frac{r_0}{4} \le r_0.$$

Indeed, if $r_0 \ge \dfrac{4}{3}$ then

$$\frac{1}{2} \le r_0 - \frac{5}{8}r_0 \longrightarrow \frac{1}{2} + \frac{5}{8}r_0 \le r_0 \longrightarrow \frac{1}{2} + \frac{1}{8}r_0 + 2 \times \frac{r_0}{4} \le r_0 \Longrightarrow b + kr_0 + 2h(r_0) + 2M \le r_0$$

It is easily seen that the last inequalities have a positive solution. For example $r_0 = 2$. We see that all assumptions of Theorem 4.1 are satisfied. Consequently from Theorem 4.1 the integral equation (11) has at least one solution in the space $C(I)$.

Example 4.3. *Let us consider now the following integral equation*

$$\begin{aligned} x(t) = \frac{1}{5}t^3 + \frac{2t(x(t)+1)}{5(1+t)} + \arctan\Big(\int_0^t \Big[ts + \frac{1}{8}(t^3+1)x(s)\Big]ds\Big) \\ + \frac{2t^2}{5(1+t^4)}\ln(|x(t)|+1) + \ln\Big(1 + \int_0^1 \Big[t^2 + \frac{1}{16}(3s^2+1)x(s)\Big]ds\Big). \end{aligned} \tag{12}$$

It can be easily seen that equation (12) is a particular case of the equation (1), where

$$\begin{aligned} f_1(t,x,y) =& \frac{2t(x+1)}{5(1+t)} + \arctan(y),\ f_2(t,x,y) = \frac{2t^2}{5(1+t^4)}\ln(|x|+1) + \ln(|y|+1), \\ u(t,s,x) =& ts + \frac{1}{8}(t^3+1)x,\ v(t,s,x) = t^2 + \frac{1}{16}(3s^2+1)x,\ g(t) = \frac{1}{5}t^3,\ h(x) = 1 + \frac{1}{4}x. \end{aligned}$$

The function g satisfies assumption (A_1) and $b = \frac{1}{5}$. Moreover, the functions f_1 and f_2 satisfy hypothesis (A_4), (A_5) and

$$\begin{aligned}|f_1(t,x,y) - f_1(t,z,w)| &= \Big|\frac{2t(x+1)}{5(1+t)} + \arctan(y) - \frac{2t(z+1)}{5(1+t)} - \arctan(w)\Big| \\ &\leq \frac{2t}{5(1+t)}|x-z| + |\arctan(y) - \arctan(w)| \\ &\leq \frac{1}{5}|x-z| + |y-w|,\end{aligned}$$

and

$$\begin{aligned}&|f_2(t,x,y) - f_2(t,z,w)| \\ &= \Big|\frac{2t^2}{5(1+t^4)}\ln(|x|+1) + \ln(|y|+1) - \frac{2t^2}{5(1+t^4)}\ln(|z|+1) - \ln(|w|+1)\Big| \\ &\leq \frac{2t^2}{5(1+t^4)}|\ln(|x|+1) - \ln(|z|+1)| + |\ln(|y|+1) - \ln(|w|+1)| \\ &\leq \frac{1}{5}\ln\Big(\frac{|x|+1}{|z|+1}\Big) + \ln\Big(\frac{|y|+1}{|w|+1}\Big) \\ &= \frac{1}{5}\ln\Big(1 + \frac{|x-z|}{|z|+1}\Big) + \ln\Big(1 + \frac{|y-w|}{|w|+1}\Big) \\ &\leq \frac{1}{5}\ln(1+|x-z|) + \ln(1+|y-w|) \\ &\leq \frac{1}{5}|x-z| + |y-w|, \forall\, x, y \in \mathbb{R} \text{ and } t \in I.\end{aligned}$$

Here we have $M = \frac{1}{5}$. Also for all $t, s \in I$ and $x \in \mathbb{R}$, we get

$$|u(t,s,x)| = \Big|ts + \frac{1}{8}(t^3+1)x\Big| \leq 1 + \frac{1}{4}|x| = h(|x|),$$

$$|v(t,s,x)| = \Big|t^2 + \frac{1}{16}(3s^2+1)x\Big| \leq 1 + \frac{1}{4}|x| = h(|x|).$$

Since $\frac{1}{5} + \frac{2}{5}r_0 + 2(1 + \frac{1}{4}r_0) + 2 \times \frac{1}{5} < r_0$, for small $r_0 > 0$, assumption (A_6) holds true for sufficiently small r_0. Hence, applying Theorem 4.1 we infer that Eq. (12) has a solution $x = x(t)$ in the space $C(I)$.

5. Conclusions

In the current work, we investigated the existence and solutions for integral equations. Also, some examples are presented to show the efficiency of our results.

Competing interests

The authors declare that they have no competing interests regarding this manuscript.

Authors' contributions

All authors contributed equally to the writing of this manuscript. All authors read and approved the final version.

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References

[1] M.A. Abdou, *On the solution of linear and nonlinear integral equation*, Appl. Math. Comput. **146**(2003), 857–871.
[2] R. P. Agarwal, D. O'Regan, *Singular Volterra integral equations*, Appl. Math. Letters **13**(2000), 115–120.
[3] A. Aghajani, J. Banaś, N. Sabzali, *Some generalizations of Darbo fixed point theorem and applications*, Bull. Belg. Math.Soc. Simon Stevin **20**(2)(2013), 345–358.
[4] A. Aghajani, J. Banaś, Y. Jalilian, *Existence of solution for a class of nonlinear Volterra singular integral equations*, Comput. Math. Appl. **62**(2011), 1215–1227.
[5] A. Aghajani, R. Allahyari, M. Mursaleen, *A generalization of Darbo's theorem with application to the solvability of systems of integral equations*, J. Comput. Appl. Math. **260** (2014), 68–77.
[6] R. Allahyari, R. Arab and A. Shole Haghighi, *Existence of solutions for some classes of integro-differential equations via measure of non-compactness*, Electron. J. Qual. Theory Differ. Equ. **41**(2015), 1-18.
[7] R. Arab, *Some generalizations of Darbo fixed point theorem and its application*, Miskolc Math. Notes **18**(2)(2017), 595-610.
[8] R. Arab, M. Rabbani, R. Mollapourasl, *On solution of a nonlinear integral equation with deviating argument based the on fixed point technique*, Appl. Comput. Math. **14**(1)(2015), 38–49.
[9] R. Arab, R. Allahyari, A. Shole Haghighi, *Construction of a measure of non-compactness on $BC(\Omega)$ and its application to Volterra integral equations*, Mediter. J. Math. **13**(3)(2016), 1197–1210.
[10] R. Arab, *The existence of fixed points via the measure of noncompactness and its application to functional-integral equations*, Mediter. J. Math. **13**(2)(2016), 759-773.
[11] I.K. Argyros, *Quadratic equations and applications to Chandrasekhars and related equations*, Bull. Austral. Math. Soc. **32** (1985), 275–292.
[12] J. Banaś, *Measures of noncompactness in the space of continuous tempered functions*, Demonstratio Math. **14** (1981), 127–133.
[13] J. Banaś, L. Olszowy, *Measures of noncompactness related to monotonicity*, Comment. Math. **41**(2001), 13-23.
[14] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, Dekker, New York, **60**(1980).
[15] J. Banaś, B. Rzepka, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl. **284** (2003), 165–173.
[16] C. Corduneanu, *Integral Equations and Applications*, Cambridge Univ. Press, New York, 1973.
[17] A. Das, B. Hazarika, R. Arab, M. Mursaleen, *Solvability of the infinite system of integral equations in two variables in the sequence spaces c_0 and ℓ_1*, J. Comput. Appl. Math. **326**(2017), 183-192.
[18] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
[19] B. Hazarika, E. Karapınar, R. Arab, M. Rabbani, *Metric-like spaces to prove existence of solution for nonlinear quadratic integral equation and numerical method to solve it*, J. Comput. Appl. Math. **328**(2018), 302-313.
[20] B. Hazarika, H. M. Srivastava, R. Arab, M. Rabbani, *Existence of solution for an infinite system of nonlinear integral equations via measure of noncompactness and homotopy perturbation method to solve it*, J. Comput. Appl. Math. **343**(2018), 341-352.
[21] F. Khojasteh, S. Shukla, S. Radenovic, *A new approach to the study of fixed point theorems via simulation functions*, Filomat **29**(6)(2015), 1189-1194.
[22] M. Mursaleen, S. A. Mohiuddine, *Applications of noncompactness to the infinite system of differential equations in l_p spaces*, Nonlinear Anal. (TMA) **75**(4)(2012), 2111-2115.
[23] H. K. Nashine, R. Arab, *Existence of solutions to nonlinear functional-integral equations via the measure of noncompactness*, J. Fixed Point Theory Appl. **20**(2)(2018), doi.org/10.1007/s11784-018-0546-1
[24] S. Reich, *Fixed points of condensing functions*. J. Math. Anal. Appl. **41** (1973), 460–467.