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Almost Periodic Generalized Ultradistributions

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Abstract. We first introduce an algebra of almost periodic generalized ultradistributions containing classical almost periodic ultradistributions as well as the algebra of almost periodic generalized functions, and then we study the fundamental properties of this algebra.

1. Introduction

The concept of an almost periodic distribution extending the classical Bohr and Stepanoff almost periodic functions is due to L. Schwartz [13]. In view of the problem of multiplication of distributions, algebras of generalized functions containing different classes of distributions have been introduced and developed, see [11]. Consequently, an algebra of almost periodic generalized functions containing almost periodic functions and almost periodic distributions has been introduced and studied in [3], and an application to systems of ordinary differential equations is given in [4].

It is well known that ultradistributions are generalization of distributions, they are useful for concrete problems, for example differential equations, see [8]. However they are also less adapted to non linear operations. Algebras of generalized ultradistributions containing ultradistributions are nowadays an important subject of research, see [2] and [7].

The almost periodicity of Beurling ultradistributions in the sense of Komatsu [10] is tackled in the paper [6].

Therefore, this work mainly investigates the concept of almost periodicity in the setting of algebras of generalized functions containing almost periodic ultradistributions. So we have a new reservoir of mathematical objects that could be useful for studying problems with nonlinear operations on ultradistributions. The paper aims to introduce an algebra of almost periodic generalized ultradistributions containing almost periodic Beurling ultradistributions of [6], as well as the algebra of almost periodic generalized functions of [3], and then it studies fundamental properties of this algebra.

The paper is organised as follows, this introduction is followed by a second section of preliminaries where we recall the mathematical notions needed in the sequel. In the third section, we first introduce the algebra of almost periodic generalized ultradistributions $\mathcal{G}_{pp}^{M}$ and then we show some properties they satisfy. The fourth section is aimed to study generalized numbers with asymptotic behavior connected to the punctual values of generalized ultradistributions. In section five, examples of almost periodic generalized ultradistributions are given, in particular we propose the embedding of the space of Beurling

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ultradistributions into $\mathcal{G}^M_{pp}$. In section six, we show how the algebra $\mathcal{G}^M_{pp}$ is stable under a nonlinear operation as the composition with tempered generalized ultradistributions. The last section is aimed to show that the extension of the classical generalized point value and the mean value to the case of almost periodic generalized ultradistributions, and then it gives a result connected with the classical Bohl-Bohr Theorem.

2. Preliminaries

We consider functions and distributions defined on the whole space of real numbers $\mathbb{R}$. Let C_{ub} denotes the space of continuous, bounded and complex valued functions defined on $\mathbb{R}$, endowed with the norm $\|.\|_\infty$ of uniform convergence on $\mathbb{R}$, the space $(C_{ub}, \|.\|_\infty)$ is a Banach algebra.

Definition 2.1. *A continuous function on $\mathbb{R}$ is said to be almost periodic if it satisfies one of the following equivalent assertions :*
i) $\forall \varepsilon > 0$ the set

$$E(\varepsilon, f) := \left\{\tau \in \mathbb{R} : \left\|f(. + \tau) - f(.)\right\|_\infty < \varepsilon\right\}$$

is relatively dense in $\mathbb{R}$, i.e. there exists a number l such that every interval of length l contains at least a number of $E(\varepsilon, f)$.
ii) For any sequence of real numbers $(h_n)_n$ one can extract a subsequence $(h_{n_k})_k$ such that $(f(. + h_{n_k}))_k$ converges uniformly on $\mathbb{R}$.
iii) $\forall \varepsilon > 0$ there exists P a trigonometric polynomial such that

$$\left\|f - P\right\|_\infty < \varepsilon$$

We denote by C_{pp} the space of almost periodic functions on $\mathbb{R}$.

We summarize some properties of the space C_{pp}, for the proofs see [9].

Proposition 2.2. *1. $C_{pp} \subset C_{ub}$.*
2. $C_{pp} \times C_{pp} \subset C_{pp}$.
3. $C_{pp} * L^1 \subset C_{pp}$.
4. If $f \in C_{pp}$ then $f' \in C_{pp}$ if and only if f' is uniformly continuous on $\mathbb{R}$.

The following result called Bohl-Bohr's Theorem gives the almost periodicity of a primitive.

Proposition 2.3. *A primitive of $f \in C_{pp}$ is almost periodic if and only if it is bounded.*

One of the important properties of almost periodic functions is the association of Fourier series to them.

Proposition 2.4. *If $f \in C_{pp}$ and $\lambda \in \mathbb{R}$ then*
i. $a_f(\lambda) := \lim\limits_{X\to+\infty} \frac{1}{X}\int\limits_0^X f(x) e^{-i\lambda x} dx$ exists.
ii. There exists at most a countable set of λ's for which $a_f(\lambda) \neq 0$.
iii. If f is positive then $M(f) := \lim\limits_{X\to+\infty} \frac{1}{X}\int\limits_0^X f(x) dx$ is zero if and only if $f \equiv 0$.
iv. $\forall \varphi \in L^1$,

$$M(f * \varphi) = M(f) \int\limits_{-\infty}^{+\infty} \varphi(x) dx. \tag{1}$$

Definition 2.5. *The numbers* $A_n := a_f(\lambda_n), n \in \mathbb{N}$, *are called Fourier coefficients of* f. *The formal series* $\sum_{n=1}^{+\infty} A_n e^{i\lambda_n x}$ *is called the Fourier series associated to* f.

Denoting by $\mathcal{E}$ the space of infinitely derivable functions, we recall the space of almost periodic infinitely derivable functions and the space of infinitely derivable bounded functions, see [13] and [3], are denoted and defined respectively by

$$\mathcal{B}_{pp} := \left\{\varphi \in \mathcal{E} : \forall j \in \mathbb{Z}_+, \varphi^{(j)} \in C_{pp}\right\}.$$

$$\mathcal{B} := \left\{\varphi \in \mathcal{E} : \forall j \in \mathbb{Z}_+, \varphi^{(j)} \in L^\infty\right\}.$$

It is well known that $\mathcal{B}$ is a Frechet algebra.

Proposition 2.6. *i)* $\mathcal{B}_{pp}$ *is a closed differential subalgebra of* $\mathcal{B}$ *stable under derivation .*
ii) $\mathcal{B}_{pp} * L^1 \subset \mathcal{B}_{pp}$.
iii) $\mathcal{B}_{pp} = \mathcal{B} \cap C_{pp}$.

Let $M = (M_k)_{k=0}^{+\infty}$ be a sequence of positive numbers, we define the following properties
Logarithmic Convexity

$$M_k^2 \leq M_{k-1}M_{k+1}, \forall k \in \mathbb{Z}_+. \tag{H_1}$$

Stability under ultraderivation

$$\exists A > 0, \exists H > 0, M_{k+q} \leq AH^{k+q}M_kM_q, \forall k, q \in \mathbb{Z}_+. \tag{H_2}$$

Non quasi-analyticity

$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty. \tag{H_3'}$$

Definition 2.7. *The associated function of the sequence M is the function defined by*

$$M(t) = \sup_k \ln \frac{t^k M_0}{M_k}, 0 < t < \infty.$$

Example 2.8. *If* $M_k = (k!)^\sigma, \sigma > 1$, *then* $M(t)$ *is equivalent to* $t^{\frac{1}{\sigma}}$.

An important properties of the associated function is given by the following result, see [10].

Proposition 2.9. *i) The sequence M satisfies* (H_1) *if and only if* $\forall k \in \mathbb{Z}_+$

$$M_k = \sup_{t>0} \frac{t^k M_0}{e^{M(t)}}.$$

ii) Let M satisfying condition (H_1), *then it satisfies* (H_2) *if and only if* $\exists A, H > 0, \forall t > 0$,

$$2M(t) \leq M(Ht) + \ln(AM_0).$$

The function M being increasing and by Proposition 2.9- ii) we have $\exists A, H > 0, \forall t_1, t_2 > 0$,

$$M(t_1) + M(t_2) \leq M(H\max(t_1, t_2)) + \ln(AM_0). \tag{2}$$

The general case is proved further in Lemma 4.2.

Remark 2.10. *We will always assume that the sequence M satisfies conditions (H_1), (H_2) and $\left(H_3'\right)$.*

We need some function spaces, see [12].

Definition 2.11. *The space $\mathcal{D}_{L^p}^{(M)}$, $1 \le p \le \infty$, is the set of $\varphi \in \mathcal{E}$ such that $\forall h > 0, \exists c > 0, \forall j \in \mathbb{Z}_+$,*

$$\left\|\partial^j \varphi\right\|_{L^p} \le c h^j M_j.$$

The space $\mathcal{D}_{L^p}^{(M)}$ is endowed with the projective limit topology of Banach spaces, i.e.

$$\mathcal{D}_{L^p}^{(M)} := \lim_{h \to 0} proj \mathcal{D}_{L^p}^{(M),h},$$

where the space

$$\mathcal{D}_{L^p}^{(M),h} := \left\{\varphi \in \mathcal{E} : \|\varphi\|_{p,h,M} < \infty\right\}, h > 0, \tag{3}$$

provided with the norm

$$\|\varphi\|_{p,h,M} := \sup_j \frac{\left\|\partial^j \varphi\right\|_{L^p}}{h^j M_j} \tag{4}$$

is a Banach space.

Remark 2.12. *The space*

$$\mathcal{D}^{(M)} := \left\{\varphi \in \mathcal{E} : \forall K \text{ compact of } \mathbb{R}, \forall h > 0, \exists c > 0, \forall j \in \mathbb{Z}_+, \sup_{x \in K} \left|\partial^j \varphi(x)\right| \le c h^j M_j\right\}$$

of Beurling ultradifferentiable functions is dense in $\mathcal{D}_{L^p}^{(M)}$ for $p \in [1, +\infty[$. We denote by $\dot{\mathcal{B}}^{(M)}$ the closure of the space $\mathcal{D}^{(M)}$ in $\mathcal{D}_{L^\infty}^{(M)} =: \mathcal{B}^{(M)}$.

Definition 2.13. *Let $p \in [1, +\infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$, the topological dual of $\mathcal{D}_{L^p}^{(M)}$, denoted by $\mathcal{D}'_{L^q,(M)}$, is called the space of L^p- Beurling ultradistributions. We denote $\mathcal{D}'_{L^1,(M)}$ the topological dual of $\dot{\mathcal{B}}^{(M)}$. The space denoted by $\mathcal{B}'_{(M)} := \left(\mathcal{D}_{L^1}^{(M)}\right)'$ is called the space of bounded ultradistributions .*

3. Almost periodic generalized ultradistributions

Let $I :=]0, 1]$, if $(f_\varepsilon)_{\varepsilon \in I}$ is a sequence of functions the notation

$$\|f_\varepsilon\|_\infty = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0,$$

means that $\exists c > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \le \varepsilon_0, \|f_\varepsilon\|_\infty \le c e^{M\left(\frac{k}{\varepsilon}\right)}$.

Definition 3.1. *i) The space of almost periodic moderate elements is the space defined by*

$$\mathcal{M}_{pp}^M := \left\{(f_\varepsilon)_{\varepsilon \in I} \in \left(\mathcal{B}_{pp}\right)^I : \forall j \in \mathbb{Z}_+, \exists k > 0, \left\|f_\varepsilon^{(j)}\right\|_\infty = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0\right\}.$$

ii) The space of almost periodic null elements is the space defined by

$$\mathcal{N}_{pp}^M := \left\{(f_\varepsilon)_{\varepsilon \in I} \in \left(\mathcal{B}_{pp}\right)^I : \forall j \in \mathbb{Z}_+, \forall k > 0, \left\|f_\varepsilon^{(j)}\right\|_\infty = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0\right\}.$$

We have the null characterization of $\mathcal{N}_{pp}^{M}$.

Proposition 3.2. *An element $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^{M}$ is null if and only if $\forall k > 0$,*

$$\left\|f_\varepsilon\right\|_\infty = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0. \tag{5}$$

Proof. The proof is based on the following classical Landau-Kolmogorov inequality

$$\left\|f^{(p)}\right\|_\infty \le 2\pi \left\|f\right\|_\infty^{1-\frac{p}{m}} \left\|f^{(m)}\right\|_\infty^{\frac{p}{m}},$$

where $0 < p < m \in \mathbb{Z}_+$ and the function f is of class C^m.

Let $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^{M}$, i.e. $\forall i \in \mathbb{Z}_+, \exists k_i > 0, \exists c_i > 0, \exists \varepsilon_i \in I, \forall \varepsilon \le \varepsilon_i$,

$$\left\|f_\varepsilon^{(i)}\right\|_\infty \le c_i \exp\left(M\left(\frac{k_i}{\varepsilon}\right)\right). \tag{6}$$

Suppose that $(f_\varepsilon)_\varepsilon$ satisfies (5), i.e. $\forall k > 0, \exists c > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \le \varepsilon_0, \left\|f_\varepsilon\right\|_\infty \le c e^{M\left(\frac{k}{\varepsilon}\right)}$. The Landau-Kolmogorov inequality for $m = 2i$, (5), (6) and (2) give

$$\left\|f_\varepsilon^{(i)}\right\|_\infty \le 2\pi \left\|f_\varepsilon\right\|_\infty^{1-\frac{1}{2}} \left\|f_\varepsilon^{(2i)}\right\|_\infty^{\frac{1}{2}} \le 2\pi \left(c e^{-M\left(\frac{k}{\varepsilon}\right)}\right)^{\frac{1}{2}} \left(c_i e^{M\left(\frac{k_i}{\varepsilon}\right)}\right)^{\frac{1}{2}} \le C_i e^{-\frac{1}{2}M\left(\frac{k}{\varepsilon}\right)+\frac{1}{2}M\left(\frac{k_i}{\varepsilon}\right)},$$

where $C_i = 2\pi A M_0 c^{\frac{1}{2}} c_i^{\frac{1}{2}}$. Let $k_0 > 0$ using the inequality (2) with $\frac{k}{\varepsilon} = H \max\left(\frac{k_i}{\varepsilon}, \frac{k_0}{\varepsilon}\right)$ we obtain the result, i.e. $\forall i \in \mathbb{Z}_+, \forall k_0 > 0$,

$$\left\|f_\varepsilon^{(i)}\right\|_\infty \le C_i e^{-M\left(\frac{k_0}{\varepsilon}\right)}.$$

□

The main properties of the spaces $\mathcal{M}_{pp}^{M}$ and $\mathcal{N}_{pp}^{M}$ are shown in the following Proposition.

Proposition 3.3. *1) The space $\mathcal{M}_{pp}^{M}$ is an algebra stable under derivation.*

2) The space $\mathcal{N}_{pp}^{M}$ is an ideal of $\mathcal{M}_{pp}^{(M)}$.

Proof. 1) The stability with respect to the derivation is obvious. Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^{M}$, then they satisfy (6) and we have $\forall\, n \in \mathbb{Z}_+$,

$$\left\|\partial^n (f_\varepsilon g_\varepsilon)\right\|_\infty \le \sum_{i+j=n} \frac{n!}{i!j!} \left|f^{(j)}(x)\right| \left|g_\varepsilon^{(i)}(x)\right| \le \sum_{i+j=n} \frac{n!}{i!j!} c_j e^{M\left(\frac{k_j}{\varepsilon}\right)} c_i e^{M\left(\frac{k_i}{\varepsilon}\right)}.$$

From (2) with $t_1 = \dfrac{k_j}{\varepsilon}, t_2 = \dfrac{k_i}{\varepsilon}, k = H\max\limits_{i+j=n}\left(k_j, k_i\right)$ and $\varepsilon \le \min\limits_{i+j=n}\left(\varepsilon_j, \varepsilon_i\right)$, we obtain

$$M\left(\frac{k_j}{\varepsilon}\right) + M\left(\frac{k_i}{\varepsilon}\right) \le M\left(\frac{k}{\varepsilon}\right) + \ln(AM_0),$$

and consequently

$$\left\|\partial^n (f_\varepsilon g_\varepsilon)\right\|_\infty \le \left(AM_0 \sum_{i+j=n} \frac{n!}{i!j!} c_j c_i\right) e^{M\left(\frac{k}{\varepsilon}\right)},$$

which gives $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^M$.

2) Let $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^M$ and $(g_\varepsilon)_\varepsilon \in \mathcal{N}_{pp}^M$, i.e. $\forall i \in \mathbb{Z}_+, \exists k_i > 0, \exists c_i > 0, \exists \varepsilon_i \in I, \forall \varepsilon \leq \varepsilon_i$,

$$\left\| f_\varepsilon^{(i)} \right\|_\infty \leq c_i \exp\left(M\left(\frac{k_i}{\varepsilon} \right) \right)$$

and $\forall i \in \mathbb{Z}_+, \forall k_1 > 0, \exists c_i' > 0, \exists \varepsilon_i' \in I, \forall \varepsilon \leq \varepsilon_i'$,

$$\left\| g_\varepsilon^{(i)} \right\|_\infty \leq c_i' e^{-M\left(\frac{k_1}{\varepsilon}\right)}, \tag{7}$$

since $\mathcal{N}_{pp}^M \subset \mathcal{M}_{pp}^M$, then $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^M$. It remains to prove (5) . Indeed

$$\left\| f_\varepsilon g_\varepsilon \right\|_\infty \leq \left\| f_\varepsilon \right\|_\infty \left\| g_\varepsilon \right\|_\infty \leq c_0 c_0' e^{M\left(\frac{k_0}{\varepsilon}\right)} e^{-M\left(\frac{k_1}{\varepsilon}\right)}.$$

Due to (2) with $t_1 = \dfrac{k_0}{\varepsilon}, t_2 = \dfrac{k}{\varepsilon}, k_1 = H \max(k_0, k)$ and $\varepsilon \leq \min\left(\varepsilon_0, \varepsilon_0'\right)$, we obtain

$$M\left(\frac{k_0}{\varepsilon}\right) - M\left(\frac{k_1}{\varepsilon}\right) \leq -M\left(\frac{k}{\varepsilon}\right) + \ln(AM_0),$$

then

$$\left\| f_\varepsilon g_\varepsilon \right\|_\infty \leq c_0 c_0' A M_0 e^{-M\left(\frac{k}{\varepsilon}\right)},$$

according to Proposition 3.2, we have $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{N}_{pp}^M$. □

The main definition of this work is the following one.

Definition 3.4. *The algebra of almost periodic generalized ultradistributions, denoted by $\mathcal{G}_{pp}^M$, is the quotient algebra*

$$\mathcal{G}_{pp}^M := \frac{\mathcal{M}_{pp}^M}{\mathcal{N}_{pp}^M}.$$

In order to obtain the properties of $\mathcal{G}_{pp}^M$ in the spirit of the space B_{pp} given in Proposition 2.6, we recall the following algebras. Let $p \in [1, \infty]$, the algebra of L^p-generalized ultradistributions is defined by

$$\mathcal{G}_{L^p}^M := \frac{\mathcal{M}_{L^p}^M}{\mathcal{N}_{L^p}^M}, \tag{8}$$

where

$$\mathcal{M}_{L^p}^M := \left\{ (f_\varepsilon)_{\varepsilon \in I} \in \mathcal{E}^I : \forall j \in \mathbb{Z}_+, \exists k > 0, \left\| f_\varepsilon^{(j)} \right\|_{L^p} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)} \right), \varepsilon \to 0 \right\},$$

and

$$\mathcal{N}_{L^p}^M := \left\{ (f_\varepsilon)_{\varepsilon I} \in \mathcal{E}^I : \forall j \in \mathbb{Z}_+, \forall k > 0, \left\| f_\varepsilon^{(j)} \right\|_{L^p} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)} \right), \varepsilon \to 0 \right\}.$$

Remark 3.5. *The elements of $\mathcal{G}_{L^\infty}^M$ are said bounded generalized ultradistributions.*

The algebra of $\mathcal{C}_{pp}-$generalized functions is defined by

$$\mathcal{G}^{M}_{\mathcal{C}_{pp}} := \frac{\mathcal{M}^{M}_{\mathcal{C}_{pp}}}{\mathcal{N}^{M}_{\mathcal{C}_{pp}}}, \tag{9}$$

where

$$\mathcal{M}^{M}_{\mathcal{C}_{pp}} := \left\{ (f_\varepsilon)_{\varepsilon\in I} \in \left(\mathcal{C}_{pp}\right)^I : \exists k > 0, \left\| f_\varepsilon \right\|_\infty = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0 \right\},$$

and

$$\mathcal{N}^{M}_{\mathcal{C}_{pp}} := \left\{ (f_\varepsilon)_{\varepsilon\in I} \in \left(\mathcal{C}_{pp}\right)^I : \forall k > 0, \left\| f_\varepsilon \right\|_\infty = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0 \right\}.$$

If $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{L^\infty}$ and $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{L^1}$, it is not difficult to show that the convolution $\widetilde{u} * \widetilde{v}$ given by

$$\left(\widetilde{u} * \widetilde{v}\right)(x) = \left(\int u_\varepsilon (x-y)\, v_\varepsilon (y)\, dy \right)_\varepsilon + \mathcal{N}^M_{L^\infty}$$

is a well-defined element of $\mathcal{G}^M_{L^\infty}$.

Proposition 3.6. *We have*
1. $\mathcal{G}^M_{pp}$ is a subalgebra of $\mathcal{G}^M_{L^\infty}$ stable under derivation and translation.
2. $\mathcal{G}^M_{pp} * \mathcal{G}^M_{L^1} \subset \mathcal{G}^M_{pp}$.
3. $\mathcal{G}^M_{pp} = \mathcal{G}^M_{L^\infty} \cap \mathcal{G}^M_{\mathcal{C}_{pp}}$.

Proof. 1. The stability with respect to derivation and translation is obvious. Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{pp}$, i.e. $(u_\varepsilon)_\varepsilon$ satisfies (6), and as $u_\varepsilon \in \mathcal{B}_{pp} = \mathcal{C}_{pp} \cap \mathcal{B} \subset \mathcal{B}$, $\forall \varepsilon > 0$, then $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^\infty}$. In the same way, if $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{pp}$, then $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{L^\infty}$.

2. Let $(u_\varepsilon)_\varepsilon$ be a representative of $\widetilde{u} \in \mathcal{G}^M_{pp}$ and $(v_\varepsilon)_\varepsilon$ be a representative of $\widetilde{v} \in \mathcal{G}^M_{L^1}$, due to Proposition 2.6-ii) we have $u_\varepsilon * v_\varepsilon \in \mathcal{B}_{pp}$ $\forall \varepsilon > 0$, and since $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ satisfy estimates as moderate elements of respectivelly $\mathcal{M}^M_{L^\infty}$ and $\mathcal{M}^M_{L^1}$, then we have $\forall j \in \mathbb{Z}_+$,

$$\left\| \left(u_\varepsilon * v_\varepsilon\right)^{(j)} \right\|_\infty \leq \|v_\varepsilon\|_{L^1} \left\| u^{(j)}_\varepsilon \right\|_\infty = O\left(\exp M\left(\frac{k}{\varepsilon}\right)\right),$$

consequently $(u_\varepsilon * v)_\varepsilon \in \mathcal{M}^M_{pp}$. One shows easily that the result is independent on representatives.

3. We clearly have $\mathcal{G}^M_{pp} \subset \mathcal{G}^M_{L^\infty} \cap \mathcal{G}^M_{\mathcal{C}_{pp}}$. Let $\widetilde{u} \in \mathcal{G}^M_{L^\infty} \cap \mathcal{G}^M_{\mathcal{C}_{pp}}$, there are two representatives of $\widetilde{u}$, namelly $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^\infty}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathcal{C}_{pp}}$ such that $(u_\varepsilon)_\varepsilon = (v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^\infty} \cap \mathcal{M}^M_{\mathcal{C}_{pp}}$, so we have that $u_\varepsilon \in \mathcal{B} \cap \mathcal{C}_{pp} = \mathcal{B}_{pp}$, $\forall \varepsilon > 0$, and satisfying (6) which give $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{pp}$. In the same way, if $(u_\varepsilon)_\varepsilon = (v_\varepsilon)_\varepsilon \in \mathcal{N}^M_{L^\infty} \cap \mathcal{N}^M_{\mathcal{C}_{pp}}$, then $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{L^\infty} \cap \mathcal{N}^M_{\mathcal{C}_{pp}}$, so we obtain $u_\varepsilon \in \mathcal{B} \cap \mathcal{C}_{pp} = \mathcal{B}_{pp}$, $\forall \varepsilon > 0$, and satisfying (7) which give that $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{pp}$. We may have also that $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^\infty}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{N}^M_{\mathcal{C}_{pp}}$ or $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{L^\infty}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathcal{C}_{pp}}$, then in the both cases we obtain that $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{pp}$. Consequently $\widetilde{u} \in \mathcal{G}^M_{pp}$. □

The following consequence of the last Proposition is a lifting to the generalized setting of the characterisation of a classical almost periodic Beurling ultradistribution, see Definition 5.3.

Corollary 3.7. *Let $\widetilde{u} \in \mathcal{G}^M_{L^\infty}$, the following statements are equivalent :*
i) $\widetilde{u} \in \mathcal{G}^M_{pp}$.
ii) $\widetilde{u} * \varphi \in \mathcal{G}^M_{\mathcal{C}_{pp}}$, $\forall \varphi \in D^M$.

4. Generalized numbers of type M

In this section, we study the algebra of generalized numbers with an asymptotic behavior of type M.

Definition 4.1. *The ring of generalized numbers of type M is defined by*

$$\widetilde{\mathbb{K}}^M := \frac{\mathcal{M}^M[\mathbb{K}]}{\mathcal{N}^M[\mathbb{K}]},$$

where

$$\mathcal{M}^M[\mathbb{K}] := \left\{ (z_\varepsilon)_\varepsilon \in \mathbb{K}^{]0,1]}, \exists k \in \mathbb{Z}_+, |z_\varepsilon| = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0 \right\}$$

and

$$\mathcal{N}^M[\mathbb{K}] := \left\{ (z_\varepsilon)_\varepsilon \in \mathbb{K}^{]0,1]}, \forall k \in \mathbb{Z}_+, |z_\varepsilon| = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0 \right\}.$$

Here $\mathbb{K}$ is the field $\mathbb{C}$ or $\mathbb{R}$.

Let us remark that it is easy to prove $\mathcal{M}^M[\mathbb{K}]$ is an algebra and $\mathcal{N}^M[\mathbb{K}]$ is an ideal of $\mathcal{M}^M[\mathbb{K}]$.

Lemma 4.2. *If the sequence M satisfies condition (H_2) then $\forall\ t_1, ..., t_n > 0, \forall n \in \mathbb{N}$, we have*

$$M(t_1) + ... + M(t_n) \leq M\left(H^{\frac{(n+2)(n-1)}{2n}} \max(t_1, ..., t_n)\right) + (n-1)\ln AM_0. \tag{10}$$

Proof. According to (H_2) and repeating this property, we obtain $\exists A > 0, \exists H > 0, \forall n \in \mathbb{N}$,

$$\begin{aligned} M_{np} &= M_{(n-1)p+p} \leq AH^{np} M_{(n-1)p} M_p \\ M_{np} &\leq A^2 H^{np} H^{(n-1)p} M_{(n-2)p} M_p^2 \\ &\ \ \vdots \\ M_{np} &\leq A^{n-1} H^{\frac{(n+2)(n-1)}{2}p} M_p^n \end{aligned}$$

So $\forall t > 0$

$$\frac{t^{np} M_0^n}{M_p^n} \leq A^{n-1} \left(H^{\frac{(n+2)(n-1)}{2n}}\right)^{np} M_0^n \frac{t^{np}}{M_{np}}$$

and then

$$\sup_p \ln \frac{t^{np} M_0^n}{M_p^n} \leq \sup_p \ln\left(\frac{t^{np} H^{\frac{(n+2)(n-1)}{2n} np} M_0}{M_{np}} A^{n-1} M_0^{n-1}\right),$$

hence

$$nM(t) \leq M\left(tH^{\frac{(n+2)(n-1)}{2n}}\right) + (n-1)\ln AM_0.$$

Let $t_1, ..., t_n > 0$, we have

$$nM(t_i) \leq M\left(t_i H^{\frac{(n+2)(n-1)}{2n}}\right) + (n-1)\ln AM_0, 1 \leq i \leq n,$$

hence

$$M(t_1) + ... + M(t_n) \leq M\left(H^{\frac{(n+2)(n-1)}{2n}} \max(t_1, t_2, ..., t_n)\right) + (n-1)\ln AM_0.$$

□

Lemma 4.3. *Let* $\widetilde{\lambda} \in \widetilde{\mathbb{K}}^M$, *then* $\left(\widetilde{\lambda}\right)^j \in \widetilde{\mathbb{K}}^M, \forall j \in \mathbb{N}$.

Proof. If $\widetilde{\lambda} = [(\lambda_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{K}}^M$ and $j \in \mathbb{N}$, as $(\lambda_\varepsilon)_\varepsilon \in \mathcal{M}^M[\mathbb{K}]$ we have $\exists k \in \mathbb{Z}_+, \exists c > 0, \exists \varepsilon_0 \in]0,1], \forall \varepsilon < \varepsilon_0, |\lambda_\varepsilon| \leq ce^{M\left(\frac{k}{\varepsilon}\right)}$, then $|\lambda_\varepsilon|^j \leq c^j e^{jM\left(\frac{k}{\varepsilon}\right)}$. Due to Lemma 4.2, for $t_1 = ... = t_n = \frac{k}{\varepsilon}$, we obtain

$$e^{jM\left(\frac{k}{\varepsilon}\right)} \leq (AM_0)^{(j-1)}\, e^{M\left(\frac{k}{\varepsilon}H^{\frac{(j+2)(j-1)}{2j}}\right)},$$

hence

$$|\lambda_\varepsilon|^j \leq c^j\,(AM_0)^{(j-1)}\, e^{M\left(\frac{k'}{\varepsilon}\right)},$$

with $k' = kH^{\frac{(j+2)(j-1)}{2j}}$, i.e. $(\lambda_\varepsilon)_\varepsilon^j \in \mathcal{M}^M[\mathbb{K}]$.

If $(\lambda_\varepsilon)_\varepsilon \in \mathcal{N}^M[\mathbb{K}]$, then $\forall k \in \mathbb{Z}_+, \exists c > 0, \exists \varepsilon_0 \in]0,1], \forall \varepsilon < \varepsilon_0, |\lambda_\varepsilon|^j \leq c^j e^{-jM\left(\frac{k}{\varepsilon}\right)}$. From (H_1) we have

$$M_{pj} \geq \frac{\left(M_p\right)^j}{M_0^{j-1}},$$

then

$$e^{-M\left(\frac{k}{\varepsilon}\right)} \geq \frac{e^{-jM\left(\frac{k}{\varepsilon}\right)}}{M_0^{j-1}},$$

so

$$c^j M_0^{j-1} e^{-M\left(\frac{k}{\varepsilon}\right)} \geq c^j e^{-jM\left(\frac{k}{\varepsilon}\right)} \geq |\lambda_\varepsilon|^j,$$

i.e. $\left(\widetilde{\lambda}\right)^j \in \widetilde{\mathbb{K}}^M$. □

Corollary 4.4. *The set of generalized numbers* $\widetilde{\mathbb{K}}^M$ *is an algebra.*

Example 4.5. *The generalized number* $\left[\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right)_\varepsilon\right] \in \widetilde{\mathbb{K}}^M, k > 0$.

Remark 4.6. *The algebra* $\widetilde{\mathbb{K}}^M$ *is not a field.*

5. Examples of almost periodic generalized ultradistributions

The algebra of almost periodic generalized functions of [3] is the quotient algebra

$$\mathcal{G}_{pp} := \frac{\mathcal{M}_{pp}}{\mathcal{N}_{pp}},$$

where

$$\mathcal{M}_{pp} = \left\{(f_\varepsilon)_{\varepsilon\in I} \in \left(\mathcal{B}_{pp}\right)^I : \forall j \in \mathbb{Z}_+, \exists k > 0, \left\|f_\varepsilon^{(j)}\right\|_\infty = O\left(\varepsilon^{-k}\right), \varepsilon \to 0\right\}$$

and

$$\mathcal{N}_{pp} = \left\{(f_\varepsilon)_{\varepsilon\in I} \in \left(\mathcal{B}_{pp}\right)^I : \forall j \in \mathbb{Z}_+, \forall k > 0, \left\|f_\varepsilon^{(j)}\right\|_\infty = O\left(\varepsilon^{k}\right), \varepsilon \to 0\right\}.$$

Proposition 5.1. *The algebra $\mathcal{G}_{pp}$ is embedded into $\mathcal{G}_{pp}^{M}$.*

Proof. Defining

$$\begin{array}{cccc} I: & \mathcal{G}_{pp} & \to & \mathcal{G}_{pp}^{M} \\ & [(u_\varepsilon)_\varepsilon] & \to & (u_\varepsilon)_\varepsilon + \mathcal{N}_{pp}^{M} \end{array},$$

to show that I is an embedding it suffices to prove that $\mathcal{M}_{pp} \subset \mathcal{M}_{pp}^{M}$ and $\mathcal{M}_{pp} \cap \mathcal{N}_{pp}^{M} \subset \mathcal{N}_{pp}$.
Due to Proposition 2.9-i),we have

$$M_k = \sup_{t>0} \frac{t^k M_0}{e^{M(t)}}, k \in \mathbb{Z}_+,$$

then

$$\forall p \in \mathbb{Z}_+, e^{M\left(\frac{k}{\varepsilon}\right)} \geq \frac{\left(\frac{k}{\varepsilon}\right)^p M_0}{M_p} \geq \left(\frac{k^p M_0}{M_p}\right) \varepsilon^{-p}, \tag{11}$$

Let $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}$, i.e. $\forall i \in \mathbb{Z}_+, \exists k > 0, \exists c_i > 0, \exists \varepsilon_i \in I, \forall \varepsilon \leq \varepsilon_i$,

$$\left\| f_\varepsilon^{(i)} \right\|_\infty \leq c_i \varepsilon^{-k},$$

in (11) , we take $p = k$, then

$$\left\| f_\varepsilon^{(i)} \right\|_\infty \leq c_i \varepsilon^{-k} \leq c_i \frac{M_k}{k^k M_0} e^{M\left(\frac{k}{\varepsilon}\right)},$$

i.e. $\forall i \in \mathbb{Z}_+, \exists k > 0, \exists C_i = c_i \frac{M_k}{k^k M_0} > 0, \exists \varepsilon_i \in I, \forall \varepsilon \leq \varepsilon_i$,

$$\left\| f_\varepsilon^{(i)} \right\|_\infty \leq C_i e^{\left(M\left(\frac{k_i}{\varepsilon}\right)\right)}$$

i.e. $(f_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^{M}$, which gives $\mathcal{M}_{pp} \subset \mathcal{M}_{pp}^{M}$.
Let $(f_\varepsilon)_\varepsilon \in \mathcal{N}_{pp}^{M}$, i.e. $\forall i \in \mathbb{Z}_+, \forall k > 0, \exists c_i' > 0, \exists \varepsilon_i' \in I, \forall \varepsilon \leq \varepsilon_i'$,

$$\left\| f_\varepsilon^{(i)} \right\|_\infty \leq c_i' e^{-M\left(\frac{k}{\varepsilon}\right)},$$

The estimate

$$\varepsilon^k \geq \left(\frac{k^k M_0}{M_k}\right) e^{-M\left(\frac{k}{\varepsilon}\right)}, \forall k \in \mathbb{Z}_+,$$

gives $\forall i \in \mathbb{Z}_+, \forall k > 0, \exists c_i' > 0, \exists \varepsilon_i' \in I, \forall \varepsilon \leq \varepsilon_i'$,

$$\left\| f_\varepsilon^{(i)} \right\|_\infty \leq c_i' \frac{M_k}{k^k M_0} \varepsilon^k$$

which gives $\mathcal{N}_{pp}^{M} \subset \mathcal{N}_{pp}$. Consequentely we have $\mathcal{M}_{pp} \cap \mathcal{N}_{pp}^{M} \subset \mathcal{N}_{pp}$ as $\mathcal{N}_{pp} \subset \mathcal{M}_{pp}^{M}$. □

A generalized trigonometric polynomial $\widetilde{P}$ is defined as

$$\widetilde{P}(x) := \sum_{k=1}^{m} \widetilde{c_k} e^{i \widetilde{\lambda}_k x}, x \in \mathbb{R},$$

where $\widetilde{c_k} \in \widetilde{\mathbb{C}}^M$ and $\widetilde{\lambda}_k \in \widetilde{\mathbb{R}}^M$.

Proposition 5.2. *Every generalized trigonometric polynomial is an almost periodic generalized ultradistribution.*

Proof. As for every $j \in \mathbb{N}$ we have $\widetilde{P}^{(j)}(x) = \sum_{k=1}^{m} \left(i\widetilde{\lambda}\right)^j \widetilde{c_k} e_{\varepsilon}^{i\widetilde{\lambda}_k x}$ then the result is obtained from the fact that $\widetilde{\mathbb{K}}^M$ is an algebra. □

We now show that as important examples of almost periodic generalized ultradistributions are almost periodic Beurling ultradistributions. Let us give the definition and the fundamental result of [6].

Definition 5.3. *A bounded ultradistribution T is an almost periodic Beurling ultradistribution if it satisfies one of the following equivalent statements :*
i) $T * \varphi \in C_{pp}, \forall \varphi \in \mathcal{D}^{(M)}$.
ii) There exist $P(D)$ an $(M)-$ultradifferential operator and $f, g \in C_{pp}$ such that $T = P(D) f + g$.
The space of almost periodic Beurling ultradistributions is denoted by $B'_{pp,(M)}$.

Recall that an $(M)-$ultradifferential operator $P(D) := \sum_j a_j D^j$ is a differential operator of infinite order satisfying $\exists h > 0\ \exists c > 0, \forall j \in \mathbb{Z}_+$,

$$\left|a_j\right| \leq c\frac{h^j}{M_j}. \tag{12}$$

If $N = \left(N_p\right)_{p=0}^{\infty}$ is a sequence of positive real numbers, the space $\mathcal{D}_{L^1}^{\{N\}}$ defined by the inductive limite topology of Banach spaces

$$\mathcal{D}_{L^1}^{\{N\}} := \lim_{h\to 0} ind \mathcal{D}_{L^1}^{(N),h},$$

where the Banach space $\mathcal{D}_{L^1}^{(N),h}$ is already defined in (3).
For $\varepsilon > 0$ and $\varphi \in \mathcal{D}_{L^1}^{\{N\}}$ define $\varphi_\varepsilon(.) := \frac{1}{\varepsilon}\varphi\left(\frac{\cdot}{\varepsilon}\right)$.

Theorem 5.4. *Let M, N be two sequence satisfying $(H_1), (H_2)$ and $\left(H_3'\right)$, the map*

$$\begin{array}{cccc} J: & B'_{pp,(MN)} & \to & \mathcal{G}_{pp}^{M} \\ & T & \mapsto & T = \left[\left(T * \varphi_\varepsilon\right)_\varepsilon\right] \end{array}$$

is a linear embedding.

Proof. Let $T \in B'_{pp,(MN)}$, according to Definition 5.3, there exist $P(D) = \sum_j a_j D^j$ an (MN)-ultradifferential operator and $f, g \in C_{pp}$ such that $T = P(D) f + g$. Moreover, we have $\forall\ \gamma \in \mathbb{Z}_+$,

$$(T * \varphi_\varepsilon)^{(\gamma)} = P(D)\left(f * \varphi_\varepsilon^{(\gamma)}\right) + g * \varphi_\varepsilon^{(\gamma)},$$

from (12) and the fact that $\varphi \in \mathcal{D}_{L^1}^{\{N\}}$, i.e. $\exists b > 0$ such that $\left\|\varphi\right\|_{b,N} < \infty$, we have

$$\begin{aligned} \left|(T * \varphi_\varepsilon)^{(\gamma)}(x)\right| &\leq \left|\left(P(D) f * \varphi_\varepsilon^{(\gamma)}\right)(x)\right| + \left|\left(g * \varphi_\varepsilon^{(\gamma)}\right)(x)\right|, \\ &\leq c\left\|f\right\|_\infty \sum_j \frac{h^j}{M_j N_j} \frac{1}{\varepsilon^{j+\gamma}} b^{j+\gamma} \int \frac{\left|\varphi^{(j+\gamma)}(y)\right|}{b^{j+\gamma}} dy + \frac{1}{\varepsilon^\gamma} \left\|g\right\|_\infty \int \left|\varphi^{(\gamma)}(y)\right| dy. \end{aligned}$$

Due to (H_2) : $\exists A, A' > 0, \exists H, H' > 0, N_{j+\gamma} \leq AH^{j+\gamma} N_j N_\gamma$ and $M_{j+\gamma} \leq A'H'^{j+\gamma} M_j M_\gamma$,

$$\begin{aligned}\left|\partial^\gamma (T * \varphi_\varepsilon)(x)\right| &\leq cAA' \left\|f\right\|_\infty \sum_j \frac{b^{j+\gamma} h^j (HH')^{j+\gamma}}{M_{j+\gamma}} \frac{1}{\varepsilon^{j+\gamma}} N_\gamma M_\gamma \int \frac{\left|\partial^{j+\gamma}\varphi(y)\right|}{b^{j+\gamma} N_{j+\gamma}} dy + \\ &+ \frac{1}{\varepsilon^\gamma} \left\|g\right\|_\infty \int \left|\partial^\gamma \varphi(y)\right| dy,\end{aligned}$$

i.e. we have

$$\begin{aligned}\frac{(2h)^\gamma}{N_\gamma M_\gamma}\left|\partial^\gamma (T * \varphi_\varepsilon)(x)\right| &\leq cAA' \left\|f\right\|_\infty \sum_j \frac{b^{j+\gamma} h^j (HH')^{j+\gamma} (2h)^\gamma}{M_{j+\gamma}} \frac{1}{\varepsilon^{j+\gamma}} \int \frac{\left|\partial^{j+\gamma}\varphi(y)\right|}{b^{j+\gamma} N_{j+\gamma}} dy + \\ &+ \frac{1}{\varepsilon^\gamma}\frac{(2h)^\gamma}{N_\gamma M_\gamma} b^\gamma \left\|g\right\|_\infty \int \frac{\left|\partial^\gamma \varphi(y)\right|}{b^\gamma} dy, \\ &\leq cAA' \left\|f\right\|_\infty \sum_j 2^{-j} \frac{(2hbHH')^{j+\gamma}}{M_{j+\gamma}} \frac{1}{\varepsilon^{j+\gamma}} \int \frac{\left|\partial^{j+\gamma}\varphi(y)\right|}{b^{j+\gamma} N_{j+\gamma}} dy + \\ &+ \frac{1}{\varepsilon^\gamma}\frac{(2h)^\gamma}{M_\gamma} b^\gamma \left\|g\right\|_\infty \int \frac{\left|\partial^\gamma \varphi(y)\right|}{b^\gamma N_\gamma} dy, \\ &\leq cAA' \left\|f\right\|_\infty \sum_j 2^{-j} \frac{\left(\frac{2hbHH'}{\varepsilon}\right)^{j+\gamma}}{M_{j+\gamma}} \left\|\varphi\right\|_{1,b,N} + \left\|g\right\|_\infty \left\|\varphi\right\|_{1,b,N} \frac{\left(\frac{2hb}{\varepsilon}\right)^\gamma}{M_\gamma},\end{aligned}$$

according to Proposition 2.9, we obtain

$$\begin{aligned}\frac{(2h)^\gamma}{N_\gamma M_\gamma}\left|\partial^\gamma (T * \varphi_\varepsilon)(x)\right| &\leq cAA' \left\|f\right\|_\infty \left\|\varphi\right\|_{1,b,N} \sum_j 2^{-j} e^{M\left(\frac{2hbHH'}{\varepsilon}\right)} + \left\|\varphi\right\|_{1,b,N} \left\|g\right\|_\infty e^{M\left(\frac{2hb}{\varepsilon}\right)}, \\ &\leq 2cAA' \left\|f\right\|_\infty \left\|\varphi\right\|_{1,b,N} e^{M\left(\frac{2hbHH'}{\varepsilon}\right)} + \left\|\varphi\right\|_{1,b,N} \left\|g\right\|_\infty e^{M\left(\frac{2hb}{\varepsilon}\right)}, \\ &\leq Ce^{M\left(\frac{k}{\varepsilon}\right)},\end{aligned}$$

where $k = H \max(2hbHH', 2hb)$ and $C = \max\left(2cAA' \left\|f\right\|_\infty \left\|\varphi\right\|_{1,b,N}, \left\|\varphi\right\|_{1,b,N} \left\|g\right\|_\infty\right)$, i.e. we proved

$$\exists k > 0, \exists C > 0, \forall \gamma \in \mathbb{Z}_+, \left|\partial^\gamma (T * \varphi_\varepsilon)(x)\right| \leq C \frac{N_\gamma M_\gamma}{(2h)^\gamma} e^{M\left(\frac{k}{\varepsilon}\right)}, \tag{13}$$

which gives $(T * \varphi_\varepsilon)_\varepsilon \in \mathcal{M}^M_{pp}$.

Let $\varphi \in \mathcal{D}^{\{N\}}_{L^1}$ and $\int_{\mathbb{R}} \varphi(x)\, dx = 1$. If $(T * \varphi_\varepsilon)_\varepsilon \in \mathcal{N}^M_{pp}$, then $\forall \varepsilon > 0$, $T * \varphi_\varepsilon \in B_{pp}$, and $\forall k > 0, \exists c > 0, \exists \varepsilon_0 \in]0, 1], \forall \varepsilon \leq \varepsilon_0$,

$$\left\|T * \varphi_\varepsilon\right\|_\infty \leq ce^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{14}$$

Let $\psi \in D^{(MN)}_{L^1}$, we have $\langle T, \psi\rangle = \lim_{\varepsilon\to 0} \int (T * \varphi_\varepsilon)(x)\,\psi(x)\, dx$, from (14), we obtain $\exists c' > 0, \forall \varepsilon \leq \varepsilon_0$,

$$\left|\int (T * \varphi_\varepsilon)(x)\,\psi(x)\, dx\right| \leq c' e^{-M\left(\frac{k}{\varepsilon}\right)},$$

the limit $\varepsilon \to 0$, gives $\langle T, \psi\rangle = 0, \forall \psi \in D^{(MN)}_{L^1}$. Hence J is injective. □

Remark 5.5. *The Theorem indicates explicitly the contribution of the mollifier φ when embedding the space of almost periodic ultradistributions into the algebra $\mathcal{G}^M_{pp}$. However, the authors of [7] propose a way to avoid the contribution of the mollifier by imposing an inevitable condition between the ultradifferentiability order (N) of the mollifier φ and the ultradifferentiability order (M) of the embedded ultradistributions, but this is valid only in the local case .*

6. A nonlinear operation

The following algebra of tempered generalized ultradistributions defined on $\mathbb{C}$ and denoted by $\mathcal{G}_{\mathcal{T}}^{M}(\mathbb{C})$ plays the same role as the algebra of tempered generalized functions, see [11] for more details, its definition is the quotient algebra

$$\mathcal{G}_{\mathcal{T}}^{M}(\mathbb{C}) := \frac{\mathcal{M}_{\mathcal{T}}^{M}(\mathbb{C})}{\mathcal{N}_{\mathcal{T}}^{M}(\mathbb{C})},$$

where the space of tempered moderate elements is defined by

$$\mathcal{M}_{\mathcal{T}}^{M} := \left\{ (f_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}^I : \forall j \in \mathbb{Z}_+, \exists k > 0, \left\| (1+|.|)^{-k} f_\varepsilon^{(j)}(.) \right\|_\infty = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0 \right\}$$

and the space of tempered null elements is

$$\mathcal{N}_{\mathcal{T}}^{M} := \left\{ (f_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}^I : \forall j \in \mathbb{Z}_+, \exists p > 0, \forall k > 0, \left\| (1+|.|)^{-p} f_\varepsilon^{(j)}(.) \right\|_\infty = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right), \varepsilon \to 0 \right\}.$$

Proposition 6.1. *Let* $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{pp}^{M}$ *and* $F = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathcal{T}}^{M}(\mathbb{C})$ *then*

$$F \circ u := [(f_\varepsilon \circ u_\varepsilon)_\varepsilon]$$

is a well-defined element of $\mathcal{G}_{pp}^{M}$.

Proof. Let $(u_\varepsilon)_\varepsilon, (f_\varepsilon)_\varepsilon$ be representatives of u and F respectively, then

$$\forall r \in \mathbb{Z}_+, \exists k_r > 0, \exists c_r > 0, \exists \varepsilon_r \in I, \forall \varepsilon \le \varepsilon_r, \left\| u_\varepsilon^{(r)} \right\|_\infty \le c_r e^{M\left(\frac{k_r}{\varepsilon}\right)}, \tag{15}$$

$$\forall j \in \mathbb{Z}_+, \exists k_j > 0, \exists c_j > 0, \exists \varepsilon_j \in I, \forall \varepsilon \le \varepsilon_j, \left\| (1+|.|)^{-k_j} f_\varepsilon^{(j)}(.) \right\|_\infty \le c_j e^{M\left(\frac{k_j}{\varepsilon}\right)} \tag{16}$$

Let us show that $(f_\varepsilon \circ u_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^{M}$. Due to the classical formula of Faà di Bruno, we have

$$(f_\varepsilon \circ u_\varepsilon)^{(n)} = \sum_{1\le r\le n} \frac{n!}{r!} f_\varepsilon^{(r)}(u_\varepsilon) \prod_{\substack{n_1+n_2+\ldots+n_r=n \\ n_j\ge 1}} \frac{u_\varepsilon^{(n_j)}}{n_j!}$$

and from Lemma 4.2, (15) and (16) , we have

$$
\begin{aligned}
e^{-M\left(\frac{k}{\varepsilon}\right)}\left\|\left(f_\varepsilon\circ u_\varepsilon\right)^{(n)}\right\|_\infty &\leq e^{-M\left(\frac{k}{\varepsilon}\right)}\sum_{1\leq r\leq n}\frac{n!}{r!}c_r\left\|1+u_\varepsilon\right\|_\infty^{k(r)}e^{M\left(\frac{k(r)}{\varepsilon}\right)}\prod_{\substack{n_1+n_2+\ldots+n_r=n\\ n_j\geq 1}}\frac{1}{n_j!}c_je^{M\left(\frac{k_j}{\varepsilon}\right)}\\
&\leq e^{-M\left(\frac{k}{\varepsilon}\right)}\sum_{1\leq r\leq n}\frac{n!}{r!}c_r\left(1+ce^{M\left(\frac{k'}{\varepsilon}\right)}\right)^{k(r)}e^{M\left(\frac{k(r)}{\varepsilon}\right)}\times\\
&\quad\times\prod_{\substack{n_1+n_2+\ldots+n_r=n\\ n_j\geq 1}}\frac{1}{n_j!}c_je^{M\left(\frac{k_j}{\varepsilon}\right)}\\
&\leq e^{-M\left(\frac{k}{\varepsilon}\right)}\sum_{1\leq r\leq n}\frac{n!}{r!}c_r\left(1+c\right)^{k(r)}e^{([k(r)]+1)M\left(\frac{k'}{\varepsilon}\right)}e^{M\left(\frac{k(r)}{\varepsilon}\right)}\times\\
&\quad\times\prod_{\substack{n_1+n_2+\ldots+n_r=n\\ n_j\geq 1}}\frac{1}{n_j!}c_je^{M\left(\frac{k_j}{\varepsilon}\right)}\\
&\leq e^{-M\left(\frac{k}{\varepsilon}\right)}\sum_{1\leq r\leq n}\frac{n!}{r!}c_r\left(1+c\right)^{k(r)}\left(AM_0\right)^{m-1}e^{M\left(\frac{k'}{\epsilon}H^{\frac{(m+2)(m-1)}{2m}}\right)}e^{M\left(\frac{k(r)}{\varepsilon}\right)}\\
&\quad\times\prod_{\substack{n_1+n_2+\ldots+n_r=n\\ n_j\geq 1}}\frac{1}{n_j!}c_je^{M\left(\frac{k_j}{\varepsilon}\right)},
\end{aligned}
$$

where $m=[k(r)]+1$. According to Lemma 4.2 , for $l=H\max\left(k(r),k'H^{\frac{(m+2)(m-1)}{2m}}\right)$, $\varepsilon\leq\min\left(\varepsilon_1,\varepsilon_2\right)$, we obtain

$$
e^{M\left(\frac{k(r)}{\varepsilon}\right)}e^{M\left(\frac{k'}{\varepsilon}H^{\frac{(m+2)(m-1)}{2m}}\right)}=e^{M\left(\frac{k(r)}{\varepsilon}\right)+M\left(\frac{k'}{\varepsilon}H^{\frac{(m+2)(m-1)}{2m}}\right)}\leq AM_0e^{M\left(\frac{l}{\varepsilon}\right)}.
$$

Consequently

$$
\begin{aligned}
e^{-M\left(\frac{k}{\varepsilon}\right)}\left\|\left(f_\varepsilon\circ u_\varepsilon\right)^{(n)}\right\|_\infty &\leq e^{-M\left(\frac{k}{\varepsilon}\right)}\sum_{1\leq r\leq n}\frac{n!}{r!}c_r\left(1+c\right)^{k(r)}\left(AM_0\right)^{m-1}AM_0e^{M\left(\frac{l}{\varepsilon}\right)}e^{nM\left(\frac{l'}{\varepsilon}\right)}\times\\
&\quad\times\prod_{\substack{n_1+n_2+\ldots+n_r=n\\ n_j\geq 1}}\frac{1}{n_j!}c_j\\
&\leq e^{-M\left(\frac{k}{\varepsilon}\right)}\sum_{1\leq r\leq n}\frac{n!}{r!}c_r\left(1+c\right)^{k(r)}\left(AM_0\right)^{m-1}AM_0e^{M\left(\frac{l}{\varepsilon}\right)}\left(AM_0\right)^{n-1}\times\\
&\quad\times e^{M\left(\frac{l'}{\epsilon}H^{\frac{(n+2)(n-1)}{2n}}\right)}\prod_{\substack{n_1+n_2+\ldots+n_r=n\\ n_j\geq 1}}\frac{1}{n_j!}c_j.
\end{aligned}
$$

For $k=H\max\left(l,l'H^{\frac{(n+2)(n-1)}{2n}}\right)$, $\varepsilon\leq\min\left(\varepsilon_1,\varepsilon_2\right)$, we have

$$
e^{M\left(\frac{l}{\varepsilon}\right)}e^{M\left(\frac{l''}{\varepsilon}H^{\frac{(m+2)(m-1)}{2m}}\right)}\leq AM_0e^{M\left(\frac{k}{\varepsilon}\right)},
$$

hence

$$\begin{aligned} e^{-M\left(\frac{k}{\varepsilon}\right)}\left\|\left(f_\varepsilon\circ u_\varepsilon\right)^{(n)}\right\|_\infty &\le \sum_{1\le r\le n}\frac{n!}{r!}c_r\left(1+c\right)^{k(r)}A^{k(r)}\left(AM_0\right)^m\left(AM_0\right)^n\times\\ &\times\prod_{\substack{n_1+n_2+\ldots+n_r=n\\ n_j\ge1}}\frac{1}{n_j!}c_jA^{n(r)}\\ &\le C_n \end{aligned}$$

Therefore, $\forall n\in\mathbb{Z}_+,\exists k>0,\exists C=C'_n>0,\exists\varepsilon_0\in I,\forall\varepsilon\le\varepsilon_0$

$$\left\|\left(f_\varepsilon\circ u_\varepsilon\right)^{(n)}\right\|_\infty\le C_ne^{M\left(\frac{k}{\varepsilon}\right)},$$

i.e $\left(f_\varepsilon\circ u_\varepsilon\right)_\varepsilon\in\mathcal{M}_{pp}^M$.

To show that $F\circ u$ is well defined, we first show

$$\left(f_\varepsilon\left(u_\varepsilon+n_\varepsilon\right)-f_\varepsilon\left(u_\varepsilon\right)\right)_\varepsilon\in\mathcal{N}_{pp}^M,\ \forall\ \left(n_\varepsilon\right)_\varepsilon\in\mathcal{N}_{pp}^M.$$

According to Proposition 6, it suffices to show $\forall k>0,\exists C>0,$

$$\left\|f_\varepsilon\left(u_\varepsilon+n_\varepsilon\right)-f_\varepsilon\left(u_\varepsilon\right)\right\|_\infty\le C\exp\left(-M\left(\frac{k}{\varepsilon}\right)\right).$$

Taylor's formula and Lemma 4.2 give

$$\begin{aligned} e^{M\left(\frac{k}{\varepsilon}\right)}\left|f_\varepsilon\left(\left(u_\varepsilon+n_\varepsilon\right)(x)\right)-f_\varepsilon\left(u_\varepsilon(x)\right)\right| &\le e^{M\left(\frac{k}{\varepsilon}\right)}\left|n_\varepsilon(x)\right|\int_0^1\left|f'_\varepsilon\left(\theta u_\varepsilon(x)\right)\right|d\theta,\\ &\le ce^{M\left(\frac{k}{\varepsilon}\right)}e^{-M\left(\frac{k_2}{\varepsilon}\right)}c_1\left(1+\left|\theta u_\varepsilon(x)\right|\right)^{k_1}e^{M\left(\frac{k_1}{\varepsilon}\right)}\\ &\le ce^{M\left(\frac{k}{\varepsilon}\right)}e^{-M\left(\frac{k_2}{\varepsilon}\right)}c_1\left(1+c\right)^{k_1}e^{k_1M\left(\frac{k'_1}{\varepsilon}\right)}e^{M\left(\frac{k_1}{\varepsilon}\right)}\\ &\le ce^{M\left(\frac{k}{\varepsilon}\right)}e^{-M\left(\frac{k_2}{\varepsilon}\right)}c_1\left(1+c\right)^{k_1}\left(AM_0\right)^{m'-1}\times\\ &\times e^{M\left(\frac{k_1}{\varepsilon}\right)}e^{M\left(\frac{k_1}{\varepsilon}H^{\frac{(m'-1)(m'+2)}{2m'}}\right)}, \end{aligned}$$

where $m'=[k_1]+1$.

Due to Lemma 4.2, let $k_2=H\max\left(k_1H^{\frac{(m'-1)(m'+2)}{2m'}},k_1,k\right)$, $\varepsilon\le\min\left(\varepsilon_1,\varepsilon_2,\varepsilon_3\right)$, then

$$e^{M\left(\frac{k}{\varepsilon}\right)}e^{M\left(\frac{k_1}{\varepsilon}H^{\frac{(m'-1)(m'+2)}{2m'}}\right)}e^{M\left(\frac{k_1}{\varepsilon}\right)}\le\left(AM_0\right)^2e^{M\left(\frac{k_2}{\varepsilon}\right)},$$

hence

$$e^{M\left(\frac{k}{\varepsilon}\right)}\left|f_\varepsilon\left(\left(u_\varepsilon+n_\varepsilon\right)(x)\right)-f_\varepsilon\left(u_\varepsilon(x)\right)\right|\le c_1\left(1+c\right)^{k_1}\left(AM_0\right)^{m'+1}=C$$

i.e. we obtain $\forall k>0,\exists C>0,\exists\varepsilon_0\in I,\forall\varepsilon\le\varepsilon_0,$

$$\left|f_\varepsilon\left(\left(u_\varepsilon+n_\varepsilon\right)(x)\right)-f_\varepsilon\left(u_\varepsilon(x)\right)\right|\le Ce^{-M\left(\frac{k}{\varepsilon}\right)}$$

Now let $(g_\varepsilon)_\varepsilon$ be another representative of F, so $(f_\varepsilon - g_\varepsilon)_\varepsilon = (h_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{T}}^M(\mathbb{C})$, and let show that $(f_\varepsilon - g_\varepsilon)(u_\varepsilon) = h_\varepsilon(u_\varepsilon) \in \mathcal{N}_{pp}^M$, $\forall (u_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^M$. Indeed according to Lemma 4.2,

$$\begin{aligned}
e^{M\left(\frac{k}{\varepsilon}\right)}|h_\varepsilon(u_\varepsilon)| &= e^{M\left(\frac{k}{\varepsilon}\right)}\left|h_\varepsilon(0) + u_\varepsilon(x)\int_0^1 h'_\varepsilon(\theta u_\varepsilon(x))\,d\theta\right| \\
&\le e^{M\left(\frac{k}{\varepsilon}\right)}\left(c_1 e^{-M\left(\frac{k_1}{\varepsilon}\right)} + c e^{M\left(\frac{k_0}{\varepsilon}\right)}(1+|\theta u_\varepsilon(x)|)^p e^{-M\left(\frac{k_1}{\varepsilon}\right)}\right) \\
&\le e^{M\left(\frac{k}{\varepsilon}\right)}\left(c_1 e^{-M\left(\frac{k_1}{\varepsilon}\right)} + c e^{M\left(\frac{k_0}{\varepsilon}\right)}(1+c_0)^p e^{pM\left(\frac{k''}{\varepsilon}\right)} e^{-M\left(\frac{k_1}{\varepsilon}\right)}\right) \\
&\le (c_1 + c(1+c_0)^p)\, e^{M\left(\frac{k}{\varepsilon}\right)} e^{-M\left(\frac{k_1}{\varepsilon}\right)} e^{M\left(\frac{k_0}{\varepsilon}\right)} e^{pM\left(\frac{k''}{\varepsilon}\right)} \\
&\le (c_1 + c(1+c)^p)\, e^{M\left(\frac{k}{\varepsilon}\right)} e^{-M\left(\frac{k_1}{\varepsilon}\right)} e^{M\left(\frac{k_0}{\varepsilon}\right)} (AM_0)^{p-1} e^{M\left(\frac{p}{\varepsilon}H^{\frac{(p-1)(p+2)}{2p}}\right)} \\
&\le (c_1 + c(1+c)^p)\,(AM_0)^{p-1} e^{M\left(\frac{k}{\varepsilon}\right)} e^{-M\left(\frac{k_1}{\varepsilon}\right)} e^{M\left(\frac{k_0}{\varepsilon}\right)} e^{M\left(\frac{p}{\varepsilon}H^{\frac{(p-1)(p+2)}{2p}}\right)}
\end{aligned}$$

The same reasoning as above, for $k_1 = H\max\left(k, k_0, pH^{\frac{(p-1)(p+2)}{2p}}\right)$, $\varepsilon \le \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, we get

$$e^{M\left(\frac{k}{\varepsilon}\right)}|h_\varepsilon(u_\varepsilon)| \le C$$

hence the result. □

7. More properties of $\mathcal{G}_{pp}^M$

The notion of point value is extended to the algebra $\mathcal{G}_{pp}^M$.

Proposition 7.1. *i) Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{pp}^M$ and $\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}^M$, the point value of $\widetilde{u}$ at $\widetilde{x}$ given by $\widetilde{u}(\widetilde{x}) := [(u_\varepsilon(x_\varepsilon))_\varepsilon]$ is a well- defined element of $\widetilde{\mathbb{C}}^M$.*

ii) If $\widetilde{u} \in \mathcal{G}_{pp}^M$, then

$$\widetilde{u} = \widetilde{0} \text{ in } \mathcal{G}_{pp}^M \Leftrightarrow \widetilde{u}(\widetilde{x}) = \widetilde{0} \text{ in } \widetilde{\mathbb{C}}^M, \forall \widetilde{x} \in \widetilde{\mathbb{R}}^M.$$

Proof. i) Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{pp}^M$ and $\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}^M$, it follows easily that $(u_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{M}^M[\mathbb{C}]$. Let $(y_\varepsilon)_\varepsilon$ another representative of $\widetilde{x}$, using Taylor's formula we have

$$\left|u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)\right| \le \left|x_\varepsilon - y_\varepsilon\right| \int_0^1 \left|u'_\varepsilon(x_\varepsilon + t(y_\varepsilon - x_\varepsilon))\right| dt,$$

and since $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{pp}^M$, then $\exists k \in \mathbb{Z}_+, \exists c > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \le \varepsilon_0$,

$$\left|u'_\varepsilon(x_\varepsilon + t(y_\varepsilon - x_\varepsilon))\right| \le c e^{M\left(\frac{k}{\varepsilon}\right)}.$$

On the other hand, $\forall k' \in \mathbb{Z}_+, \exists c' > 0$ such that $\left|y_\varepsilon - x_\varepsilon\right| \le c' e^{-M\left(\frac{k'}{\varepsilon}\right)}$, and therefore, taking $c'' = \max(c, c')$, we have

$$\left|u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)\right| \le c'' e^{M\left(\frac{k}{\varepsilon}\right)} e^{-M\left(\frac{k'}{\varepsilon}\right)},$$

From Lemma 4.2 with $t_1 = \dfrac{k}{\varepsilon}$, $t_2 = \frac{k''}{\varepsilon}$ and $k' = H \max(k, k'')$, we obtain

$$\left|u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)\right| \le c'' A M_0 e^{-M\left(\frac{k''}{\varepsilon}\right)},$$

which shows that $u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon) \in \mathcal{N}^M[\mathbb{C}]$.

ii) If $\widetilde{u} = \widetilde{0}$ in $\mathcal{G}^M_{pp}$, it's obvious that $\widetilde{u}(\widetilde{x}) = \widetilde{0}$ in $\widetilde{\mathbb{C}}^M$. Conversely, suppose that $\widetilde{u} \neq 0$ in $\mathcal{G}^M_{pp}$, in view of Proposition 3.2, we have

$$\exists k > 0, \forall c > 0, \forall \varepsilon_0 \in I, \exists \varepsilon \le \varepsilon_0, |u_\varepsilon(x_\varepsilon)| > c e^{-M\left(\frac{k}{\varepsilon}\right)}, \tag{17}$$

and since $\widetilde{u}(\widetilde{x}) = 0$, then $u_\varepsilon(x_\varepsilon)$ satisfies (7) for $i = 0$, i.e. $\forall k > 0, \exists c_0 > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \le \varepsilon_0$,

$$|u_\varepsilon(x_\varepsilon)| \le c_0 e^{-M\left(\frac{k}{\varepsilon}\right)},$$

which gives is a contradiction with (17). □

Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{pp}$, the generalized mean value of $\widetilde{u}$, denoted $M_g(\widetilde{u})$, is defined by

$$M_g(\widetilde{u}) := \left(\lim_{X\to+\infty} \frac{1}{X} \int_0^X u_\varepsilon(y)\,dy\right)_\varepsilon + \mathcal{N}^M(\mathbb{C}).$$

Proposition 7.2. *If $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{pp}$ then $M_g(\widetilde{u})$ is a well-defined element of $\widetilde{\mathbb{C}}^M$.*

Proof. Let $(u_\varepsilon)_\varepsilon$ a representative of $\widetilde{u}$, we have

$$\left|\lim_{X\to+\infty} \frac{1}{X} \int_0^X u_\varepsilon(y)\,dy\right| \le \lim_{X\to+\infty} \frac{1}{X} \int_0^X \left|u_\varepsilon(y)\right| dy \le c e^{M\left(\frac{k}{\varepsilon}\right)},$$

hence $\left(\lim_{X\to+\infty} \frac{1}{X} \int_0^X u_\varepsilon(y)\,dy\right)_\varepsilon \in \mathcal{M}^M[\mathbb{C}]$. If $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{pp}$, then $(u_\varepsilon)_\varepsilon$ satisfies (7), consequently

$$\left|\lim_{X\to+\infty} \frac{1}{X} \int_0^X u_\varepsilon(y)\,dy\right| \le c e^{-M\left(\frac{k}{\varepsilon}\right)},$$

i.e. $\left(\lim_{X\to\infty} \frac{1}{X} \int_0^X u_\varepsilon(y)\,dy\right)_\varepsilon \in \mathcal{N}^M[\mathbb{C}]$. □

Let $\widetilde{u} \in \mathcal{G}^M_{pp}$ and $\widetilde{\lambda} \in \widetilde{\mathbb{R}}^M$, then $\widetilde{u}e^{-i\widetilde{\lambda}x} \in \mathcal{G}^M_{pp}$, consequently the generalized mean value $M_g\left(\widetilde{u}e^{-i\widetilde{\lambda}x}\right)$ of $\widetilde{u}$ is a well-defined element of $\widetilde{\mathbb{C}}^M$. We define the generalized spectra of $\widetilde{u}$ as the set of generalized numbers defined by

$$\Lambda_g(\widetilde{u}) := \left\{\widetilde{\lambda} \in \widetilde{\mathbb{R}}^M : a_{\widetilde{\lambda}}(\widetilde{u}) \neq 0 \text{ in } \widetilde{\mathbb{C}}^M\right\}, \tag{18}$$

where

$$a_{\widetilde{\lambda}}(\widetilde{u}) := M_g\left(\widetilde{u}e^{-i\widetilde{\lambda}x}\right).$$

Example 7.3. *Let* $\widetilde{P}(.) = \sum_{k=1}^{m} \widetilde{c_k} e^{i\widetilde{\lambda_k}\cdot}$ *then*

$$\Lambda_g\left(\widetilde{P}\right) = \left\{\widetilde{\lambda_k} : k = 1, ...m\right\}.$$

Remark 7.4. *The generalized spectra of an almost periodic generalized ultradistribution* $\Lambda_g(\widetilde{u})$ *is not necessarily countable, see [3].*

In the same way as almost periodic distributions, see [13], the mean value of $T \in B'_{pp,(M)}$ is defined as the value

$$M(T) := M(T * \rho) = \lim_{X\to\infty} \frac{1}{X} \int_0^X (T * \rho)(y)\, dy,$$

where $\rho \in \mathcal{D}^{(M)}$ and $\int_{\mathbb{R}} \rho(x)\, dx = 1$.

Proposition 7.5. *If* $T \in B'_{pp,MN}$ *then* $M_g(J(T)) = M(T)$ *in* $\widetilde{\mathbb{C}}^M$.

Proof. We have for $\rho \in \mathcal{D}^{(MN)}$ and $\int_{-\infty}^{+\infty} \rho(x)\, dx = 1$,

$$M_g(J(T)) - M(T) = \left[\left(\lim_{X\to\infty} \frac{1}{X} \int_0^X T * (\varphi_\varepsilon - \rho)(y)\, dy\right)_\varepsilon\right], \varepsilon > 0,$$

where $\varphi \in \mathcal{D}^{\{N\}}_{L^1}$ is such that $\int_{\mathbb{R}} \varphi(x)\, dx = 1$. Since $T \in B'_{pp,(MN)}$, there exist $P(D) = \sum_j a_j D^j$ an $(MN)-$ultradifferential operator and $f, g \in C_{pp}$ such that $T = P(D) f + g$, hence

$$\begin{aligned} T * (\varphi_\varepsilon - \rho)(y) &= P(D) f * (\varphi_\varepsilon - \rho)(y) + g * (\varphi_\varepsilon - \rho)(y), \\ &= a_0 f * (\varphi_\varepsilon - \rho)(y) + \sum_{j\geq 1} a_j f * D^j (\varphi_\varepsilon - \rho)(y) + g * (\varphi_\varepsilon - \rho)(y), \end{aligned}$$

By the Proposition 2.4-vi), we obtain

$$\begin{aligned} \lim_{X\to\infty} \frac{1}{X} \int_0^X T * (\varphi_\varepsilon - \rho)(y)\, dy &= a_0 M(f) \int_{\mathbb{R}} (\varphi_\varepsilon - \rho)(x)\, dx + \\ &\quad + \sum_{j\geq 1} a_j M(f) \int_{\mathbb{R}} D^j (\varphi_\varepsilon - \rho)(x)\, dx + \\ &\quad + M(g) \int_{\mathbb{R}} (\varphi_\varepsilon - \rho)(x)\, dx = 0, \end{aligned}$$

as $\forall \varepsilon \in I, \int_{\mathbb{R}} \varphi_\varepsilon(x)\, dx = \int_{\mathbb{R}} \rho(x)\, dx = 1$ and $D^{j-1}(\varphi_\varepsilon - \rho)\big|_{-\infty}^{+\infty} = 0$. So the result is obtained. □

A primitive $\widetilde{U}$ of $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{pp}$ is defined by $\widetilde{U}(x) = \left(\int_{x_0}^{x} u_\varepsilon(y)\, dy\right)_\varepsilon + \mathcal{N}(\mathbb{C}), x_0 \in \mathbb{R}$. This definition does not depend on the representative. We have a generalized version of the classical Bohl-Bohr's result.

Proposition 7.6. *A primitive of an almost periodic generalized ultradistribution is almost periodic if and only if it is a bounded generalized ultradistribution.*

Proof. Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{pp}$ and suppose that $\widetilde{U} = [(U_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{L^\infty}$, i.e. $U_\varepsilon(x) := \int_{x_0}^{x} u_\varepsilon(y)\,dy \in \mathcal{B}, \forall \varepsilon > 0$, then U_ε is a bounded primitive of the almost periodic function u_ε according to the classical Bohl-Bohr's Theorem $U_\varepsilon \in \mathcal{B}_{pp}$. In addition $(U_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^\infty}$, i.e. $\forall j \in \mathbb{Z}_+, \exists k > 0, \exists c > 0, \left\| U_\varepsilon^{(j)} \right\|_{L^\infty} \leq ce^{M\left(\frac{k}{\varepsilon}\right)}$, hence $(U_\varepsilon)_\varepsilon \in \mathcal{M}^M_{pp}$, so $\widetilde{U} \in \mathcal{G}^M_{C_{pp}}$, and then $\widetilde{U} \in \mathcal{G}^M_{pp}$ by Proposition 3.6-3. Conversely, if $\widetilde{U} \in \mathcal{G}^M_{pp} = \mathcal{G}^M_{L^\infty} \cap \mathcal{G}^M_{C_{pp}} \subset \mathcal{G}^M_{L^\infty}$, then $\widetilde{U}$ is a bounded generalized ultradistribution. □

Remark 7.7. *As a generalization of the obtained result in a forthcoming work, see [5], we study systems of ordinary differential equations*

$$\dot{\widetilde{u}} = A\widetilde{u} + \widetilde{f},$$

where $\widetilde{f} = \left(\left[(f_{1,\varepsilon})_\varepsilon\right], ..., \left[(f_{n,\varepsilon})_\varepsilon\right]\right) \in \left(\mathcal{G}^M_{pp}\right)^n$ *and* $A = \left(a_{ij}\right)_{0 \leq i,j \leq n}$ *is a square matrix of order n of elements of* $\mathbb{C}$.

References

[1] Benmeriem K., Bouzar C., Generalized Gevrey ultradistributions. New York J. Math., 15, p. 37-72, (2009).
[2] Benmeriem K., Bouzar C., An algebra of generalized Roumieu ultradistributions. Rend. Sem. Mat. Pol. Torino, Vol. 70, N^{0}2, p. 101-109, (2012).
[3] Bouzar C., Khalladi M.T., Almost periodic generalized functions. Novi Sad J. Math, Vol. 41, N^{0}1, p. 33-42, (2011).
[4] Bouzar C., Khalladi M.T., Linear differential equations in the algebra of almost periodic generalized functions. Rend. Sem. Univ. Politec. Torino, Vol. 70 : 2, p. 111-120, (2012).
[5] Bouzar C., Ouikene F., Almost periodic generalized ultradistributional solutions of differential equations. Preprint, University of Oran, (2018).
[6] Cioranescu I., The characterization of the almost periodic ultradistributions of Beurling type. Proc. A.M.S., Vol. 116, N^{0}1, p. 127-134, (1992).
[7] Debrouwere A., Vernaeve H., Vindas J., Optimal embeddings of ultradistributions into differential algebras. Monatsh. Math., vol. 186 : 3, p. 407-438, (2018).
[8] Garetto C., Ruzhansky M., Hyperbolic second order equations with non-regular time dependent coefficients, Arch. Ration. Mech. Anal. 217, p. 113–154, (2015)
[9] Fink A.M. Almost periodic differential equations. Lecture Notes in Math., N$^\circ$ 377, Springer, (1974).
[10] Komatsu H., Ultradistributions I, J. Fac. Sci. Univ. Tokyo, Sect. IA, 20, p. 25-105, (1973).
[11] Nedeljkov M., Pilipovic S., Scarpalézos D., Linear theory of Colombeau generalized functions. Pitman Res. Not. Math., 385, (1998).
[12] Pilipovic S., Characterization of bounded sets in spaces of ultradistributions. Proc. Amer. Math. Soc., Vol. 120, p. 1191-1206, (1994).
[13] Schwartz L., Théorie des distributions. Herman, 2ieme Ed., (1966).