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Iterative Methods for Solving Split Feasibility Problems and Fixed Point Problems in Banach Spaces

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Abstract. In this paper, we consider a class of split feasibility problems in Banach space. By using shrinking projective method and the modified proximal point algorithm, we propose an iterative algorithm. Under suitable conditions some strong convergence theorems are proved. Our results extend a recent result of Takahashi-Xu-Yao (Set-Valued Var. Anal. 23, 205-221 (2015)) from Hilbert spaces to Banach spaces. Moreover, the method of proof is also different.

1. Introduction

Many applications of the split feasibility problem (SFP), which was first introduced by Censor and Elfving [1], have appeared in various fields of science and technology, such as in signal processing, medical image reconstruction and intensity-modulated radiation therapy (for more information, see [2,3] and the references therein). In fact, Censor and Elfving [1] studied SFP in a finite-dimensional space, by considering the problem of finding a point

$$x^* \in C \text{ such that } Ax^* \in Q \tag{1.1}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$, and A is an $n \times n$ matrix. They introduced an iterative method for solving (SFP) (1.1).

On the other hand, variational inclusion problems are being used as mathematical programming models to study a large number of optimization problems arising in finance, economics, network, transportation and engineering science. The formal form of a variational inclusion problem is to find $x^* \in H$ such that

$$0 \in Bx^* \tag{1.2}$$

where $B : H \to 2^H$ is a set-valued operator. If B is a maximal monotone operator, the elements in the solution set of the problem (1.2) are called the zero of maximal monotone operator. This problem was introduced by Martinet [4], and later it has been studied by many authors.

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It is well known that the popular iteration method that was used for solving the problem (1.2) is the following proximal point algorithm: for a given $x \in H$,

$$x_{n+1} = J^B_{\lambda_n} x_n, \forall n \in \mathbb{N},$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ is the resolvent of the considered maximal monotone operator B corresponding to λ_n (see, also [5 − 9]) for more details).

A related topic to the above variational inclusion problem is fixed point theory which has been a very powerful and important tool in the study of mathematical models. Of course, many authors were interested in and studied the approximating of a fixed point of nonlinear mappings by using iterative methods, and applied the obtained results to many important problems, such as the null point problem,variational inequality problem, optimization problems.

For solving the (SFP) and the fixed point problem, Takahashi et al. [10] considered the problem of finding a point $x^* \in H$ such that

$$0 \in Bx^*, \textit{ and } Lx^* \in Fix(T) \tag{1.3}$$

where $B : H_1 \to 2^{H_1}$ is a maximal monotone operator, $L : H_1 \to H_2$ is a bounded linear operator and $T : H_2 \to H_2$ is a nonexpansive mapping. They considered the following iterative algorithm: for any $x_1 \in H_1$,

$$x_{n+1} = J^B_{\lambda_n}(I - \gamma_n L^*(I - T)Lx_n), n \geq 1, \tag{1.4}$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ satisfy some suitable conditions, and $J^B_{\lambda_n}$ is the resolvent of a maximal monotone operator B associated to $\{\lambda_n\}$. They proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to a point $x^* \in B^{-1}(0) \bigcap L^{-1}Fix(T)$.

Motivated and inspired by [10], the purpose of this paper is to continue to study the problem (1.3) in Banach space. By using shrinking projective method and the modified proximal point algorithm, we propose an iterative algorithm. Under suitable conditions, some strong convergence theorems are proved. Our results extend the corresponding results in Takahashi et al. [10] from Hilbert spaces to Banach spaces. Moreover, the method of proof adopted in this paper is different from that one in [10].

2. Preliminaries

Throughout this article, we assume that the Banach space are real. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let E be a Banach space and E^* be the topological dual of E. For all $x \in E$ and $x^* \in E^*$, we denote by $\langle x, x^* \rangle$ the value of x^* at x. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \ x \in E \tag{2.1}$$

is called the normalized duality mapping.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$ with $x \neq y$. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}, \tag{2.2}$$

for all $\epsilon \in [0, 2]$. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. A Hilbert space is 2-uniformly convex.

The modulus of smoothness of E: $\rho_E : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in U, \|y\| \leq t\}. \tag{2.3}$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. A typical example of uniformly smooth Banach space is L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$. Let q be a fixed real number with $q > 1$, then a Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$. It is well known that every q-uniformly smooth Banach space is uniformly smooth.

It is also well known that if E is a uniformly smooth Banach space, then J is uniformly continuous from norm to norm on each bounded subset of E.

Let C be a nonempty, closed, and convex subset of a strictly convex and reflexive Banach E, then the metric projection

$$P_C x = \arg\min_{y \in C} \|x - y\|, x \in E, \tag{2.4}$$

is the unique minimizer of the norm distance.

Let E be a smooth, reflexive, and strictly convex Banach space. Consider the functional defined by [11, 12]

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \forall x, y \in E, \tag{2.5}$$

where J is the normalized duality mapping. It is clear that in a Hilbert space H, (2.5) reduces to $\phi(x, y) = \|x - y\|^2, \forall x, y \in H$.

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \forall x, y \in E, \tag{2.6}$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z), \forall x, y \in E. \tag{2.7}$$

Following Alber [11], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C x = \arg\min_{y \in C} \phi(y, x), \forall x \in E, \tag{2.8}$$

That is, $\Pi_C(x) = \overline{x}$, where $\overline{x}$ is the unique solution to the minimization problem $\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x)$. The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J(see, e.g., [11 − 14]). In Hilbert space H, $\Pi_C = P_C$.

Let $T : E \to E$ be a mapping. we say that T is nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in E. \tag{2.9}$$

A set-valued mapping $A : E \to 2^{E^*}$ is said to be monotone if for any $x, y \in E$ and any $x^* \in Ax$, $y^* \in Ay$, we have

$$\langle x^* - y^*, x - y\rangle \geq 0. \tag{2.10}$$

A monotone operator A is said to be maximal if its $Gr(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in graph of any other monotone operator.

If E is a strictly convex, reflexive, and smooth Banach space, and $B : E \to 2^{E^*}$ is a maximal monotone operator then, for any positive real number λ, we can define a single-valued mapping $J_\lambda^B : E \to E$ by

$$J_\lambda^B(x) = (J + \lambda B)^{-1} J(x), \ x \in E. \tag{2.11}$$

This mapping is called the resolvent of B for $\lambda > 0$.

It is know that $B^{-1}(0) = F(J_\lambda^B)$ for every $\lambda > 0$ and $B^{-1}(0)$ is a closed and convex subset of E. For more details, see [15-17].

Lemma 2.1 [18] Let E be a strictly convex and reflexive smooth Banach space. Let $B : E \to 2^{E^*}$ be a maximal monotone operator and J_λ^B be the resolvent of B for $\lambda > 0$, then

$$\phi(u, J_\lambda^B x) \leq \phi(u, x)$$

for any $u \in B^{-1}(0)$ and $x \in E$.

Lemma 2.2[14] Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.

Lemma 2.3[12] Let E be a smooth, strictly convex, and reflective Banach space and C be nonempty closed convex subset of E. Then, the following conclusions hold:
(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y),\ \forall x \in C, y \in E$.
(2) If $x \in E$ and $z \in C$, then $z = \Pi_C x$ iff $\langle z - y, Jx - Jz\rangle \geq 0, \forall y \in C$.
(3) For $x,\ y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Lemma 2.4[17] Let E be a smooth, strictly convex, and reflective Banach space. Let C be a nonempty, closed, and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conclusions hold:
(1) $z = P_C x_1$.
(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \forall y \in C$.

Lemma 2.5[19] For a given number $r > 0$, a real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [o, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g\|x - y\|,$$

for all $x, y \in E$ with $\|x\| \leq r$ and $\|y\| \leq r$, and $t \in [0, 1]$

Lemma 2.6[19] Let E be a 2-uniformly smooth Banach space with the best smoothness constants $\kappa > 0$. Then, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|\kappa y\|^2, \forall x, y \in E.$$

3. The Main Results

Theorem 3.1 Let E_1 be a real 2-uniformly convex and 2-uniformly smooth Banach space with the best smoothness constant $\kappa > 0$. Let E_2 be a smooth strictly convex and reflective Banach space. Let $B : E_1 \longrightarrow 2^{E_1^*}$ be a maximal monotone operator, $L : E_1 \longrightarrow E_2$ be a bounded linear operator with adjoint L^*, and $T : E_2 \longrightarrow E_2$ be a nonexpansive mappings. Let $x_1 \in E_1$, $C_1 = E_1$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_1^{-1}(J_1 x_n + \gamma L^* J_2(P_{F(T)} - I)Lx_n, n \geq 1, \\ y_n = J_1^{-1}[(1 - \alpha_n)J_1 z_n + \alpha_n J_1 J_\lambda^B z_n], \\ C_{n+1} = \{\nu \in C_n : \phi(\nu, y_n) \leq \phi(\nu, x_n); \phi(\nu, z_n) \leq \phi(\nu, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases} \tag{3.1}$$

where $J_\lambda^B = (J_1 + \lambda B)^{-1} J_1$. If $Q = \{p \in B^{-1}(0) : Lp \in F(T)\} \neq \emptyset$ and the following conditions are satisfied:
(1) $\alpha_n \in [\delta, 1), \delta > 0$,
(2) $0 < \gamma < \frac{1}{\kappa^2\|L\|^2}$,
then $\lim_{n\to\infty} x_n = x^* = \Pi_Q x_1$ which is a solution of problem (1.3).

Proof We shall divide the proof into five steps.

Step 1 First we show that C_n is a closed and convex subset for each $n \geq 1$.

Since $C_1 = E_1$, C_1 is closed and convex. Assume that C_n is closed and convex for some $n \geq 2$. Next, by induction, we prove that C_{n+1} is also closed and convex. In fact, it follows from (3.1) that for any $\nu \in C_n$, we have

$$\phi(\nu, y_n) \leq \phi(\nu, x_n) \Longleftrightarrow 2\langle \nu, Jx_n - Jy_n\rangle \leq \|x_n\|^2 - \|y_n\|^2, \tag{3.2}$$

$$\phi(v, z_n) \leq \phi(v, x_n) \Longleftrightarrow 2\langle v, Jx_n - Jz_n\rangle \leq \|x_n\|^2 - \|z_n\|^2. \tag{3.3}$$

These imply that C_{n+1} is a closed and convex subset of E_1.

Step 2 Now we prove that $Q \subseteq C_n$, for all $n \geq 1$.

Let $p \in Q$. By the assumption that E_1 is a 2-uniformly convex and 2-uniformly smooth real Banach space, so, E_1^* is 2-uniformly smooth real Banach space and $J_1 = (J_1^*)^{-1}$. It follows from (3.1) and Lemma 2.6 that

$$\begin{aligned}\phi(p, z_n) &= \phi(p, J_1^{-1}(J_1x_n + \gamma L^*J_2(P_{F(T)} - I)Lx_n)\\ &= \|p\|^2 - 2\langle p, J_1x_n + \gamma L^*J_2(P_{F(T)} - I)Lx_n\rangle\\ &\quad+\|J_1x_n + \gamma L^*J_2(P_{F(T)} - I)Lx_n\|^2\\ &\leq \|p\|^2 - 2\langle p, J_1x_n\rangle - 2\langle p, \gamma L^*J_2(P_{F(T)} - I)Lx_n\rangle + \|x_n\|^2\\ &\quad+2\gamma\langle Lx_n, J_2(P_{F(T)} - I)Lx_n\rangle + 2k^2\|L\|^2\gamma^2\|(P_{F(T)} - I)Lx_n\|^2\\ &\leq \phi(p, x_n) - 2\gamma\langle Lp - Lx_n, J_2(P_{F(T)} - I)Lx_n\rangle\\ &\quad+2k^2\|L\|^2\gamma^2\|(P_{F(T)} - I)Lx_n\|^2.\end{aligned} \tag{3.4}$$

Furthermore from Lemma 2.4 we have that

$$\begin{aligned}\langle Lp - Lx_n,\ & J_2(P_{F(T)} - I)Lx_n\rangle\\ &= \langle Lp - P_{F(T)}Lx_n + P_{F(T)}Lx_n - Lx_n, J_2(P_{F(T)} - I)Lx_n\rangle\\ &= \langle Lp - P_{F(T)}Lx_n, J_2(P_{F(T)} - I)Lx_n\rangle\\ &\quad+\langle(P_{F(T)} - I)Lx_n, J_2(P_{F(T)} - I)Lx_n\rangle\\ &= \langle Lp - P_{F(T)}Lx_n, J_2(P_{F(T)} - I)Lx_n\rangle + \|(P_{F(T)} - I)Lx_n\|^2\\ &\geq \|(P_{F(T)} - I)Lx_n\|^2.\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4), and by using the condition (2) to simplify, we have

$$\phi(p, z_n) \leq \phi(p, x_n) - 2\gamma(1 - k^2\|L\|^2\gamma)\|(P_{F(T)} - I)Lx_n\|^2 \leq \phi(p, x_n). \tag{3.6}$$

Furthermore, from Lemma 2.1, Lemma 2.5, (3.1) and (3.6), we obtain

$$\begin{aligned}\phi(p, y_n) &= \phi(p, J_1^{-1}[(1 - \alpha_n)J_1z_n + \alpha_nJ_1J_\lambda^Bz_n])\\ &= \|p\|^2 - 2\langle p, (1 - \alpha_n)J_1z_n + \alpha_nJ_1J_\lambda^Bz_n\rangle\\ &\quad+\|(1 - \alpha_n)J_1z_n + \alpha_nJ_1J_\lambda^Bz_n\|^2\\ &\leq \|p\|^2 - 2(1 - \alpha_n)\langle p, J_1z_n\rangle - 2\alpha_n\langle p, J_1J_\lambda^Bz_n\rangle\\ &\quad+(1 - \alpha_n)\|z_n\|^2 + \alpha_n\|J_\lambda^Bz_n\|^2 - \alpha_n(1 - \alpha_n)g(\|J_1z_n - J_1J_\lambda^Bz_n\|)\\ &= (1 - \alpha_n)\phi(p, z_n) + \alpha_n\phi(p, J_\lambda^Bz_n)\\ &\quad-\alpha_n(1 - \alpha_n)g(\|J_1z_n - J_1J_\lambda^Bz_n\|)\\ &\leq (1 - \alpha_n)\phi(p, z_n) + \alpha_n\phi(p, z_n) - \alpha_n(1 - \alpha_n)g(\|J_1z_n - J_1J_\lambda^Bz_n\|)\\ &\leq (1 - \alpha_n)\phi(p, x_n) + \alpha_n(p, x_n) - \alpha_n(1 - \alpha_n)g(\|J_1z_n - J_1J_\lambda^Bz_n\|)\\ &\leq \phi(p, x_n).\end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that $p \in C_{n+1}$. This implies that $Q \subseteq C_n$ for all $n \geq 1$.

Therefore, $\Pi_{C_{n+1}}x_1$ is well defined.

Step 3 Now we prove that $\{x_n\}$ is a Cauchy sequence.

Let $p \in Q$, by the definition of C_n, We have $x_n = \Pi_{C_n}x_1$ for all $n \geq 1$. Hence It follows from Lemma 2.3 that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \leq \phi(p, x_1) - \phi(p, \Pi_{C_n}x_1) \leq \phi(p, x_1), \forall n \geq 1. \tag{3.8}$$

This implies that $\{\phi(x_n, x_1)\}$ is bounded. In addition, since $x_n = \Pi_{C_n}x_1$ and

$$x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subseteq C_n,$$

we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \forall n \geq 1. \tag{3.9}$$

Therefore, $\{\phi(x_n, x_1)\}$ is nondecreasing and bounded. So the limit $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. Hence from Lemma 2.3, we have

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x_1) \le \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) = \phi(x_{n+1}, x_1) - \phi(x_n, x_1), \tag{3.10}$$

which implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.11}$$

This together with Lemma 2.2 shows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

For some positive integers m , n with $m \le n$, it follows from $x_n = \prod_{C_n} x_1 \subseteq C_m$ and Lemma 2.2 that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \textstyle\prod_{C_n} x_1) \\ &\le \phi(x_m, x_1) - \phi(\textstyle\prod_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned} \tag{3.13}$$

Since $\lim_{n\to\infty} \phi(x_n, x_1)$ exists, it follows from (3.13) and Lemma 2.2 that $\lim_{n\to\infty} \|x_n - x_m\| = 0$. Therefore, $\{x_n\}$ is a cauchy sequence.

Step 4 Now we prove that $\lim_{n\to\infty} \|z_n - J_\lambda^B z_n\| = 0$ and $\lim_{n\to\infty} \|(P_{F(T)} - I)Lx_n\| = 0$.
Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subseteq C_n$, by the definition of C_{n+1} and (3.11), we have

$$\begin{aligned} &\phi(x_{n+1}, z_n) \le \phi(x_{n+1}, x_n) \to 0 (as\ n \to \infty), \\ &\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) \to 0, (as\ n \to \infty). \end{aligned} \tag{3.14}$$

By Lemma 2.2 , we have

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0 \ \ and \ \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \tag{3.15}$$

and so

$$\lim_{n\to\infty} \|y_n - z_n\| = 0 \ \ and \ \lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.16}$$

Furthermore, it follows from (3.6) that

$$\begin{aligned} 2\gamma(1- \ & k^2\|L\|^2\gamma)\|(P_{F(T)} - I)Lx_n\|^2 \le \phi(p, x_n) - \phi(p, z_n) \\ &= \|p\|^2 - 2\langle p, J_1 x_n\rangle + \|x_n\|^2 - \|p\|^2 + 2\langle p, J_1 z_n\rangle - \|z_n\|^2 \\ &= 2\langle p, J_1 z_n - J_1 x_n\rangle + \|x_n\|^2 - \|z_n\|^2 \\ &\le 2\|p\| \cdot \|J_1 z_n - J_1 x_n\| + \|x_n - z_n\| \cdot (\|x_n\| + \|z_n\|). \end{aligned} \tag{3.17}$$

Since E_1 is a 2-uniformly convex and 2-uniformly smooth real Banach space, J_1 is uniformly continuous from norm-to-norm. By (3.16), we have $\|J_1 z_n - J_1 x_n\| \to 0$. Also, from (3.17) and the condition (2), we have

$$\lim_{n\to\infty} \|(P_{F(T)} - I)L(x_n)\| = 0. \tag{3.18}$$

Again from (3.1) we have

$$\alpha_n \|J_1 J_\lambda^B z_n - J_1 z_n\| = \|J_1 y_n - J_1 z_n\|. \tag{3.19}$$

Hence from (3.16) and condition (1) we have

$$\lim_{n\to\infty} \|J_1 J_\lambda^B z_n - J_1 z_n\| = 0, \tag{3.20}$$

which implies that

$$\lim_{n\to\infty} \|J_\lambda^B z_n - z_n\| = 0. \tag{3.21}$$

Step 5 Now we prove that $\lim_{n\to\infty} x_n = x^* = \Pi_Q x_1$.

In fact, since $\{x_n\}$ is a cauchy sequence, there exists $x^* \in E_1$ such that $\lim_{n\to\infty} x_n = x^*$. It follows from (3.16) and (3.21) that

$$\lim_{n\to\infty} z_n = x^*, \lim_{n\to\infty} J_\lambda^B z_n = x^*. \tag{3.22}$$

Since $J_\lambda^B = (J_1 + \lambda B)^{-1} J_1$, we have that $\frac{J_1 z_n - J_1 J_\lambda^B z_n}{\lambda} \in B J_\lambda^B z_n$, for all $n \in \mathbb{N}$. From the monotonicity of B, we have that

$$0 \le \langle u - J_\lambda z_n, v^* - \frac{J_1 z_n - J_1 J_\lambda z_n}{\lambda} \rangle,$$

for all $(u, v^*) \in Gr(B)$. Taking $n \to \infty$, we have from (3.21) and (3.22) that $0 \le \langle u - x^*, v^* - 0\rangle$ for all $(u, v^*) \in Gr(B)$. Since B is maximal monotone, we have $x^* \in B^{-1}(0)$. In addition, from Lemma 2.4 , we obtain

$$\begin{aligned} \|(I - P_{F(T)})Lx^*\|^2 &= \langle J_2(Lx^* - P_{F(T)}(Lx^*)), Lx^* - P_{F(T)}(Lx^*)\rangle \\ &= \langle J_2(Lx^* - P_{F(T)}(Lx^*)), Lx^* - Lx_n + Lx_n \\ &\quad - P_{F(T)}(Lx_n) + P_{F(T)}(Lx_n) - P_{F(T)}(Lx^*)\rangle \\ &= \langle J_2(Lx^* - P_{F(T)}(Lx^*)), Lx^* - Lx_n\rangle \\ &\quad + \langle J_2(Lx^* - P_{F(T)}(Lx^*)), Lx_n - P_{F(T)}(Lx_n)\rangle \\ &\quad + \langle J_2(Lx^* - P_{F(T)}(Lx^*)), P_{F(T)}(Lx_n) - P_{F(T)}(Lx^*)\rangle \\ &\le \langle J_2(Lx^* - P_{F(T)}(Lx^*)), Lx^* - Lx_n\rangle \\ &\quad + \langle J_2(Lx^* - P_{F(T)}(Lx^*)), Lx_n - P_{F(T)}(Lx_n)\rangle. \end{aligned} \tag{3.23}$$

Since L is a bounded linear operator, we have that $\lim_{n\to\infty} \|Lx_n - Lx^*\| = 0$. Hence by (3.18) we get $\|(I - P_{F(T)})(Lx^*)\| = 0$. This implies that $Lx^* \in F(T)$. Therefore we have $x^* \in Q$.

Let $z = \Pi_Q x_1, z \in Q$. From $x_n = \Pi_{C_n} x_1$ and $z \in Q \subseteq C_n$, we have

$$\phi(x_n, x_1) \le \phi(z, x_1). \tag{3.24}$$

This implies that

$$\phi(x^*, x_1) \le \lim_{n\to\infty} \phi(x_n, x_1) \le \phi(z, x_1). \tag{3.25}$$

By the definition of $z = \Pi_Q x_1$, we have $x^* = z$. Therefore, $\lim_{n\to\infty} x_n = x^* = \Pi_Q x_1$.

The proof of Theorem 3.1 is completed.

The following result can be obtained from Theorem 3.1 immediately.

Corollary 3.2 Let H_1 and H_2 be two real Hilbert space. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone operator, $L : H_1 \to H_2$ be a bounded linear operator with adjoint L^*, $T : H_2 \to H_2$ be a nonexpansive mapping.

Let $x_1 \in H$ and $C_1 = H_1$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n + \gamma L^*(P_{F(T)} - I)Lx_n, n \ge 1, \\ y_n = (1 - \alpha_n)z_n + \alpha_n J_\lambda^B z_n \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|; \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases} \tag{3.26}$$

where $P_{F(T)}$ is the metric projection of H_2 onto $F(T)$ and $P_{C_{n+1}}$ is the metric projection of H_1 onto C_{n+1}, $J_\lambda^B = (I + \lambda B)^{-1}$, for $\lambda > 0$. If $Q = \{p \in B^{-1}(0) : Lp \in F(T)\} \neq \emptyset$ and the following conditions are satisfied:

(1) $\alpha_n \in [\delta, 1), \delta > 0$;

(2) $0 < \gamma < \frac{1}{\|L\|^2}$.

then $\lim_{n\to\infty} x_n = x^* = P_Q x_1$.

Remark 3.3 Theorem 3.1 generalizes the main results in [10] from Hilbert space to Banach space and the weak convergence of the iterative extends to strong convergence. Moreover, the method of proof is different from that one in [10].

4. Application to split null point problem

Let H_1, H_2 be two real Hilbert spaces. Let $B_1 : H_1 \longrightarrow 2^{H_1}$ and $B_2 : H_2 \longrightarrow 2^{H_2}$ be two set-valued maximal monotone mappings and $L : H_1 \longrightarrow H_2$ be a bounded linear operators. The "so-called" (SNPP) is to find a point $x^* \in H_1$, such that

$$0 \in B_1(x^*) \text{ and } 0 \in B_2(Lx^*). \tag{4.1}$$

For solving problem (4.1), Byrne et al. [20] proposed the following iterative algorithm, for $\lambda > 0$ and an arbitrary $x_1 \in H_1$,

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma L^*(I - J_\lambda^{B_2})Lx_n), \forall n \in \mathbb{N}$$

where L^* is the adjoint of L, $\gamma \in (0, \frac{2}{\|L\|^2})$, $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are the resolvents of maximal monotone operators B_1 and B_2, respectively. Under suitable control conditions, they proved that $\{x_n\}$ converges weakly to a point x^* which is a solution of problem (4.1).

Since $B_2 : H_2 \to 2^{H_2}$ is a maximal monotone operator, it is well know that its resolvent operator $J_\lambda^{B_2} = (I + \lambda B_2)^{-1}$ is nonexpansive and $B_2^{-1}(0) = F(J_\lambda^{B_2})$. Then problem (4.1) is equivalent to find a point $x^* \in H$ such that

$$0 \in B_1(x^*) \text{ and } Lx^* \in B_2^{-1}(0) = F(J_\lambda^{B_2}). \tag{4.2}$$

Then the following result can be obtained from Corollary 3.2 immediately.

Theorem 4.1 Let $B_1 : H_1 \longrightarrow 2^{H_1}$ and $B_2 : H_2 \longrightarrow 2^{H_2}$ be maximal monotone operators, and $L : H_1 \longrightarrow H_2$ be a bounded linear operator with adjoint L^*. Let $x_1 \in H_1$ and $C_1 = H_1$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n + \gamma L^*(P_{F(J_\lambda^{B_2})} - I)Lx_n, n \geq 1, \\ y_n = (1 - \alpha_n)z_n + \alpha_n J_\lambda^{B_1} z_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|; \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases} \tag{4.3}$$

where $P_{F(J_\lambda^{B_2})}$ is the metric projection of H_2 onto $P_{F(J_\lambda^{B_2})}$ and $P_{C_{n+1}}$ is the metric projection of H_1 onto C_{n+1}, $J_\lambda^{B_1} = (I + \lambda B_1)^{-1}$, $J_\lambda^{B_2} = (I + \lambda B_2)^{-1}$ for $\lambda > 0$. If $Q = \{p \in B_1^{-1}(0) : Lp \in B_2^{-1}(0)\} \neq \emptyset$ and the following conditions are satisfied:

(1) $\alpha_n \in [\delta, 1), \delta > 0$;

(2) $0 < \lambda < \frac{1}{\|L\|^2}$.

Then the sequence $\{x_n\}$ converges strongly to a point $x^* = P_Q x_1$, which is a solution of problem (4.2).

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