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On the Delta Shock Wave Interactions for the Isentropic Chaplygin Gas System Consisting of Three Scalar Equations

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Abstract. The Riemann problem for the one-dimensional version of isentropic compressible Euler system for the Chaplygin gas consisting of three scalar equations is considered. It is shown that the Riemann solutions involve only two situations: the combination of three contact discontinuities or a delta shock wave. The generalized Rankine-Hugoniot conditions of delta shock wave are derived and the exact delta shock wave solution including the strength and propagation speed is obtained explicitly. The solutions to the perturbed Riemann problem are constructed globally when the initial data are taken to be the three piecewise constant initial data. The wave interaction problem is extensively investigated and some interesting phenomena are observed. It is shown that the limits of solutions to the perturbed Riemann problem converge to the corresponding ones to the Riemann problem when the perturbation parameter tends to zero.

1. Introduction

The two-dimensional isentropic compressible Euler system is shown in the conservative form [48]

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x + (\rho u v)_y = 0, \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p(\rho))_y = 0, \end{cases} \tag{1.1}$$

where ρ is the density, (u, v) is the velocity and p stands for the pressure given by $p(\rho) = A\rho^{\gamma}$ with $A > 0$ and $1 < \gamma < 3$ for the isentropic situation. It is well known that the Cauchy problem for the two-dimensional isentropic compressible Euler system (1.1) remains formidable for its complexity, even the Riemann problem which is the simplest Cauchy problem. This motivates our interest to consider the one-dimensional simplified version of the system (1.1) which is given by

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \\ (\rho v)_t + (\rho u v)_x = 0, \end{cases} \tag{1.2}$$

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which is encountered such as in [8, 15] when the solutions (ρ, u, v) with the claimed properties that are independent of the y-variable. The system (1.2) consists of three scalar equations which represent the conservation of mass and two linear momentums. The Riemann problem for the system (1.2) was considered in [15] for the situation $p(\rho) = 0$.

In the present paper, we are concerned with the Riemann problem for the following one-dimensional isentropic compressible Euler system for the Chaplygin gas [12] with the equation of state $p(\rho) = -\frac{1}{\rho}$, which can be expressed as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \frac{1}{\rho})_x = 0, \\ (\rho v)_t + (\rho u v)_x = 0, \end{cases} \tag{1.3}$$

with the Riemann initial data taken to be the two piecewise constant initial data

$$(\rho, u, v)(x, 0) = \begin{cases} (\rho_-, u_-, v_-), & x < 0, \\ (\rho_+, u_+, v_+), & x > 0. \end{cases} \tag{1.4}$$

It is assumed that all the $\rho_\pm$, $u_\pm$ and $v_\pm$ in the Riemann initial data (1.4) are given constants and should satisfy $\rho_\pm > 0$. It is shown that the system (1.3) is a strictly hyperbolic and fully linearly degenerate system, in which all the waves associated with the corresponding characteristic fields are contact discontinuities. Thus, the system (1.3) is attributed to the so-called Temple class [41] whose Riemann solutions have relatively simpler structures. In fact, we can construct the solutions to the Riemann problem (1.3) and (1.4) in completely explicit forms by using the method of characteristics. More precisely, there are only two kinds of Riemann solutions which consists of the combination of three contact discontinuities when $u_- - \frac{1}{\rho_-} < u_+ + \frac{1}{\rho_+}$ or a delta shock wave when $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+}$. In addition, the strength and propagation speed of delta shock wave and the assignments of u and v on this delta shock wave curve can be obtained by taking advantage of the so-called generalized Rankine-Hugoniot conditions of delta shock wave which are composed of a set of ordinary differential equations.

With the solutions to the Riemann problem (1.3) and (1.4) in hand, it is natural to expect the study of the wave interaction problem for the system (1.3) for the reason that the solutions to the Riemann problem (1.3) and (1.4) cannot describe the dynamic pictures of the system (1.3) in all the situations. In order to cope with it, the three piecewise constant initial data are taken as

$$(\rho, u, v)(x, 0) = \begin{cases} (\rho_-, u_-, v_-), & x < 0, \\ (\rho_m, u_m, v_m), & 0 < x < x_0, \\ (\rho_+, u_+, v_+), & x > x_0, \end{cases} \tag{1.5}$$

where $x_0 > 0$. The particular Cauchy problem (1.3) and (1.5) is the so-called perturbed (or double) Riemann problem for the reason that the three piecewise constant initial data (1.5) may be regarded as a particular small perturbation of the corresponding Riemann initial data (1.4) when x_0 is considered to be the so-called perturbation parameter. The wave interaction problem can be widely investigated for the system (1.3) when the initial data (1.5) are taken, including the interaction between the delta shock wave and the combination of three contact discontinuities and the interaction between two different combinations of three contact discontinuities. Some interesting phenomena can be captured during the process of interaction, such as the delta contact discontinuity is generated and the interaction between a forward contact discontinuity with a backward contact discontinuity gives rise to a new delta shock wave. In fact, the global solutions to the perturbed Riemann problem (1.3) and (1.5) can be constructed fully thanking to the particular structure of the system (1.3). Furthermore, it can be seen that the limits $x_0 \to 0$ of solutions to the perturbed Riemann problem (1.3) and (1.5) are identical with the corresponding ones to the Riemann problem (1.3) and (1.5) when the perturbation parameter x_0 tends to zero.

The model of Chaplygin gas was initially introduced by Chaplygin [4], Tsien [43] and von Karman [44] as an approximation to compute the lifting force on the wings of the aircraft in the gas dynamics. In some of cosmology theories [1, 2, 7, 32], the model was used to describe the dark energy in the universe, in which the formation of singularity in the solutions such as delta shock wave can be used to illustrate some physical phenomena in the evolution of universe, involving the universal inflation and boom and the formation and development of black hole. About the traditional isentropic Chaplygin gas dynamic system consisting of only the conservation of mass and one linear momentum, the concentration phenomenon of solution to the Riemann problem was obtained by Brenier [3] under the suitable assumption of Riemann initial data. The wave interaction problem was considered by Guo, Zhang and Yin [13] when the three piecewise constant initial data were taken. The generalized Riemann problem with delta initial data was also considered by Wang and Zhang [46] with the help of the generalized Rankine-Hugoniot conditions. Let us also see such as [11, 14, 17, 21, 25, 27, 28, 33, 34, 37, 39, 42, 47] for some related results. In contrast to the traditional isentropic Chaplygin gas dynamic system, the solution to the Riemann problem (1.3) and (1.4) is either the combination of three contact discontinuities when $u_- - \frac{1}{\rho_-} < u_+ + \frac{1}{\rho_+}$ or the delta shock wave solution when $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+}$, which can be obtained by using more complicated generalized Rankine-Hugoniot conditions. It is clear to see that the wave interaction problem for the system (1.3) studied here is more complicated than that in [13] for the traditional isentropic Chaplygin gas dynamic system, which is the building block to construct the solutions to the two-dimensional Riemann problem [6, 12, 18, 19, 31, 45] for the isentropic Chaplygin gas system (1.1) with the equation of state $p(\rho) = -\frac{1}{\rho}$.

The plan of the paper is displayed in the following way. In section 2, the solutions to the Riemann problem (1.3) and (1.4) are constructed explicitly, including the combination of three contact discontinuities and the delta shock wave. Furthermore, the generalized Rankine-Hugoniot conditions of delta shock wave are derived in detail. In section 3, the perturbed Riemann problem (1.3) and (1.5) is dealt with in details by investigating all kinds of wave interactions appearing during the process of construction of solutions. Moreover, one can see that the the limits $x_0 \to 0$ of solutions to the perturbed Riemann problem (1.3) and (1.5) are identical with the corresponding ones to the Riemann problem (1.3) and (1.4) in all the situations.

2. The Riemann problem

It is easily shown that the system (1.3) has three different real eigenvalues

$$\lambda_1 = u - \frac{1}{\rho}, \quad \lambda_2 = u, \quad \lambda_3 = u + \frac{1}{\rho}, \tag{2.1}$$

and three linearly independent right eigenvectors

$$\overrightarrow{r_1} = \Big(\rho, -\frac{1}{\rho}, 0\Big)^T, \quad \overrightarrow{r_2} = (0,0,1)^T, \quad \overrightarrow{r_3} = \Big(\rho, \frac{1}{\rho}, 0\Big)^T. \tag{2.2}$$

A simple calculation shows that $\nabla\lambda_i \cdot \overrightarrow{r_i} = 0$ ($i = 1, 2, 3$) in which the symbol ∇ denotes the gradient with respect to (ρ, u, v), such that all the eigenvalues of the system (1.3) are linearly degenerate. Thus, the system (1.3) is a strictly hyperbolic and fully linearly degenerate system and the wave associated with each characteristic field is contact discontinuity denoted by J. In addition, the three pairs of Riemann invariants associated with the above right eigenvectors (2.2) are taken as

$$\Big\{u - \frac{1}{\rho}, v\Big\}, \qquad \{\rho, u\}, \qquad \Big\{u + \frac{1}{\rho}, v\Big\}. \tag{2.3}$$

For all the eigenvalues are linearly degenerate, we only need to consider discontinuous solution. For a bounded discontinuity located at the position $\sigma = \xi = \frac{x}{t}$, the Rankine-Hugoniot conditions for the system

(1.3) are in the form

$$\begin{cases} \sigma[\rho] = [\rho u], \\ \sigma[\rho u] = [\rho u^2 - \frac{1}{\rho}], \\ \sigma[\rho v] = [\rho u v], \end{cases} \tag{2.4}$$

where $[\rho] = \rho_r - \rho_l$ stands for the jump of ρ across the discontinuity, etc. Through a tedious calculation, it follows from (2.4) that

$$\begin{cases} (u_l - \sigma)[\rho] + \rho_r[u] = 0, \\ \frac{1}{\rho_l^2\rho_r}[\rho] + (u_l - \sigma)[u] = 0, \\ (u_r - \sigma)[v] = 0. \end{cases} \tag{2.5}$$

It can be derived from the third equation in (2.5) that we have $\sigma = u_r$ or $v_l = v_r$. If $\sigma = u_r$ is true, then it follows from the first equation in (2.5) that we have $-[u][\rho] + \rho_r[u] = \rho_l[u] = 0$, which enables us to have $u_l = u_r$. Moreover, $\rho_l = \rho_r$ can be obtained directly by substituting $\sigma = u_l = u_r$ into the second equation in (2.5). Otherwise, if $v_l = v_r$ holds, then we arrive at $\sigma = u_l \pm \frac{1}{\rho_l}$ by combining the first and second equations in (2.5) provided that $[\rho][u] \neq 0$. In fact, one can easily get $[\rho] = [u] = 0$ from (2.5) when $[\rho][u] = 0$ holds, which corresponds to constant state. On the one hand, if $\sigma = u_l - \frac{1}{\rho_l}$, then we have $\frac{1}{\rho_l}[\rho] + \rho_r[u] = 0$ from the first equation in (2.5), which implies that $u_l - \frac{1}{\rho_l} = u_r - \frac{1}{\rho_r}$. On the other hand, if $\sigma = u_l + \frac{1}{\rho_l}$, then the first equation in (2.5) becomes $-\frac{1}{\rho_l}[\rho] + \rho_r[u] = 0$, which means that $u_l + \frac{1}{\rho_l} = u_r + \frac{1}{\rho_r}$.

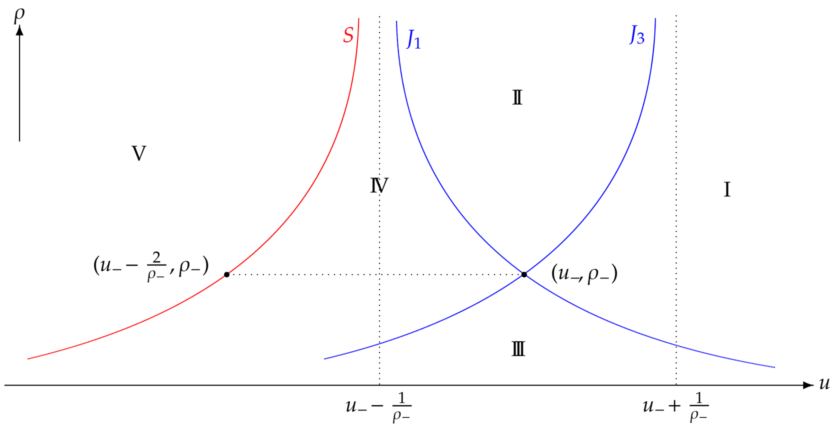


Fig.1 The elementary wave curves projected onto the upper-half (u, ρ) phase plane is shown for the Riemann problem (1.3) and (1.4), in which J_1 has the line $u = u_- - \frac{1}{\rho_-}$ and the positive $u-$axis as its asymptotes, J_3 has the line $u = u_- + \frac{1}{\rho_-}$ and the negative $u-$axis as its asymptotes, and S has the line $u = u_- - \frac{1}{\rho_-}$ and the negative $u-$axis as its asymptotes.

Summarizing up the above calculations and discussions together, the three contact discontinuities can be expressed respectively by

$$J_1: \quad \sigma = u_l - \frac{1}{\rho_l} = u_r - \frac{1}{\rho_r} \quad \text{and} \quad v_l = v_r, \tag{2.6}$$

$$J_2: \quad \sigma = u_l = u_r, \quad \rho_l = \rho_r \quad \text{and} \quad v_l \neq v_r, \tag{2.7}$$

$$J_3: \quad \sigma = u_l + \frac{1}{\rho_l} = u_r + \frac{1}{\rho_r} \quad \text{and} \quad v_l = v_r. \tag{2.8}$$

It is clear to see that the state variables ρ and u are invariant and only the state variable v varies when across J_2. Thus, we consider the elementary wave curves projected onto the upper-half (u, ρ) phase plane. For convenience, the positions of ρ and u are exchanged in the phase plane. Let the left state (ρ_-, u_-, v_-) be fixed, then it is deduced from (2.6) and (2.8) that the curve of $J_1(u_-, \rho_-)$ has two asymptotes $u = u_- - \frac{1}{\rho_-}$ and $\rho = 0$ and the curve of $J_3(u_-, \rho_-)$ has two asymptotes $u = u_- + \frac{1}{\rho_-}$ and $\rho = 0$ in the (u, ρ) phase plane. In addition, the curve S starting from the point $(u_- - \frac{2}{\rho_-}, \rho_-)$ can also be drawn to satisfy $u + \frac{1}{\rho} = u_- - \frac{1}{\rho_-}$. Let us draw Figure 1 to collect these curves together in the upper-half (u, ρ) phase plane. It is easily seen that the upper-half (u, ρ) phase plane is divided into five parts I, II, III, IV and V by these curves.

If $u_- - \frac{1}{\rho_-} < u_+ + \frac{1}{\rho_+}$ is satisfied, namely $(u_+, \rho_+) \in I \cup II \cup III \cup IV$, then the solution to the Riemann problem (1.3) and (1.4) consists of three contact discontinuities which may be represented by

$$(\rho, u, v)(x, t) = \begin{cases} (\rho_-, u_-, v_-), & x < \sigma_1 t, \\ (\rho_*, u_*, v_-), & \sigma_1 t < x < \sigma_2 t, \\ (\rho_*, u_*, v_+), & \sigma_2 t < x < \sigma_3 t, \\ (\rho_+, u_+, v_+), & x > \sigma_3 t, \end{cases} \tag{2.9}$$

in which

$$\frac{1}{\rho_*} = \frac{1}{2}(u_+ + \frac{1}{\rho_+}) - \frac{1}{2}(u_- - \frac{1}{\rho_-}), \qquad u_* = \frac{1}{2}(u_+ + \frac{1}{\rho_+}) + \frac{1}{2}(u_- - \frac{1}{\rho_-}), \tag{2.10}$$

$$\sigma_1 = u_- - \frac{1}{\rho_-}, \qquad \sigma_2 = \frac{1}{2}(u_+ + \frac{1}{\rho_+}) + \frac{1}{2}(u_- - \frac{1}{\rho_-}), \qquad \sigma_3 = u_+ + \frac{1}{\rho_+}. \tag{2.11}$$

On the other hand, if $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+}$ is satisfied, namely $(u_+, \rho_+) \in V$, then it can be seen from [3, 12] that it is necessary to introduce the delta shock wave solution to the Riemann problem (1.3) and (1.4). Let us first introduce the following definition of two-dimensional weighted Dirac delta function in the sense of distributions such as in [5, 20, 35, 38]. Also see [9, 10, 16, 22, 26, 29, 30, 36] for the more general definition of delta shock wave solution.

Definition 2.1. *Let $\Gamma = \{(x(s), t(s)) : a < s < b\}$ be a parameterized smooth curve in the (x, t) plane, then a two-dimensional weighted Dirac delta function $\beta(s)\delta_\Gamma$ supported on Γ is defined as*

$$\langle \beta(s)\delta_\Gamma, \psi(x(s), t(s)) \rangle = \int_a^b \beta(s)\psi(x(s), t(s))ds \tag{2.12}$$

for any test function $\psi \in C_0^\infty(R \times R_+)$.

With the above definition in mind, we use the following theorem to describe the delta shock wave solution to the Riemann problem (1.3) and (1.4) when $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+}$.

Theorem 2.2. *If $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+}$, then the delta shock wave solution to the Riemann problem (1.3) and (1.4) is constructed in the form*

$$(\rho, u, v)(x, t) = \begin{cases} (\rho_-, u_-, v_-), & x < x(t), \\ (\beta(t)\delta(x - x(t)), u_\delta, v_\delta), & x = x(t), \\ (\rho_+, u_+, v_+), & x > x(t), \end{cases} \tag{2.13}$$

in which $x = x(t)$ and $\beta(t)$ stand for the curve and strength of delta shock wave, u_δ and v_δ represent the assignments of u and v on this delta shock wave curve, respectively. In order to make the second equation in the system (1.3) hold in the weak sense of distributions, the term $\frac{1}{\rho}$ is defined by

$$\frac{1}{\rho} = \begin{cases} \frac{1}{\rho_-}, & x < \sigma_\delta t, \\ 0, & x = \sigma_\delta t, \\ \frac{1}{\rho_+}, & x > \sigma_\delta t, \end{cases} \tag{2.14}$$

and then

$$(\frac{1}{\rho})_x = (\frac{1}{\rho_+} - \frac{1}{\rho_-})\delta(x - \sigma_\delta t) \tag{2.15}$$

should be required. In addition, the delta shock wave solution of the form (2.13) should satisfy the following generalized Rankine-Hugoniot conditions

$$\begin{cases} \dfrac{dx}{dt} = \sigma_\delta = u_\delta, \\ \dfrac{d\beta(t)}{dt} = \sigma_\delta[\rho] - [\rho u], \\ \dfrac{d(\beta(t)u_\delta)}{dt} = \sigma_\delta[\rho u] - [\rho u^2 - \frac{1}{\rho}], \\ \dfrac{d(\beta(t)v_\delta)}{dt} = \sigma_\delta[\rho v] - [\rho u v], \end{cases} \tag{2.16}$$

and the over-compressive entropy condition

$$\lambda_1(\rho_+, u_+, v_+) \le \lambda_2(\rho_+, u_+, v_+) \le \lambda_3(\rho_+, u_+, v_+) \le \sigma_\delta \le \lambda_1(\rho_-, u_-, v_-) \le \lambda_2(\rho_-, u_-, v_-) \le \lambda_3(\rho_-, u_-, v_-). \tag{2.17}$$

More precisely, if $\rho_- \neq \rho_+$, then we have

$$\sigma_\delta = u_\delta = \frac{\rho_+ u_+ - \rho_- u_- + \mu}{\rho_+ - \rho_-}, \quad v_\delta = \frac{\rho_+ v_+ - \rho_- v_-}{\rho_+ - \rho_-} + \frac{\rho_- \rho_+ (u_+ - u_-)(v_+ - v_-)}{(\rho_+ - \rho_-)\mu}, \quad x(t) = \sigma_\delta t, \quad \beta(t) = \mu t, \tag{2.18}$$

in which the notation

$$\mu = \sqrt{\rho_- \rho_+ \Big((u_+ - u_-)^2 - (\frac{1}{\rho_+} - \frac{1}{\rho_-})^2 \Big)} \tag{2.19}$$

has been used. Otherwise, if $\rho_- = \rho_+$, then we have

$$\sigma_\delta = u_\delta = \frac{u_- + u_+}{2}, \quad v_\delta = \frac{v_- + v_+}{2}, \quad x(t) = \frac{u_- + u_+}{2} t, \quad \beta(t) = (\rho_- u_- - \rho_+ u_+)t. \tag{2.20}$$

Proof. If a delta shock wave solution to the Riemann problem (1.3) and (1.4) is shown in the form (2.13), then it should satisfy the weak form of the system (1.3)

$$\begin{cases} \displaystyle\int_0^{+\infty} \int_{-\infty}^{+\infty} \Big(\rho \psi_t + \rho u \psi_x \Big) dx dt = 0, \\ \displaystyle\int_0^{+\infty} \int_{-\infty}^{+\infty} \Big(\rho u \psi_t + \Big(\rho u^2 - \frac{1}{\rho} \Big) \psi_x \Big) dx dt = 0, \\ \displaystyle\int_0^{+\infty} \int_{-\infty}^{+\infty} \Big(\rho v \psi_t + \rho u v \psi_x \Big) dx dt = 0, \end{cases} \tag{2.21}$$

for all test functions $\psi(x, t) \in C_c^\infty(R_+^2)$ in the sense of distributions. Let us use $\sigma_\delta = \frac{dx}{dt}$ to denote the propagation speed of delta shock wave, then we have $\sigma_\delta = u_\delta$ for the reason that the concentration of ρ needs to travel at the same propagation speed of discontinuity. In fact, it follows from the third equation in (2.21) that

$$\begin{aligned} I_3 &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \Big(\rho v \psi_t + \rho u v \psi_x \Big) dx dt \\ &= \int_0^{+\infty} \int_{-\infty}^{\sigma_\delta t} \Big(\rho_- v_- \psi_t + \rho_- u_- v_- \psi_x \Big) dx dt + \int_0^{+\infty} \int_{\sigma_\delta t}^{+\infty} \Big(\rho_+ v_+ \psi_t + \rho_+ u_+ v_+ \psi_x \Big) dx dt \\ &\quad + \int_0^{+\infty} v_\delta \beta(t) \Big(\psi_t(\sigma_\delta t, t) + u_\delta \psi_x(\sigma_\delta t, t) \Big) dt. \end{aligned}$$

Without loss of generality, let us assume $\sigma_\delta > 0$, then we have

$$\begin{aligned} I_3 &= \int_{-\infty}^{0}\int_{0}^{+\infty} \rho_- v_- \psi_t dtdx + \int_{0}^{+\infty}\int_{\frac{x}{\sigma_\delta}}^{+\infty} \rho_- v_- \psi_t dtdx + \int_{0}^{+\infty}\int_{0}^{\frac{x}{\sigma_\delta}} \rho_+ v_+ \psi_t dtdx \\ &\quad + \int_{0}^{+\infty} (\rho_- u_- v_- - \rho_+ u_+ v_+)\psi(\sigma_\delta t, t)dt + \int_{0}^{+\infty} \beta(t) v_\delta d\psi(\sigma_\delta t, t) \\ &= \int_{0}^{+\infty} (\rho_+ v_+ - \rho_- v_-)\psi(x, \frac{x}{\sigma_\delta})dx + \int_{0}^{+\infty} (\rho_- u_- v_- - \rho_+ u_+ v_+)\psi(\sigma_\delta t, t)dt - \int_{0}^{+\infty} v_\delta \beta'(t)\psi(\sigma_\delta t, t)dt. \end{aligned}$$

By virtue of the substitution of variables, we have

$$I_3 = \int_{0}^{+\infty} \Big\{\sigma_\delta(\rho_+ v_+ - \rho_- v_-) + (\rho_- u_- v_- - \rho_+ u_+ v_+) - v_\delta \beta'(t)\Big\}\psi(\sigma_\delta t, t)dt, \tag{2.22}$$

such that the fourth equality in (2.16) can be established. Similarly, the second and third equalities in (2.16) can also be derived from the first and second equations in (2.21) such as in [17, 34]. Thus, the delta shock wave solution of the form (2.13) should satisfy the generalized Rankine-Hugoniot conditions (2.16).

Let us solve the system (2.16) with the initial condition $x(0) = 0$ and $\beta(0) = 0$, where u_δ and v_δ are undetermined constants. By combining the second and third equations in (2.16) and noticing that $u_\delta = \sigma_\delta$, one has

$$(\rho_+ - \rho_-)\sigma_\delta^2 - 2(\rho_+ u_+ - \rho_- u_-)\sigma_\delta + (\rho_+ u_+^2 - \rho_- u_-^2) - \Big(\frac{1}{\rho_+} - \frac{1}{\rho_-}\Big) = 0. \tag{2.23}$$

On the other hand, the over-compressive entropy condition (2.17) becomes

$$u_+ + \frac{1}{\rho_+} \leq \sigma_\delta \leq u_- - \frac{1}{\rho_-}. \tag{2.24}$$

Therefore, σ_δ is determined uniquely from (2.23) together with (2.24) and subsequently $x(t)$, $\beta(t)$ and v_δ can be obtained from (2.16) directly. Thus, the results of (2.18), (2.19) and (2.20) can be achieved by a trivial calculation. The proof is completed. □

3. Interaction between delta shock wave and combination of three contact discontinuities

The propose of this section is to construct the global solutions to the perturbed Riemann problem (1.3) and (1.5) entirely by virtue of the method of characteristics to study all the possible wave interactions during the process of constructions of solutions. In particular, we are devoted to studying the interaction problem of delta shock wave which has been paid extensive attention such as in [13, 14, 23, 24, 40] recently. In order to contain all the cases fully, our discussion should be divided into the following four cases according to the different combinations of Riemann solutions originating from the initial points $(0, 0)$ and $(x_0, 0)$. Without loss of generality, we assume that ρ_-, ρ_m and ρ_+ are different from each other. Otherwise, the other special situations can also be dealt with by employing the same method adopted here.

Case 1. Interaction between two delta shock waves

In this case, the interaction between two delta shock waves originating from $(0, 0)$ and $(x_0, 0)$ is given rise to when both the conditions $u_- - \frac{1}{\rho_-} > u_m + \frac{1}{\rho_m}$ and $u_m - \frac{1}{\rho_m} > u_+ + \frac{1}{\rho_+}$ are satisfied. Easy calculations show that the two delta shock waves meet in finite time and subsequently they coalesce into one delta shock wave for the reason that $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+}$ can be obtained directly. The details are omitted due to the fact that the global solution to the perturbed Riemann problem (1.3) and (1.5) has a relatively simpler structure here.

Case 2. Interaction between delta shock wave and combination of three contact discontinuities

The interaction between the delta shock wave emitting from $(0,0)$ and the combination of three contact discontinuities emanating from $(x_0,0)$ happens if and only if both $u_- - \frac{1}{\rho_-} > u_m + \frac{1}{\rho_m}$ and $u_m - \frac{1}{\rho_m} < u_+ + \frac{1}{\rho_+}$ are satisfied. For the sufficiently small time t, the solution to the perturbed Riemann problem (1.3) and (1.5) may be represented by the following symbols (see Fig.2):

$$(\rho_-,u_-,v_-)+S_\delta+(\rho_m,u_m,v_m)+J_1+(\rho_1,u_1,v_m)+J_2+(\rho_1,u_1,v_+)+J_3+(\rho_+,u_+,v_+), \tag{3.1}$$

in which

$$(\rho_1,u_1)=\Big(\frac{2}{u_++\frac{1}{\rho_+}-u_m+\frac{1}{\rho_m}},\frac{1}{2}(u_++\frac{1}{\rho_+})+\frac{1}{2}(u_m-\frac{1}{\rho_m})\Big). \tag{3.2}$$

The propagation speeds of $S_{\delta 1}$ and J_1 are

$$\sigma_{\delta 1}=\frac{\rho_m u_m-\rho_-u_-+\sqrt{\rho_-\rho_m\Big((u_m-u_-)^2-(\frac{1}{\rho_m}-\frac{1}{\rho_-})^2\Big)}}{\rho_m-\rho_-},\qquad \sigma_1=u_m-\frac{1}{\rho_m}, \tag{3.3}$$

respectively. In view of over-compressive entropy condition of $S_{\delta 1}$, one deduces that $\sigma_{\delta 1}\geq u_m+\frac{1}{\rho_m}>\sigma_1$. Thus, $S_{\delta 1}$ catches up with J_1 in finite time and the intersection point (x_1,t_1) can be calculated by

$$\begin{cases} x_1=\sigma_{\delta 1}t_1, \\ x_1-x_0=\sigma_1 t_1, \end{cases} \tag{3.4}$$

which yields

$$(x_1,t_1)=\Big(\frac{\sigma_{\delta 1}x_0}{\sigma_{\delta 1}-\sigma_1},\frac{x_0}{\sigma_{\delta 1}-\sigma_1}\Big). \tag{3.5}$$

In addition, the strength of $S_{\delta 1}$ before the time t_1 can be calculated by

$$\beta(t)=\sqrt{\rho_-\rho_m\Big((u_m-u_-)^2-(\frac{1}{\rho_m}-\frac{1}{\rho_-})^2\Big)}\,t \quad \text{for} \quad 0\leq t\leq t_1. \tag{3.6}$$

It can be seen that a new Riemann problem for the system (1.3) is formed at the point (x_1,t_1) where the state on the left-hand side is (ρ_-,u_-,v_-) and the state on the right-hand side is (ρ_1,u_1,v_m). It follows from (3.2) that

$$u_1+\frac{1}{\rho_1}=u_++\frac{1}{\rho_+}. \tag{3.7}$$

Thus, our discussion should be divided into the following two subcases according to $u_++\frac{1}{\rho_+}<u_--\frac{1}{\rho_-}$ or not.

Subcase 2.1. $u_++\frac{1}{\rho_+}<u_--\frac{1}{\rho_-}$

Let us first consider the situation when $u_++\frac{1}{\rho_+}<u_--\frac{1}{\rho_-}$. If $u_++\frac{1}{\rho_+}<u_--\frac{1}{\rho_-}$, then the interaction between $S_{\delta 1}$ and J_1 gives rise to a new delta shock wave denoted by $S_{\delta 2}$. The propagation speeds of $S_{\delta 2}$ and J_2 are given respectively by

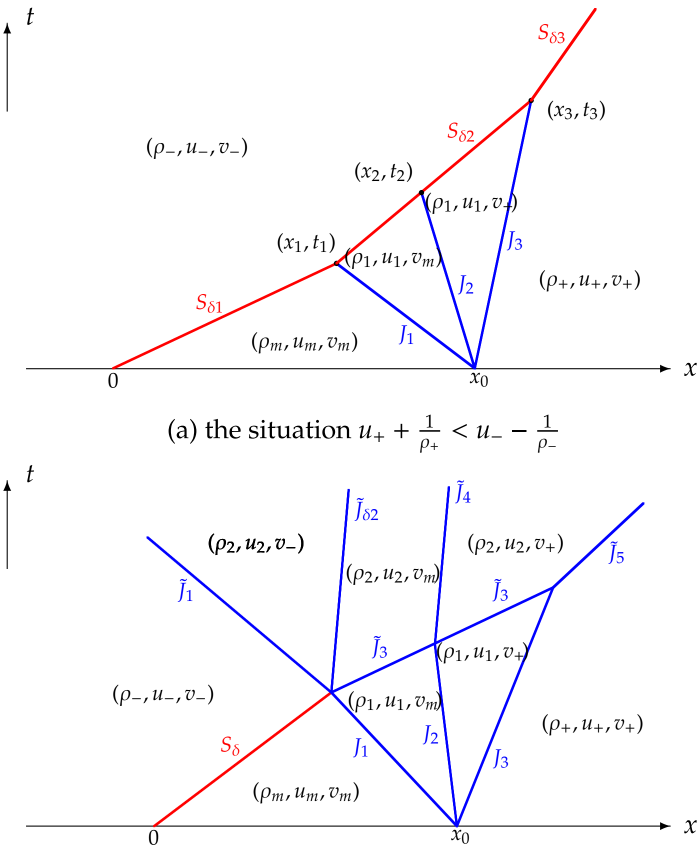
$$\sigma_{\delta 2}=\frac{\rho_1 u_1-\rho_-u_-+\sqrt{\rho_-\rho_1\Big((u_1-u_-)^2-(\frac{1}{\rho_1}-\frac{1}{\rho_-})^2\Big)}}{\rho_1-\rho_-},\qquad \sigma_2=u_1. \tag{3.8}$$

Subsequently, $S_{\delta 2}$ meets J_2 at the intersection point (x_2, t_2) which may be computed by

$$\begin{cases} x_2 - x_1 = \sigma_{\delta 2}(t_2 - t_1), \\ x_2 - x_0 = \sigma_2 t_2, \end{cases} \tag{3.9}$$

in which (x_1, t_1) is given by (3.5), such that we have

$$(x_2, t_2) = \Big(x_0 + \frac{\sigma_2(\sigma_{\delta 2} t_1 + x_0 - x_1)}{\sigma_{\delta 2} - \sigma_2}, \frac{\sigma_{\delta 2} t_1 + x_0 - x_1}{\sigma_{\delta 2} - \sigma_2}\Big). \tag{3.10}$$



(a) the situation $u_+ + \frac{1}{\rho_+} < u_- - \frac{1}{\rho_-}$

(b) the situations $u_+ + \frac{1}{\rho_+} > u_- - \frac{1}{\rho_-}$

Fig.2 The interaction between the delta shock wave and the combination of three contact discontinuities is displayed for two different situations when both $u_- - \frac{1}{\rho_-} > u_m + \frac{1}{\rho_m}$ and $u_m - \frac{1}{\rho_m} < u_+ + \frac{1}{\rho_+}$ are satisfied.

It can be seen that $S_{\delta 2}$ passes through J_2 without changing its direction and only the value of v_δ on the delta shock wave curve changes. This is due to the fact that the propagation speed of delta shock wave only depends on the state variables ρ and u on both sides of delta shock curve which remain unchanged when $S_{\delta 2}$ passes through J_2. Later, $S_{\delta 2}$ also meets J_3 in finite time whose intersection point (x_3, t_3) can also be computed by

$$\begin{cases} x_3 - x_1 = \sigma_{\delta 2}(t_3 - t_1), \\ x_3 - x_0 = \sigma_3 t_3, \end{cases} \tag{3.11}$$

in which (x_1, t_1) is also given by (3.5) and $\sigma_3 = u_+ + \frac{1}{\rho_+}$, such that we also have

$$(x_3, t_3) = \Big(x_0 + \frac{\sigma_3(\sigma_{\delta2}t_1 + x_0 - x_1)}{\sigma_{\delta2} - \sigma_3}, \frac{\sigma_{\delta2}t_1 + x_0 - x_1}{\sigma_{\delta2} - \sigma_3}\Big). \tag{3.12}$$

In the end, the interaction between $S_{\delta2}$ and J_3 gives rise to a new delta shock wave denoted by $S_{\delta3}$ with the invariant propagation speed $\sigma_{\delta3}$ which may be calculated by

$$\sigma_{\delta3} = \frac{\rho_+u_+ - \rho_-u_- + \sqrt{\rho_-\rho_+\Big((u_+ - u_-)^2 - (\dfrac{1}{\rho_+} - \dfrac{1}{\rho_-})^2\Big)}}{\rho_+ - \rho_-}. \tag{3.13}$$

Thus, if the conditions $u_- - \frac{1}{\rho_-} > u_m + \frac{1}{\rho_m}$ and $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+} > u_m - \frac{1}{\rho_m}$ are satisfied simultaneously, then the global solution to the perturbed Riemann problem (1.3) and (1.5) can be illustrated in Figure 2(a). In addition, the strength of delta shock wave after the time t_1 can also be calculated by

$$\beta(t) = \sqrt{\rho_-\rho_1\Big((u_1 - u_-)^2 - (\frac{1}{\rho_1} - \frac{1}{\rho_-})^2\Big)}(t - t_1) + \beta(t_1) \quad \text{for} \quad t_1 < t \le t_3, \tag{3.14}$$

$$\beta(t) = \sqrt{\rho_-\rho_+\Big((u_+ - u_-)^2 - (\frac{1}{\rho_+} - \frac{1}{\rho_-})^2\Big)}(t - t_3) + \beta(t_3) \quad \text{for} \quad t > t_3. \tag{3.15}$$

Subcase 2.2. $u_+ + \frac{1}{\rho_+} > u_- - \frac{1}{\rho_-}$

On the other hand, if $u_+ + \frac{1}{\rho_+} > u_- - \frac{1}{\rho_-}$, then the interaction between $S_{\delta1}$ and J_1 generates three contact discontinuities denoted by $\widetilde{J_1}$, $\widetilde{J_2}$ and $\widetilde{J_3}$ which can be described in Fig.2(b), in which (ρ_2, u_2) is given by

$$(\rho_2, u_2) = \Big(\frac{2}{u_1 + \frac{1}{\rho_1} - u_- + \frac{1}{\rho_-}}, \frac{1}{2}(u_1 + \frac{1}{\rho_1}) + \frac{1}{2}(u_- - \frac{1}{\rho_-})\Big), \tag{3.16}$$

namely we have

$$(\rho_2, u_2) = \Big(\frac{2}{u_+ + \frac{1}{\rho_+} - u_- + \frac{1}{\rho_-}}, \frac{1}{2}(u_+ + \frac{1}{\rho_+}) + \frac{1}{2}(u_- - \frac{1}{\rho_-})\Big). \tag{3.17}$$

The strength of delta shock wave at the point (x_1, t_1) can be calculated by the formula (3.6), which is the mass accumulation for the density ρ at that time. In other words, we need to deal with the generalized Riemann problem for the system (1.3) with the delta-type initial data

$$\rho|_{t=t_1} = \left\{\begin{array}{ll} \rho_-, & x < x_1 \\ \rho_1, & x > x_1 \end{array}\right\} + \beta(t_1)\delta_{(x_1,t_1)}, \quad (u, v)|_{t=t_1} = \left\{\begin{array}{ll} (u_-, v_-), & x < x_1, \\ (u_1, v_m), & x > x_1. \end{array}\right. \tag{3.18}$$

By using the method developed in [33, 46], one of the admissible solutions to the generalized Riemann problem (1.3) and (3.18) can be constructed in the following form

$$\rho(x,t) = \left\{\begin{array}{ll} \rho_-, & x - x_1 < (u_- - \frac{1}{\rho_-})(t - t_1) \\ \rho_2, & (u_- - \frac{1}{\rho_-})(t - t_1) < x - x_1 < (u_1 + \frac{1}{\rho_1})(t - t_1) \\ \rho_1, & x - x_1 > (u_1 + \frac{1}{\rho_1})(t - t_1) \end{array}\right\} + \beta(t_1)\delta(x - x_1 - u_2(t - t_1)), \tag{3.19}$$

$$(u, v)(x,t) = \left\{\begin{array}{ll} (u_-, v_-), & x - x_1 < (u_- - \frac{1}{\rho_-})(t - t_1), \\ (u_2, v_-), & (u_- - \frac{1}{\rho_-})(t - t_1) < x - x_1 < u_2(t - t_1), \\ (u_2, v_m), & u_2(t - t_1) < x - x_1 < (u_1 + \frac{1}{\rho_1})(t - t_1), \\ (u_1, v_m), & x - x_1 > (u_1 + \frac{1}{\rho_1})(t - t_1). \end{array}\right. \tag{3.20}$$

This kind of delta type wave expressed in the form (3.19) together with (3.20) was called as the so-called delta contact discontinuity in [24]. The above constructed solution shows that the mass accumulation for the density ρ cannot disappear abruptly which may be supported on $\widetilde{J}_2$. In order to show accurately, we use the symbol $\widetilde{J}_{\delta 2}$ instead of $\widetilde{J}_2$ in Figure 2.(b).

In what follows, we need to check that the system (1.3) is satisfied in the sense of distributions in the neighborhood of $\widetilde{J}_{\delta 2}$ if $\beta(t_1)\delta(x-x_1-u_2(t-t_1))$ is involved in the solution of ρ, namely the singular measure is introduced into the line of $\widetilde{J}_{\delta 2}$. In fact, for any $\varphi \in C_0^\infty(R\times R_+)$, it is obvious to see that the weak form of the system (1.3)

$$\begin{cases} \langle \rho_t + (\rho u)_x, \varphi\rangle = 0, \\ \langle (\rho u)_t + (\rho u^2 - \frac{1}{\rho})_x, \varphi\rangle = 0, \\ \langle (\rho v)_t + (\rho u v)_x, \varphi\rangle = 0, \end{cases} \tag{3.21}$$

holds provided that $\mathrm{supp}\varphi \cap \{(x,t)|x = x_1 + u_2(t-t_1)), t > t_1\} = \emptyset$. Otherwise, if $\mathrm{supp}\varphi \cap \{(x,t)|x = x_1 + u_2(t-t_1)), t > t_1\} \neq \emptyset$, then we need to prove that the solution (3.19) and (3.20) should satisfy the system (1.3) near the support of the delta function. In the local neighbourhood of $\widetilde{J}_{\delta 2}$, if we substitute (3.19) and (3.20) into the first equation in the system (1.3), then the following equality

$$\rho_t + (\rho u)_x = -u_2\beta(t_1)\delta'(x-x_1-u_2(t-t_1)) + u_2\beta(t_1)\delta'(x-x_1-u_2(t-t_1)) = 0 \tag{3.22}$$

holds in the weak sense of distributions.

As in [13], in the local neighbourhood of $\widetilde{J}_{\delta 2}$, $\frac{1}{\rho}$ is defined by

$$\frac{1}{\rho} = \begin{cases} \frac{1}{\rho_2}, & x \neq x_1 + u_2(t-t_1), \\ 0, & x = x_1 + u_2(t-t_1), \end{cases} \tag{3.23}$$

thus it can be seen from Theorem 2.2 that the generalized derivative $(\frac{1}{\rho})_x = 0$ holds in the sense of distributions in the local neighbourhood of $\widetilde{J}_{\delta 2}$. Analogously, we also have

$$(\rho u)_t + (\rho u^2 - \frac{1}{\rho})_x = -(u_2)^2\beta(t_1)\delta'(x-x_1-u_2(t-t_1)) + (u_2)^2\beta(t_1)\delta'(x-x_1-u_2(t-t_1)) = 0. \tag{3.24}$$

In the end, we draw our attention on the third equation in the system (1.3). Let us use v_δ to denote the value of v on the line $\widetilde{J}_{\delta 2}$ and notice that u is a constant u_2 in the local neighbourhood of $\widetilde{J}_{\delta 2}$. If we substitute (3.19) and (3.20) into the third equation in the system (1.3), then we obtain the following generalized derivatives

$$(\rho v)_t = -u_2(\rho_2 v_m - \rho_2 v_-)\delta(x-x_1-u_2(t-t_1)) - u_2 v_\delta \beta(t_1)\delta'(x-x_1-u_2(t-t_1)), \tag{3.25}$$

$$(\rho u v)_x = u_2(\rho_2 v_m - \rho_2 v_-)\delta(x-x_1-u_2(t-t_1)) + u_2 v_\delta \beta(t_1)\delta'(x-x_1-u_2(t-t_1)), \tag{3.26}$$

such that the following equality

$$(\rho v)_t + (\rho u v)_x = 0 \tag{3.27}$$

also holds in the weak sense of distributions.

Subsequently, $\widetilde{J}_3$ will catch up with J_2 in finite time and the interaction between $\widetilde{J}_3$ and J_2 only gives rise to 2-wave and 3-wave for the reason that the relation $u_1 + \frac{1}{\rho_1} = u_2 + \frac{1}{\rho_2}$ holds. As before, $\widetilde{J}_3$ keeps the propagation speed $u_1 + \frac{1}{\rho_1}$ invariant and thus cannot change its direction, and while the propagation speed of J_2 varies from u_1 to u_2 when J_2 passes through $\widetilde{J}_3$ and thus is denoted with $\widetilde{J}_4$ after penetration. The state between $\widetilde{J}_3$ and $\widetilde{J}_4$ is (ρ_2, u_2, v_+) after the interaction is finished. Obviously, $\widetilde{J}_{\delta 2}$ and $\widetilde{J}_4$ share the same propagation speed u_2 and thus no interaction happens between them. Finally, $\widetilde{J}_3$ and J_3 coalesce into a new contact discontinuity denoted with $\widetilde{J}_5$. Thus, if the conditions $u_- - \frac{1}{\rho_-} > u_m + \frac{1}{\rho_m}$ and $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+} > u_m - \frac{1}{\rho_m}$ are satisfied simultaneously, then we can also draw Figure 2(b) to describe the global solution to the perturbed Riemann problem (1.3) and (1.5).

Remark 3.1. If the Dirac delta function is supported on the 1-contact discontinuity curve $\widetilde{J}_{\delta 1}$ or the 3-contact discontinuity curve $\widetilde{J}_{\delta 3}$, then we can also see that the system (1.3) is still satisfied in the sense of distributions by using the similar calculation as above. As a consequence, we can see that the mass of Dirac delta function $\beta(t_1)\delta$ at the point (x_1, t_1) can also travel with the 1-contact discontinuity curve or the 3-contact discontinuity curve. Thus, $\beta(t_1)\delta(x - x_1 - u_2(t - t_1))$ in the formula (3.19) can be substituted by the more general form $c_1\delta(x - x_1 - (u_- - \frac{1}{\rho_-})(t - t_1)) + c_2\delta(x - x_1 - u_2(t - t_1)) + c_3\delta(x - x_1 - (u_1 + \frac{1}{\rho_1})(t - t_1))$ for any non-negative constants c_1, c_2 and c_3 satisfying the requirement $c_1 + c_2 + c_3 = \beta(t_1)$. Here we only construct one admissible solution (3.19) together with (3.20) to the generalized Riemann problem (1.3) and (3.18). Thus, the uniqueness of the generalized Riemann problem (1.3) and (3.18) is still an open problem and may be obtained by providing more strictly restrictive condition. Fortunately, it is clear to see that the conclusion of Theorem 3.1 below is still true due to $\beta(t_1) \to 0$ as $x_0 \to 0$.

Case 3. Interaction between combination of three contact discontinuities and delta shock wave

The interaction between the combination of three contact discontinuities departing from $(0, 0)$ and the delta shock wave departing from $(x_0, 0)$ occurs when both $u_- - \frac{1}{\rho_-} < u_m + \frac{1}{\rho_m}$ and $u_m - \frac{1}{\rho_m} > u_+ + \frac{1}{\rho_+}$ are satisfied. The details are also omitted here for the reason that this case can be dealt with similarly to that for Case 2.

Case 4. Interaction between two combinations of three contact discontinuities

In the end, we draw our attention on the interaction between two combinations of three contact discontinuities starting from $(0, 0)$ and $(x_0, 0)$ when both the conditions $u_- - \frac{1}{\rho_-} < u_m + \frac{1}{\rho_m}$ and $u_m - \frac{1}{\rho_m} < u_+ + \frac{1}{\rho_+}$ are satisfied. For the sufficiently small time t, the solution to the perturbed Riemann problem (1.3) and (1.5) may also be expressed by the following symbols (see Figs.3 and 4):

$$\begin{aligned}&(\rho_-, u_-, v_-) + J_1 + (\rho_1, u_1, v_-) + J_2 + (\rho_1, u_1, v_m) + J_3 + (\rho_m, u_m, v_m) + J_4 + (\rho_2, u_2, v_m) + J_5\\ &+(\rho_2, u_2, v_+) + J_6 + (\rho_+, u_+, v_+),\end{aligned} \tag{3.28}$$

in which

$$(\rho_1, u_1) = \Big(\frac{2}{u_m + \frac{1}{\rho_m} - u_- + \frac{1}{\rho_-}}, \frac{1}{2}(u_m + \frac{1}{\rho_m}) + \frac{1}{2}(u_- - \frac{1}{\rho_-})\Big), \tag{3.29}$$

$$(\rho_2, u_2) = \Big(\frac{2}{u_+ + \frac{1}{\rho_+} - u_m + \frac{1}{\rho_m}}, \frac{1}{2}(u_+ + \frac{1}{\rho_+}) + \frac{1}{2}(u_m - \frac{1}{\rho_m})\Big). \tag{3.30}$$

The propagation speeds of J_3 and J_4 can be calculated respectively by

$$\sigma_3 = u_m + \frac{1}{\rho_m} \qquad \sigma_4 = u_m - \frac{1}{\rho_m}. \tag{3.31}$$

It is clear to see that J_3 and J_4 meet in finite time and the intersection point (x_1, t_1) can be calculated by

$$\begin{cases} x_1 = \sigma_3 t_1 = (u_m + \frac{1}{\rho_m})t_1,\\ x_1 - x_0 = \sigma_4 t_1 = (u_m - \frac{1}{\rho_m})t_1,\end{cases} \tag{3.32}$$

such that we have

$$(x_1, t_1) = \Big(\frac{x_0(\rho_m u_m + 1)}{2}, \frac{x_0\rho_m}{2}\Big). \tag{3.33}$$

Then, a new Riemann problem for the system (1.3) will be formulated at the intersection point (x_1, t_1) with the initial data given by

$$(\rho, u, v)(x, t_1) = \begin{cases} (\rho_1, u_1, v_m), & x < x_1, \\ (\rho_2, u_2, v_m), & x > x_1. \end{cases} \tag{3.34}$$

It follows from (3.29) and (3.30) that

$$u_1 - \frac{1}{\rho_1} = u_- - \frac{1}{\rho_-}, \qquad u_2 + \frac{1}{\rho_2} = u_+ + \frac{1}{\rho_+}. \tag{3.35}$$

Thus, our discussion should also be divided into the following two subcases according to $u_- - \frac{1}{\rho_-} < u_+ + \frac{1}{\rho_+}$ or not.

Subcase 4.1. $u_- - \frac{1}{\rho_-} < u_+ + \frac{1}{\rho_+}$

If $u_- - \frac{1}{\rho_-} < u_+ + \frac{1}{\rho_+}$, then the interaction between J_3 and J_4 gives rise to only 1-wave and 3-wave for the reason that the left state is the same as the right one for the state variable v in the Riemann initial data (3.34). Let us use J_7 and J_8 to denote the 1-wave and the 3-wave respectively after the interaction between J_3 and J_4 is finished (see Fig.3). The state between J_7 and J_8 can be expressed as (ρ_3, u_3, v_m), which can be calculated by

$$\begin{cases} \sigma_7 = u_3 - \frac{1}{\rho_3} = u_1 - \frac{1}{\rho_1} = u_- - \frac{1}{\rho_-}, \\ \sigma_8 = u_3 + \frac{1}{\rho_3} = u_2 + \frac{1}{\rho_2} = u_+ + \frac{1}{\rho_+}, \end{cases} \tag{3.36}$$

which gives rise to

$$(\rho_3, u_3) = \Big(\frac{2}{u_+ + \frac{1}{\rho_+} - u_- + \frac{1}{\rho_-}}, \frac{1}{2}(u_+ + \frac{1}{\rho_+}) + \frac{1}{2}(u_- - \frac{1}{\rho_-})\Big). \tag{3.37}$$

On the one hand, it is clear to see that J_7 is parallel to J_1 for the reason that they share the same propagation speed $u_- - \frac{1}{\rho_-}$. Subsequently, J_2 will also meet with J_7 in finite time and the interaction between J_2 and J_7 only gives rise to 1-wave and 2-wave for the reason that we have the relation $u_1 - \frac{1}{\rho_1} = u_3 - \frac{1}{\rho_3}$. More precisely, J_7 keeps the propagation speed $u_- - \frac{1}{\rho_-}$ invariant and thus cannot change its direction, and while the propagation speed of J_2 varies from u_1 to u_3 when J_2 passes through J_7 and thus it is denoted with J_9 after penetration. The state between J_7 and J_9 is (ρ_3, u_3, v_-) after the interaction is finished.

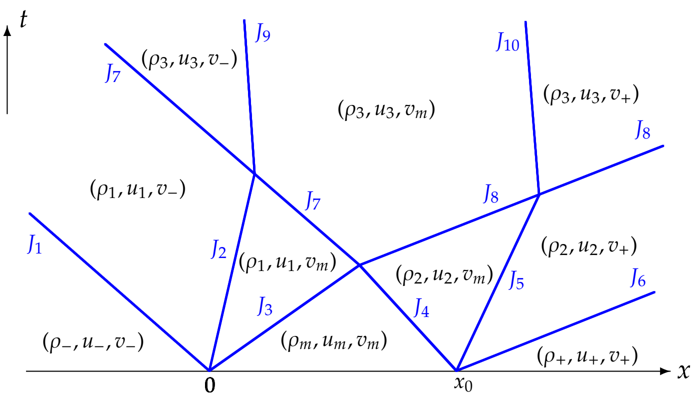


Fig.3 The interaction between two combinations of three contact discontinuities is shown when both $u_- - \frac{1}{\rho_-} < u_m + \frac{1}{\rho_m}$ and $max(u_- - \frac{1}{\rho_-}, u_m - \frac{1}{\rho_m}) < u_+ + \frac{1}{\rho_+}$ are satisfied.

On the other hand, it can also be seen that J_8 is parallel to J_6 for the reason that they have the same propagation speed $u_+ + \frac{1}{\rho_+}$. Later, J_8 will also meet with J_5 in finite time and the interaction between J_8 and J_5 only gives rise to 2-wave and 3-wave for the reason that we have the relation $u_3 + \frac{1}{\rho_3} = u_2 + \frac{1}{\rho_2}$. As before, J_8 also keeps the propagation speed $u_+ + \frac{1}{\rho_+}$ invariant and thus cannot change its direction, and while the propagation speed of J_5 varies from u_2 to u_3 when J_5 passes through J_8 and thus is denoted with J_{10} after penetration. The state between J_{10} and J_8 is (ρ_3, u_3, v_+) after the interaction is completed.

Up to now, all the interactions have been completed due to the fact that J_1 is parallel to J_7, J_9 is parallel to J_{10}, and J_8 is parallel to J_6. Summarizing up the above results together, it can be concluded that if the conditions $u_- - \frac{1}{\rho_-} < u_m + \frac{1}{\rho_m}$, $u_m - \frac{1}{\rho_m} < u_+ + \frac{1}{\rho_+}$ and $u_- - \frac{1}{\rho_-} < u_+ + \frac{1}{\rho_+}$ are available, then we can use Figure 3 to illustrate the structure of the global solution to the perturbed Riemann problem (1.3) and (1.5) for this situation.

Subcase 4.2. $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+}$

If $u_- - \frac{1}{\rho_-} > u_+ + \frac{1}{\rho_+}$, then we have $u_m - \frac{1}{\rho_m} < u_+ + \frac{1}{\rho_+} < u_- - \frac{1}{\rho_-} < u_m + \frac{1}{\rho_m}$. In this subcase, the interaction between J_3 and J_4 gives rise to a new delta shock wave $S_{\delta 1}$ whose propagation speed is

$$\sigma_{\delta 1} = \frac{\rho_2 u_2 - \rho_1 u_1 + \sqrt{\rho_1 \rho_2 \Big((u_2 - u_1)^2 - (\dfrac{1}{\rho_2} - \dfrac{1}{\rho_1})^2 \Big)}}{\rho_2 - \rho_1}, \tag{3.38}$$

which satisfies the over-compressive entropy condition

$$u_2 + \frac{1}{\rho_2} \le \sigma_{\delta 1} \le u_1 - \frac{1}{\rho_1}. \tag{3.39}$$

On the other hand, the propagation speeds of J_2 and J_5 are computed respectively by

$$\sigma_2 = u_1 = \frac{1}{2}(u_m + \frac{1}{\rho_m}) + \frac{1}{2}(u_- - \frac{1}{\rho_-}), \tag{3.40}$$

$$\sigma_5 = u_2 = \frac{1}{2}(u_+ + \frac{1}{\rho_+}) + \frac{1}{2}(u_m - \frac{1}{\rho_m}). \tag{3.41}$$

It follows from (3.39) and (3.40) that

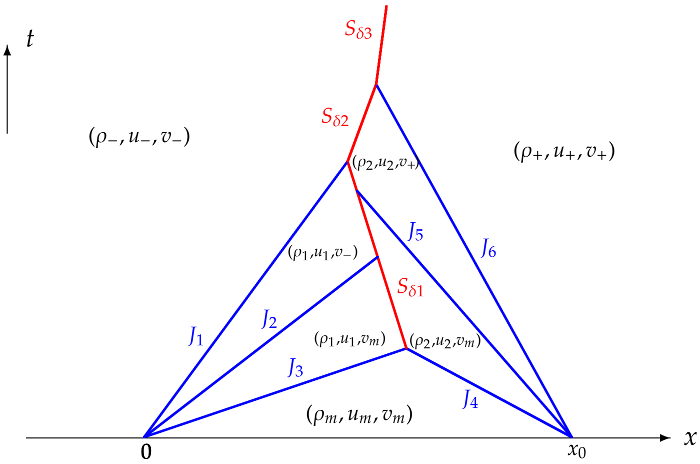
$$\sigma_2 - \sigma_{\delta 1} \ge \frac{1}{2}(u_m + \frac{1}{\rho_m}) + \frac{1}{2}(u_- - \frac{1}{\rho_-}) - (u_1 - \frac{1}{\rho_1}) = \frac{1}{2}(u_m + \frac{1}{\rho_m}) - \frac{1}{2}(u_- - \frac{1}{\rho_-}) > 0, \tag{3.42}$$

in which the relation $u_1 - \frac{1}{\rho_1} = u_- - \frac{1}{\rho_-}$ in (3.35) has been used. Analogously, taking into account the other relation $u_2 + \frac{1}{\rho_2} = u_+ + \frac{1}{\rho_+}$ in (3.35), it can also be derived from (3.39) and (3.41) that

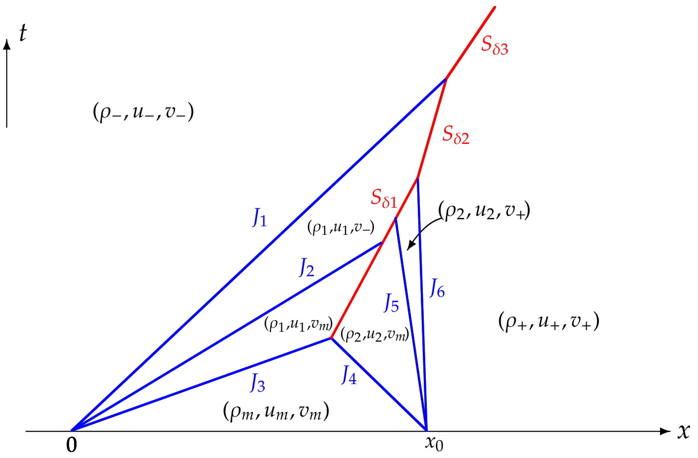
$$\sigma_{\delta 1} - \sigma_5 \ge u_2 + \frac{1}{\rho_2} - \frac{1}{2}(u_+ + \frac{1}{\rho_+}) - \frac{1}{2}(u_m - \frac{1}{\rho_m}) = \frac{1}{2}(u_+ + \frac{1}{\rho_+}) - \frac{1}{2}(u_m - \frac{1}{\rho_m}) > 0. \tag{3.43}$$

Thus we have $\sigma_2 > \sigma_{\delta 1} > \sigma_5$, which means that the delta shock wave $S_{\delta 1}$ will pass through J_2 and J_5 in finite time.

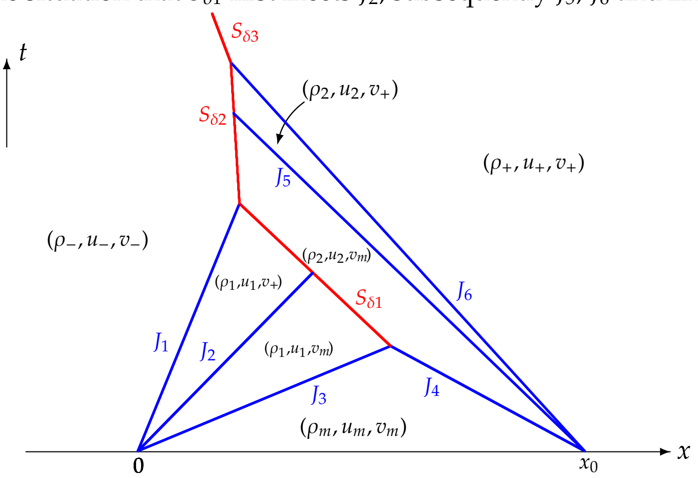
It is remarkable to notice that the delta shock wave cannot change its movement direction when it passes through J_2 and J_5 for the reason that the propagation of delta shock wave only depends on the state variables ρ and u which do not change when across J_2 and J_5. In fact, only the state variable v changes when the delta shock wave $S_{\delta 1}$ passes through J_2 and J_5. It is expected to know that $S_{\delta 1}$ first passes through J_2 or J_5 depending on the detailed choice of initial data (1.5). If $S_{\delta 1}$ does not change its direction before it passes through J_2, then the intersection point of $S_{\delta 1}$ and J_2 can be calculated by



(a) the situation that $S_{\delta 1}$ first meets J_2, subsequently J_5, J_1 and finally J_6



(b) the situation that $S_{\delta 1}$ first meets J_2, subsequently J_5, J_6 and finally J_1



(c) the situation that $S_{\delta 1}$ first meets J_2, subsequently J_1, J_5 and finally J_6

Fig.4 The interaction between two combinations of three contact discontinuities is displayed for three different situations when $u_m - \frac{1}{\rho_m} < u_+ + \frac{1}{\rho_+} < u_- - \frac{1}{\rho_-} < u_m + \frac{1}{\rho_m}$ is satisfied and furthermore the assumption that $S_{\delta 1}$ first intersects with J_2 is made.

$$\begin{cases} x_2 - x_1 = \sigma_{\delta 1}(t_2 - t_1), \\ x_2 = \sigma_2 t_2, \end{cases} \tag{3.44}$$

where (x_1, t_1) is given by (3.33), which yields

$$(x_2, t_2) = \Big(\frac{\sigma_2(x_1 - \sigma_{\delta 1} t_1)}{\sigma_2 - \sigma_{\delta 1}}, \frac{x_1 - \sigma_{\delta 1} t_1}{\sigma_2 - \sigma_{\delta 1}}\Big). \tag{3.45}$$

Similarly, if $S_{\delta 1}$ does not change its direction before it passes through J_5, then the intersection point of $S_{\delta 1}$ and J_5 can also be calculated by

$$\begin{cases} \bar{x}_2 - x_1 = \sigma_{\delta 1}(\bar{t}_2 - t_1), \\ \bar{x}_2 - x_0 = \sigma_5 \bar{t}_2, \end{cases} \tag{3.46}$$

which also enables us to have

$$(\bar{x}_2, \bar{t}_2) = \Big(x_0 + \frac{\sigma_5(x_1 - x_0 - \sigma_{\delta 1} t_1)}{\sigma_5 - \sigma_{\delta 1}}, \frac{x_1 - x_0 - \sigma_{\delta 1} t_1}{\sigma_5 - \sigma_{\delta 1}}\Big). \tag{3.47}$$

Thus, it can be concluded that $S_{\delta 1}$ first meets J_2 if $t_2 < \bar{t}_2$, otherwise $S_{\delta 1}$ first meets J_5 if $t_2 > \bar{t}_2$.

Without loss of generality, we assume that $t_2 < \bar{t}_2$, namely $S_{\delta 1}$ first meets J_2. With the similar discussion as above, we can also see that there are still three possible situations to occur. Let us draw Figure 4 to illustrate the three possible situations and take the situation in Figure 4(a) to give a detailed explanation. In Figure 4(a), the delta shock wave first passes through J_2 and subsequently passes through J_5 without changing its direction. Then, it meets J_1 at the intersection point (x_3, t_3) determined by

$$\begin{cases} x_3 - x_1 = \sigma_{\delta 1}(t_3 - t_1), \\ x_3 = \sigma_1 t_3, \end{cases} \tag{3.48}$$

in which $\sigma_1 = u_- - \frac{1}{\rho_-}$ and (x_1, t_1) is also given by (3.33), such that we have

$$(x_3, t_3) = \Big(\frac{\sigma_2(x_1 - \sigma_{\delta 1} t_1)}{\sigma_1 - \sigma_{\delta 1}}, \frac{x_1 - \sigma_{\delta 1} t_1}{\sigma_1 - \sigma_{\delta 1}}\Big). \tag{3.49}$$

The delta shock wave changes its direction when it passes through J_1. Let us use $S_{\delta 2}$ to denote the delta shock wave after interaction whose propagation speed is

$$\sigma_{\delta 2} = \frac{\rho_2 u_2 - \rho_- u_- + \sqrt{\rho_- \rho_2\Big((u_2 - u_-)^2 - (\dfrac{1}{\rho_2} - \dfrac{1}{\rho_-})^2\Big)}}{\rho_2 - \rho_-}. \tag{3.50}$$

Then, the delta shock wave $S_{\delta 2}$ will meet J_6 at the intersection point (x_4, t_4) determined by

$$\begin{cases} x_4 - x_3 = \sigma_{\delta 2}(t_4 - t_3), \\ x_4 - x_0 = \sigma_6 t_4, \end{cases} \tag{3.51}$$

in which $\sigma_6 = u_+ + \frac{1}{\rho_+}$ and (x_3, t_3) is given by (3.49), such that we have

$$(x_4, t_4) = \Big(x_0 + \frac{\sigma_6(x_3 - x_0 - \sigma_{\delta 2} t_3)}{\sigma_6 - \sigma_{\delta 2}}, \frac{x_3 - x_0 - \sigma_{\delta 2} t_3}{\sigma_6 - \sigma_{\delta 2}}\Big). \tag{3.52}$$

Finally, the delta shock wave is denoted with $S_{\delta 3}$ when it penetrates through J_6, whose propagation speed is

$$\sigma_{\delta 3}=\frac{\rho_+u_+-\rho_-u_-+\sqrt{\rho_-\rho_+\Big((u_+-u_-)^2-(\dfrac{1}{\rho_+}-\dfrac{1}{\rho_-})^2\Big)}}{\rho_+-\rho_-}. \tag{3.53}$$

In addition, the strength of delta shock wave can be calculated respectively by

$$\beta(t)=\sqrt{\rho_1\rho_2\Big((u_2-u_1)^2-(\frac{1}{\rho_2}-\frac{1}{\rho_1})^2\Big)}(t-t_1) \quad \text{for} \quad t_1<t\le t_3, \tag{3.54}$$

$$\beta(t)=\sqrt{\rho_-\rho_2\Big((u_2-u_-)^2-(\frac{1}{\rho_2}-\frac{1}{\rho_-})^2\Big)}(t-t_3)+\beta(t_3) \quad \text{for} \quad t_3<t\le t_4, \tag{3.55}$$

$$\beta(t)=\sqrt{\rho_-\rho_+\Big((u_+-u_-)^2-(\frac{1}{\rho_+}-\frac{1}{\rho_-})^2\Big)}(t-t_4)+\beta(t_4) \quad \text{for} \quad t>t_4. \tag{3.56}$$

On the other hand, if $t_2>\bar{t}_2$, namely $S_{\delta 1}$ first meets J_5, then there are also three possible situations to occur. The details are omitted here for the reason that the process of discussion is completely similar to the above one for $t_2<\bar{t}_2$.

Let us take Figure 2(b) as an example to illustrate the limit $x_0\to 0$ of solution to the perturbed Riemann problem (1.3) and (1.5). If the limit $x_0\to 0$ is taken, then it can be derived from (3.5) and (3.6) that $\lim_{x_0\to 0}(x_1,t_1)=(0,0)$ and $\lim_{x_0\to 0}\beta(t_1)=0$. Moreover, we can see that the intersection point of $\widetilde{J_3}$ and J_3 also tends to the origin as x_0 tends to zero. In addition, the delta contact discontinuity $\widetilde{J_{\delta 2}}$ is degenerated to be a contact discontinuity and coincides with $\widetilde{J_4}$ in the limit $x_0\to 0$ situation. Thus, we can conclude that the limit $x_0\to 0$ of solution to the perturbed Riemann problem (1.3) and (1.5) in Figure 2(b) is identical with the corresponding one to the Riemann problem (1.3) and (1.4) which is the combination of three contact discontinuities. With the similar discussion as above, we can see that if $u_--\frac{1}{\rho_-}<u_++\frac{1}{\rho_+}$, then the limit $x_0\to 0$ of solution for the perturbed Riemann problem (1.3) and (1.5) is also the combination of three contact discontinuities. On the other hand, if $u_--\frac{1}{\rho_-}>u_++\frac{1}{\rho_+}$, then only a delta shock wave is left when all the interactions have been finished. Thus, it can be concluded that the large-time asymptotic behaviors of solutions to the perturbed Riemann problem (1.3) and (1.5) are identical with the corresponding ones to the Riemann problem (1.3) and (1.4) for all the situations.

From the above detailed calculations and discussions for the perturbed Riemann problem (1.3) and (1.5), we can use the theorem to describe the main result of this paper as below.

Theorem 3.1. *If $u_--\frac{1}{\rho_-}<u_++\frac{1}{\rho_+}$, then the limit $x_0\to 0$ of solution to the perturbed Riemann problem (1.3) and (1.5) is the combination of three contact discontinuities. Otherwise, if $u_--\frac{1}{\rho_-}>u_++\frac{1}{\rho_+}$, then the limit $x_0\to 0$ of solution to the perturbed Riemann problem (1.3) and (1.5) is a delta shock wave. More precisely, the solutions to the perturbed Riemann problem (1.3) and (1.5) converge to the corresponding ones to the Riemann problem (1.3) and (1.4) when the limit $x_0\to 0$ is taken.*

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