Filomat 33:16 (2019), 5331–5343
https://doi.org/10.2298/FIL1916331S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Solution to Nonlinear Volterra Integro-Dynamic Equations via Fixed Point Theory

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Abstract. In this paper we discuss the existence and uniqueness of solutions of a certain type of nonlinear Volterra integro-dynamic equations on time scales. We investigate the problem in the setting of a complete b-metric space and apply a fixed point theorem with a contractive condition involving b-comparison function. We use the theorem to show the existence of a unique solution of some particular integro-dynamic equations.

1. Introduction and Preliminaries

Many problems in science and engineering are modeled by differential or integral equations and in some cases by integro-differential equations. The integro-differential equations contain both derivatives and integrals of the unknown function. On the other hand, the studies related with unification of continuous and discrete problems, in other words, problems defined on time scales have gained a lot of attention recently. Accordingly, the integro-differential equations have been generalized on an arbitrary time scale as integro-dynamic equations [15].

The problem of existence and uniqueness of solutions of differential, difference, and integral equations is one of the main application areas of the fixed point theory. In addition, the problems involving fractional derivatives are studied as fixed point problems [1]. The question of existence and uniqueness of integro-differential equations has also been investigated by many authors, see e.g. [16, 22, 23]. However, there are only few studies regarding the existence and uniqueness of solutions of integro-dynamic equations.

We start with a brief introduction of the basic concepts on time scales.

Definition 1.1. *([10],[13])*

1. *A time scale is an arbitrary nonempty closed subset of the real numbers. A time scale is usually denoted by the symbol* $\mathbb{T}$.
2. *For* $t \in \mathbb{T}$ *the forward jump operator* $\sigma : \mathbb{T} \longmapsto \mathbb{T}$ *is defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

2010 *Mathematics Subject Classification*. 45B05; 47H10,54H25
Keywords. integro-dynamic equation, fixed point, comparison function, time scales
Received: 28 March 2019; Revised: 20 July 2019; Accepted: 07 September 2019
Communicated by Naseer Shahzad
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3. *For $t \in \mathbb{T}$ the backward jump operator $\rho : \mathbb{T} \longmapsto \mathbb{T}$ is defined as*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

4. *We set*

$$\inf \emptyset = \sup \mathbb{T}, \quad \sup \emptyset = \inf \mathbb{T}.$$

Remark 1.2. *It is easy to see that for any $t \in \mathbb{T}$ we have $\sigma(t) \geq t$ and $\rho(t) \leq t$.*

Let $\mathbb{T}$ be a time scale with forward jump operator and backward jump operator σ and ρ, respectively.

Definition 1.3. *([10],[13]) We define the set*

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} \quad \text{otherwise.} \end{cases}$$

Definition 1.4. *([10],[13]) The graininess function $\mu : \mathbb{T} \longmapsto [0, \infty)$ is defined as*

$$\mu(t) = \sigma(t) - t.$$

Definition 1.5. *([10],[13]) Let $f : \mathbb{T} \longmapsto \mathbb{R}$ be a function and let $t \in \mathbb{T}^{\kappa}$. We define $f^{\Delta}(t)$ to be the number, provided it exists, as follows: for any $\epsilon > 0$ there is a neighborhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s| \quad \text{for} \quad \text{all} \quad s \in U, \quad s \neq \sigma(t).$$

$f^{\Delta}(t)$ is called the delta or Hilger derivative of f at t.

f is delta or Hilger differentiable or shortly, differentiable, in T^{κ} if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

The function $f^{\Delta} : \mathbb{T} \longmapsto \mathbb{R}$ is said to be delta derivative or Hilger derivative or shortly, the derivative, of f in T^{κ}.

Remark 1.6. *If $\mathbb{T} = \mathbb{R}$, then the delta derivative coincides with the classical derivative.*

Note that the delta derivative is well-defined. For the properties of the delta derivative we refer the reader to [10] and [13].

It should be mentioned that there is another type of derivative defined on time scales known as nabla derivative. Its definition reads as follows.

Definition 1.7. *([10]) Let $f : \mathbb{T} \longmapsto \mathbb{R}$ be a function and let $t \in \mathbb{T}_{\kappa}$, where $\mathbb{T}_{\kappa} = \mathbb{T} - \{m\}$, if $\mathbb{T}$ has right-scattered minimum m and $\mathbb{T}_{\kappa} = \mathbb{T}$ otherwise. We define $f^{\nabla}(t)$ to be the number, provided it exists, as follows: for any $\epsilon > 0$ there is a neighborhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that*

$$|f(\rho(t)) - f(s) - f^{\nabla}(\rho(t) - s)| \leq \epsilon |\rho(t) - s| \quad \text{for} \quad \text{all} \quad s \in U, \quad s \neq \rho(t).$$

$f^{\nabla}(t)$ is called the nabla derivative of f at t.

We refer the readers to some very recent studies related with these two types of derivatives of fractional order [2, 3]. In this study, we only deal with the delta derivative.

Definition 1.8. *([10],[13]) A function $f : \mathbb{T} \longmapsto \mathbb{R}$ is called regulated provided that its right-sided limits exist(finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at all left-dense points in $\mathbb{T}$.*

Definition 1.9. *A continuous function $f : \mathbb{T} \longmapsto \mathbb{R}$ is called pre-differentiable with region of differentiation D, provided that*

1. *$D \subset \mathbb{T}^{\kappa}$,*
2. *$\mathbb{T}^{\kappa} \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$,*
3. *f is differentiable at each $t \in D$.*

Theorem 1.10. *([10], [13]) Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}$, $f : \mathbb{T}^\kappa \longmapsto \mathbb{R}$ be a given regulated map. Then there exists exactly one pre-differentiable function F satisfying*

$$F^\Delta(t) = f(t) \quad \text{for} \quad \text{all} \quad t \in D, \quad F(t_0) = x_0.$$

Definition 1.11. *([10],[13]) Assume that $f : \mathbb{T} \longmapsto \mathbb{R}$ is a regulated function. Any function F by Theorem 1.10 is called a pre-antiderivative of f. The indefinite integral of the regulated function f is defined as*

$$\int f(t)\Delta t = F(t) + c,$$

where c is an arbitrary constant and F is a pre-antiderivative of f. The Cauchy integral is defined as

$$\int_\tau^s f(t)\Delta t = F(s) - F(\tau) \quad \text{for} \quad \text{all} \quad \tau, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \longmapsto \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \longmapsto \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad \text{holds} \quad \text{for} \quad \text{all} \quad t \in \mathbb{T}^\kappa.$$

Finally, we give the definition and some properties of the monomials on time scales.

Definition 1.12. *([13, 14]) Monomials on time scales are defined recursively as follows.*

$$\begin{aligned} h_0(t,\alpha) &= 1, \\ h_1(t,\alpha) &= \int_\alpha^t h_0(t,\alpha)\Delta t = t - \alpha, \\ h_k(t,\alpha) &= \int_\alpha^t h_{k-1}(t,\alpha)\Delta t, \end{aligned} \tag{1}$$

for $k \geq 2$.

A useful property related with the upper bound of time scale monomials is given in [14].

Theorem 1.13. *[14] For each $k \in \mathbb{N}_0$ the inequality*

$$h_k(t,\alpha) \leq \frac{(t-\alpha)^k}{k!} \tag{2}$$

holds for each $t, \alpha \in \mathbb{T}$, $t \geq \alpha$.

For detailed information on basic calculus on time scales we refer the reader to [10] and [13]. In what follows, we define the Volterra integro-dynamic equation of the second kind which will be discussed in the next section.

Let $\mathbb{T}$ be a time scale with delta differential operator Δ and forward jump operator σ, respectively.

Definition 1.14. *A Volterra integro-dynamic equation of the second kind is given as*

$$\phi^{\Delta^n}(x) = u(x) + \int_{x_0}^x F(s, x, \sigma(s), \sigma(x), \phi(s))\Delta s, \quad x \in [x_0, A]_\mathbb{T}. \tag{3}$$

Here $u : [x_0, A]_\mathbb{T} \to \mathbb{R}$ and $F : ([x_0, A]_\mathbb{T})^4 \times \mathbb{R} \to \mathbb{R}$ are given functions, and ϕ is the unknown function.

In this study, we consider the special case where the equation (3) contains a first order Δ-derivative, that is, an equation of the form

$$\phi^{\Delta}(x) = u(x) + \int_{x_0}^{x} F(s, x, \sigma(s), \sigma(x), \phi(s))\Delta s, \quad x \in [x_0, A]_{\mathbb{T}}. \tag{4}$$

We will investigate the existence and uniqueness of the solution of the equation (4) on a b-metric space. The concepts of b-metric and b-metric spaces have been thoroughly employed in connection with fixed point theory and its applications. We first recall the definition of b-metric space.

Definition 1.15. *Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function satisfying*

1. $d(x, y) = 0$ *if and only if* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$,

for all $x, y, z \in X$ and some constant $s > 1$. Then the function d and the space (X, d) are called a b-metric and a b-metric space with a constant s, respectively.

For a detailed overview on the subject we refer the reader to [7, 12].

The theoretical ground of our application is an existence-uniqueness theorem for contractive mappings defined on b-metric spaces via b-comparison functions. Berinde [8] and Rus [20] defined first the comparison functions and later the b-comparison functions in order to generalize the Banach contraction mapping principle. Below we briefly recall the basic notions on comparison and b-comparison functions.

Definition 1.16. *([8],[9],[20])*

1. *Comparison function is an increasing mapping $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the condition $\varphi^n(t) \to 0$, as $n \to \infty$ for any $t \in [0, \infty)$ where φ^n is the n-th iterate of φ.*
2. *For a real number $s \geq 1$ a b-comparison function is a function $\varphi_b : [0, +\infty) \to [0, +\infty)$ satisfying the conditions*
   (b_1) *φ_b is increasing,*
   (b_2) *there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi_b^{k+1}(t) \leq a s^k \varphi_b^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.*

In the sequel, we denote the class of b-comparison functions by Φ_b. Obviously, every b-comparison function is a comparison function.

We will need the following essential properties in our further discussion.

Lemma 1.17. *([8],[20]) If $\varphi : [0, +\infty) \to [0, +\infty)$ is a comparison function (or a b-comparison function), then the following hold:*

(1) *each iterate φ^k of φ $k \geq 1$, is also a comparison (b-comparison) function;*
(2) *φ is continuous at 0;*
(3) *$\varphi(t) < t$, for any $t > 0$.*

Lemma 1.18. *[9] For a b-comparison function $\varphi_b : [0, +\infty) \to [0, +\infty)$ the following hold:*

(1) *the series $\sum_{k=0}^{\infty} s^k \varphi_b^k(t)$ converges for any $t \in [0, +\infty)$;*
(2) *the function $b_s : [0, +\infty) \to [0, +\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k \varphi_b^k(t)$, $t \in [0, \infty)$ is increasing and continuous at 0.*

Example 1.19. *The following functions are comparison functions (respectively b-comparison functions with $s > 1$).*

1. $\varphi(t) = kt$ *(respectively* $\varphi_b(t) = \frac{k}{s}t$, *where* $k \in [0, 1)$*).*
2. $\varphi(t) = \ln(1 + kt)$ *(respectively* $\varphi_b(t) = \ln\left(1 + \frac{k}{s}t\right)$ *, where* $k \in [0, 1)$*).*
3. $\varphi(t) = \frac{t}{1+t}$ *(respectively* $\varphi_b(t) = \frac{t}{s(1+t)}$*).*

For more details on comparison functions and examples we refer the reader to [8, 9, 20].

One of the interesting fixed point results, by using comparison function was given by Bota *et al.* [11].

Theorem 1.20. *Let T be a continuous self mapping on a complete b-metric space (X, d) with constant $s \geq 1$ and let $\alpha : X \times X \to [0, \infty)$ be an auxiliary function such that*

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Assume that $\varphi_b : [0, +\infty) \to [0, +\infty)$ is a b-comparison function and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If the inequality

$$\alpha(x, y)d(Tx, Ty) \leq \varphi_b(d(x, y)),$$

is satisfied for all $x, y \in X$, then T possess a fixed point. In addition, if for any pair $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then we guarantee the uniqueness of the obtained fixed point.

Contraction mappings defined via the function α in the statement of the Theorem 1.20 are known as the α-admissible mappings and can be regarded as generalizations of the usual contraction mappings. Admissible mappings are also an attractive concept in fixed point theory [4, 19]. It is easy to deduce several consequences of Theorem 1.20, by setting the auxiliary function α and also setting the b-metric constant $s = 1$, see e.g. [17]. Among them, by letting $\alpha(x, y) = 1$, we state the following corollary of Theorem 1.20 that was reported by Pacurar [18] as follows.

Theorem 1.21. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and let $T : X \to X$ be a self mapping on X. Assume that $\varphi_b : [0, +\infty) \to [0, +\infty)$ is a b-comparison function. If for all $x, y \in X$ we have*

$$d(Tx, Ty) \leq \varphi_b(d(x, y)),$$

then T has a unique fixed point.

For more results related with contraction mappings on b-metric spaces and comparison functions we refer the reader to [6, 11].

2. Existence and uniqueness theorem

In this section we consider an initial value problem associated with a nonlinear Volterra integro-dynamic equations and discuss the existence and uniqueness of its solution in the setting of b-metric spaces. Now we state our application.

Let $\mathbb{T}$ be a time scale with delta differential operator Δ and forward jump operator σ, respectively. Consider the initial value problem

$$\begin{aligned} \phi^{\Delta}(x) &= u(x) + \int_{x_0}^{x} K(s, x, \sigma(s), \sigma(x))F(\phi(s))\Delta s, \quad x \in [x_0, A]_{\mathbb{T}}, \\ \phi_(x_0) &= \alpha, \end{aligned} \tag{5}$$

where $u : [x_0, A]_{\mathbb{T}} \to \mathbb{R}$ and $K : ([x_0, A]_{\mathbb{T}})^4 \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ are given functions. Let $C[x_0, A]_{\mathbb{T}}$ be the space of continuous functions on $[x_0, A]_{\mathbb{T}}$ and let $d : [x_0, A]_{\mathbb{T}} \times [x_0, A]_{\mathbb{T}} \to [0, \infty)$ be defined as

$$d(x, y) = \sup_{t \in C[x_0, A]_{\mathbb{T}}} |x(t) - y(t)|^2. \tag{6}$$

Then, d is a b-metric on $C[x_0, A]_{\mathbb{T}}$ with $s = 2$ and the space $(C[x_0, A]_{\mathbb{T}}, d)$ is a complete b-metric space [11].

First, we notice that the initial value problem (5) can be transformed to the form

$$\phi(t) = \alpha + \int_{x_0}^{t} u(x)\Delta x + \int_{x_0}^{t}\int_{x_0}^{x} K(s, x, \sigma(s), \sigma(x))F(\phi(s))\Delta s\Delta x, \quad t \in [x_0, A]_{\mathbb{T}}, \tag{7}$$

upon taking the delta integral of both sides on $[x_0, t]$, where $t \in [x_0, A]_{\mathbb{T}}$. It is easy to see that if the functions u, K and F are delta integrable, then the right-hand-side of (7) is a continuous function on $[x_0, A]_{\mathbb{T}}$. Define the mapping $T : C[x_0, A]_{\mathbb{T}} \to C[x_0, A]_{\mathbb{T}}$ as

$$T\phi(t) = \alpha + \int_{x_0}^{t} u(x)\Delta x + \int_{x_0}^{t}\int_{x_0}^{x} K(s, x, \sigma(s), \sigma(x))F(\phi(s))\Delta s\Delta x, \quad , t \in [x_0, A]_{\mathbb{T}}. \tag{8}$$

Obviously, a solution of the problem (5) (equivalently (7)) is a fixed point of T.

In what follows, we propose the following existence-uniqueness theorem for the solution of (5).

Theorem 2.1. *Let $\mathbb{T}$ be a time scale and $[x_0, A]_{\mathbb{T}}$ be a finite interval for some $x_0, A \in \mathbb{T}$. Assume that for any $\phi, \psi \in C[x_0, A]_{\mathbb{T}}$ the following conditions are satisfied.*

1. *$|\phi(t)| < C$, that is, ϕ is bounded on $[x_0, A]_{\mathbb{T}}$.*
2. *the functions u and K are delta integrable on $[x_0, A]_{\mathbb{T}}$,*
3. *the function F is delta integrable on $[x_0, A]_{\mathbb{T}}$ and satisfies*

$$|F(\phi(s)) - F(\psi(s))|^2 \le \varphi_b(|\phi(s) - \psi(s)|^2), \quad s \in [x_0, A]_{\mathbb{T}}, \tag{9}$$

for some b-comparison function φ_b.

4. *we have*

$$\int_{x_0}^{t}\int_{x_0}^{x} |K(s, x, \sigma(s), \sigma(x))|^2 \Delta s\Delta x \le L, \quad x, t \in [x_0, A]_{\mathbb{T}}, \tag{10}$$

for some $L < \dfrac{2}{(A - x_0)^2}$.

Then, the map T defined in (8) *has a unique fixed point, that is, the integral equation* (5) *has a unique solution in $C[x_0, A]_{\mathbb{T}}$.*

Proof. By the definition of the map T in (8) and the Cauchy-Schwarz inequality for integrals on time scales [5]

$$\left(\int_a^b f(x)g(x)\Delta x\right)^2 \le \left(\int_a^b (f(x))^2\Delta x\right)\left(\int_a^b (g(x))^2\Delta x\right),$$

we get

$$\begin{aligned} |T\phi(t) - T\psi(t)|^2 &= \left|\int_{x_0}^{t}\int_{x_0}^{x} K(s, x, \sigma(s), \sigma(x))\left[F(\phi(s)) - F(\psi(s))\right]\Delta s\Delta x\right|^2 \\ &\le \left(\int_{x_0}^{t}\int_{x_0}^{x} |K(s, x, \sigma(s), \sigma(x))|^2\Delta s\Delta x\right)\left(\int_{x_0}^{t}\int_{x_0}^{x} |F(\phi(s)) - F(\psi(s))|^2\Delta s\Delta x\right). \end{aligned}$$

Employing the conditions (9) and (10) we obtain

$$|T\phi(t) - T\psi(t)|^2 \le L\left(\int_{x_0}^{t}\int_{x_0}^{x} \varphi_b(|\phi(s) - \psi(s)|^2)\Delta s\Delta x\right),$$

where the function φ_b is a b-comparison function. Taking the supremum over $[x_0, A]_{\mathbb{T}}$ together with the definition of the metric (6), we get

$$d(T\phi, T\psi) \leq L\varphi_b(d(\phi, \psi)) \int_{x_0}^{t} \int_{x_0}^{x} \Delta s \Delta x. \tag{11}$$

Notice that

$$\int_{x_0}^{t} \int_{x_0}^{x} \Delta s \Delta x = \int_{x_0}^{t} (x - x_0) \Delta x = \int_{x_0}^{t} h_1(x, x_0) \Delta x = h_2(t, x_0), \tag{12}$$

where h_1 and h_2 are the time scale monomials defined in (1). It follows from the condition 4. of the theorem, the inequality (11) and the Theorem 1.13 that

$$\begin{aligned} d(T\phi, T\psi) &\leq Lh_2(t, x_0)\varphi_b(d(\phi, \psi)) \\ &\leq L\frac{(t - x_0)^2}{2}\varphi_b(d(\phi, \psi)) \\ &\leq L\frac{(A - x_0)^2}{2}\varphi_b(d(\phi, \psi)) \\ &\leq \varphi_b(d(\phi, \psi)). \end{aligned}$$

Then, by the Theorem 1.21, the map T defined in (8) has a unique fixed point, that is, the integral equation (5) has a unique solution in $C[x_0, A]_{\mathbb{T}}$. □

This theorem has several consequences which we give below.

Corollary 2.2. *Let $\mathbb{T}$ be a time scale and $[x_0, A]_{\mathbb{T}}$ be a finite interval for some $x_0, A \in \mathbb{T}$. Assume that for any $\phi, \psi \in C[x_0, A]_{\mathbb{T}}$ the following conditions are satisfied.*

1. *$|\phi(t)| < C$, that is, ϕ is bounded on $[x_0, A]_{\mathbb{T}}$.*
2. *the functions u and K are delta integrable on $[x_0, A]_{\mathbb{T}}$,*
3. *the function F is delta integrable on $[x_0, A]_{\mathbb{T}}$ and satisfies*

$$|F(\phi(s)) - F(\psi(s))|^2 \leq \frac{1}{2}|\phi(s) - \psi(s)|^2, \quad s \in [x_0, A]_{\mathbb{T}}. \tag{13}$$

4. *we have*

$$\int_{x_0}^{t} \int_{x_0}^{x} |K(s, x, \sigma(s), \sigma(x))|^2 \Delta s \Delta x \leq L, \quad x, t \in [x_0, A]_{\mathbb{T}} \tag{14}$$

for some $L < \dfrac{2}{(A - x_0)^2}$.

Then, the map T defined in (8) *has a unique fixed point, that is, the integral equation* (5) *has a unique solution in $C[x_0, A]_{\mathbb{T}}$.*

Proof. By taking the b-comparison function φ_b in Theorem 2.1 as $\varphi_b(t) = \dfrac{t}{2}$, the proof follows immediately. □

As another consequence, we give an existence-uniqueness result on the metric space $C[x_0, A]_{\mathbb{T}}$ with the usual metric $d(\phi, \psi) = \sup_{t \in [x_0, A]_{\mathbb{T}}} |\phi(t) - \psi(t)|$.

Corollary 2.3. *Let $\mathbb{T}$ be a time scale and $[x_0, A]_{\mathbb{T}}$ be a finite interval for some $x_0, A \in \mathbb{T}$. Assume that for any $\phi, \psi \in C[x_0, A]_{\mathbb{T}}$ the following conditions are satisfied.*

1. *$|\phi(t)| < C$, that is, ϕ is bounded on $[x_0, A]_{\mathbb{T}}$.*

2. *the functions u and K are delta integrable on $[x_0, A]_{\mathbb{T}}$,*
3. *the function F is delta integrable on $[x_0, A]_{\mathbb{T}}$ and satisfies*

$$|F(\phi(s)) - F(\psi(s))| \le \varphi(|\phi(s) - \psi(s)|), \quad s, x \in [x_0, A]_{\mathbb{T}}, \tag{15}$$

for some comparison function φ.

4. *we have*

$$\int_{x_0}^{t} \int_{x_0}^{x} |K(s, x, \sigma(s), \sigma(x))| \Delta s \Delta x \le L \tag{16}$$

for some $L < \dfrac{2}{(A - x_0)^2}$.

Then, the map T defined in (8) *has a unique fixed point, that is, the integral equation* (5) *has a unique solution in $C[x_0, A]_{\mathbb{T}}$.*

Proof. Let $(C[x_0, A]_{\mathbb{T}}, d)$ be the metric space with the usual metric $d(\phi, \psi) = \sup_{t \in [x_0, A]_{\mathbb{T}}} |\phi(t) - \psi(t)|$ and φ be a given comparison function. The proof follows from Theorem 1.21 with $s = 1$. □

3. Applications

In this section we apply the result in Theorem 2.1 to particular examples of Volterra integro-dynamic equations of the second type.

Example 3.1. *Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the following nonlinear Volterra integro-dynamic equation*

$$\phi^{\Delta}(t) \;=\; -\frac{3}{4t^3} - \frac{t^3}{140} + \frac{1}{140} + \int_1^t \frac{1 + s^2}{20} \frac{1}{1 + |\phi(s)|} \Delta s, \quad , x \in [1, 4]_{\mathbb{T}}, \tag{17}$$

together with the initial condition $\phi(1) = 1$. Note that here we have ,

$$\sigma(t) = \inf\{s \in \mathbb{T} = 2^{\mathbb{N}_0} : s > t\} = \inf\{2t, 4t, ...\} = 2t, \quad t \in \mathbb{T}.$$

In fact, it is easy to see that $\phi(t) = \dfrac{1}{t^2}$ is a solution of given nonlinear Volterra integro-dynamic equation (17)*. Indeed,*

$$\phi^{\Delta}(t) = \frac{\dfrac{1}{\sigma^2(t)} - \dfrac{1}{t^2}}{\sigma(t) - t} = -\frac{\sigma(t) + t}{t^2 \sigma^2(t)} = -\frac{3}{4t^3}, \quad t \in [1, 4].$$

Let $f(t) = \dfrac{t^3}{140} - \dfrac{1}{140}, \quad t \in [1, 4]$. Then, the delta derivative of $f(t)$ can be computed as

$$f^{\Delta}(t) = \frac{1}{140}\left(\sigma^2(t) + t\sigma(t) + t^2\right) = \frac{1}{140}(4t^2 + 2t^2 + t^2) = \frac{t^2}{20}, \quad t \in [1, 4].$$

Therefore, the right hand side of the nonlinear Volterra integro-dynamic equation (17) *becomes*

$$\begin{aligned} -\frac{3}{4t^3} - \frac{t^3}{140} + \frac{1}{140} + \int_1^t \frac{1+s^2}{20} \frac{1}{1+|\phi(s)|}\Delta s &= -\frac{3}{4t^3} - \frac{t^3}{140} + \frac{1}{140} + \int_1^t \frac{1+s^2}{20} \frac{s^2}{1+s^2}\Delta s \\ &= -\frac{3}{4t^3} - \frac{t^3}{140} + \frac{1}{140} + \int_1^t \frac{s^2}{20}\Delta s \\ &= -\frac{3}{4t^3} - \frac{t^3}{140} + \frac{1}{140} + \int_1^t f^{\Delta}(s)\Delta s \\ &= -\frac{3}{4t^3} - \frac{t^3}{140} + \frac{1}{140} + f(s)\Big|_{s=1}^t \\ &= -\frac{3}{4t^3} - \frac{t^3}{140} + \frac{1}{140} + (\frac{s^3}{140} - \frac{1}{140})\Big|_{s=1}^t \\ &= -\frac{3}{4t^3} - \frac{t^3}{140} + \frac{1}{140} + \frac{t^3}{140} - \frac{1}{140} \\ &= -\frac{3}{4t^3} = \phi^{\Delta}(t), \quad t \in [1,4]. \end{aligned}$$

For the given example the map T in (8) *is defined by*

$$T\phi(t) = 1 + \int_1^t \left(-\frac{3}{4x^3} - \frac{x^3}{140} + \frac{1}{140}\right)\Delta x + \int_1^t \int_1^x \frac{1+s^2}{20} \frac{1}{1+|\phi(s)|}\Delta s\Delta x, \quad t \in [1,4]_{\mathbb{T}} \tag{18}$$

so that $K(s,x,\sigma(s),\sigma(x)) = \dfrac{1+s^2}{10}$, $F(\phi(s)) = \dfrac{1}{2}\dfrac{1}{1+|\phi(s)|}$. *By the assumptions of theorem, let* $|\phi(t)| < C = 4$, *Then, from the definition of the map T in* (18) *and the Cauchy-Schwarz inequality, it follows that*

$$\begin{aligned} |T\phi - T\psi|^2 &= \left|\int_1^t \int_1^x \frac{1+s^2}{10}\frac{1}{2}(\frac{1}{1+|\phi(s)|} - \frac{1}{1+|\psi(s)|})\Delta s\Delta x\right|^2 \\ &\leq \left(\int_1^t \int_1^x \frac{(1+s^2)^2}{100}\Delta s\Delta x\right).\left(\int_1^t \int_1^x \frac{1}{4}\left|\frac{1}{1+|\phi(s)|} - \frac{1}{1+|\psi(s)|}\right|^2 \Delta s\Delta x\right). \end{aligned}$$

Observe that

$$\begin{aligned} \frac{1}{4}\left|\frac{1}{1+|\phi(s)|} - \frac{1}{1+|\psi(s)|}\right|^2 &\leq \frac{\frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2}{\Big|1+|\phi(s)|+|\psi(s)|+|\phi(s)||\psi(s)|\Big|^2} \\ &\leq \frac{\frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2}{1+|\phi(s)|^2+|\psi(s)|^2} \\ &\leq \frac{\frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2}{1+\frac{1}{4}|\phi(s)|^2+\frac{1}{4}|\psi(s)|^2} \\ &\leq \frac{\frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2}{1+\frac{1}{4}|\phi(s)|^2+\frac{1}{4}|\psi(s)|^2-\frac{2}{4}|\phi(s)||\psi(s)|} \\ &= \frac{\frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2}{1+\frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2}. \end{aligned}$$

Using the inequality $\dfrac{r}{1+r} \leq \ln(1+r)$ *for* $r > -1$, *leads to*

$$|T\phi - T\psi|^2 \leq \left(\frac{1}{100}\int_1^t \int_1^x (1+s^2)^2\Delta s\Delta x\right).\left(\int_1^t \int_1^x \ln(1+\frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2)\Delta s\Delta x\right),$$

for $|\phi(s)| - |\psi(s)| > -4$, *which is valid by the assumption* $|\phi(s)| < 4$. *Then, evaluating the delta integral and using* $|x| \leq 4$ *we get*

$$\begin{aligned} |T\phi - T\psi|^2 &\leq \frac{1}{100}\left(\frac{t^6}{1953} + \frac{2t^4}{105} + \frac{t^2}{3} - \frac{286}{217}t + \frac{9424}{9765}\right)\left(\int_1^t \int_1^x \ln(1 + \frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2)\Delta s\Delta x\right), \\ &\leq L\left(\int_1^t \int_1^x \ln(1 + \frac{1}{4}\Big||\phi(s)| - |\psi(s)|\Big|^2)\Delta s\Delta x\right), \end{aligned}$$

where $L = \frac{12}{100}$. *Taking the supremum over* $t, s \in [1,4]_{\mathbb{T}}$ *together with the definition of the metric* (6), *one gets*

$$\begin{aligned} d(T\phi, T\psi) &\leq L\ln(1 + \tfrac{1}{4}d(\phi,\psi))\left(\int_1^t \int_1^x \Delta s\Delta x\right) \\ &= L\ln(1 + \tfrac{1}{4}d(\phi,\psi))h_2(t,1), \end{aligned}$$

whereupon by the Theorem 1.13 we get

$$\begin{aligned} d(T\phi, T\psi) &\leq L\ln(1 + \frac{1}{4}d(\phi,\psi))\frac{(t-1)^2}{2} \\ &\leq L\frac{(A-1)^2}{2}\ln(1 + \frac{1}{4}d(\phi,\psi)). \end{aligned}$$

Since $L = \frac{12}{100}$ *and* $t, x \in [1,4]_{\mathbb{T}}$, *that is,* $|x| \leq 4$ *and* $|t| \leq 4$, *then* $L\frac{(A-1)^2}{2} = \frac{54}{100} < 1$, *and hence,*

$$d(T\phi, T\psi) \leq L\frac{(A-1)^2}{2}\ln(1 + \frac{1}{4}d(\phi,\psi)) \leq \ln(1 + \frac{1}{4}d(\phi,\psi)).$$

Choosing the b-comparison function as $\varphi_b(t) = \ln(1 + \frac{1}{4}t)$ *(see the Example 1.19), we have*

$$d(T\phi, T\psi) \leq \varphi_b(d(\phi,\psi)).$$

Therefore, by the Theorem 2.1 the map T defined in (18) *has a unique fixed point, that is, the integral equation* (17) *given in the example has a unique solution in* $C[1,4]_{\mathbb{T}}$.

Example 3.2. *Let* $\mathbb{T} = \mathbb{Z}$. *Consider the following nonlinear Volterra integro-dynamic equation on* $\mathbb{T} = \mathbb{Z}$.

$$\phi^{\Delta}(t) \quad = \quad 2t + 1 - \frac{t^3}{60\sqrt{2}} + \frac{1}{60\sqrt{2}} + \int_1^t \frac{s}{20\sqrt{2}}\left[\sqrt{\phi(s)} + \frac{1}{3\sqrt{\phi(s)}} + 1\right]\Delta s, \quad x \in [1,4]_{\mathbb{T}}, \tag{19}$$

together with the initial condition $\phi(1) = 1$. *On* $\mathbb{T} = \mathbb{Z}$ *we have*

$$\sigma(t) = \inf\{s \in \mathbb{T} = \mathbb{Z} : s > t\} = \inf\{t+1, t+2, ...\} = t+1, \quad t \in \mathbb{Z}.$$

It is easy to see that $\phi(t) = t^2$ *is a solution of given nonlinear Volterra integro-dynamic equation* (19). *Clearly,*

$$\phi^{\Delta}(t) = \frac{\sigma^2(t) - t^2}{\sigma(t) - t} = \sigma(t) + t = 2t + 1, \quad t \in [1,4].$$

Let $f(t) = \dfrac{t^3}{60\sqrt{2}} - \dfrac{1}{60\sqrt{2}}$, $\quad t \in [1,4]$. *Then, the delta derivative of* $f(t)$ *is computed as follows*

$$f^{\Delta}(t) = \frac{1}{60\sqrt{2}}\left(\sigma^2(t) + t\sigma(t) + t^2\right) = \frac{1}{60\sqrt{2}}((t+1)^2 + t(t+1) + t^2) = \frac{t^2}{20\sqrt{2}} + \frac{t}{20\sqrt{2}} + \frac{1}{60\sqrt{2}},$$

for $t \in [1,4]$. *Therefore, the right hand side of the nonlinear Volterra integro-dynamic equation in* (19) *becomes*

$$
\begin{aligned}
& 2t+1-\frac{t^3}{60\sqrt{2}}+\frac{1}{60}\sqrt{2}+\int_1^t \frac{s}{20\sqrt{2}}(s+\frac{1}{3s}+1)\Delta s \\
&= 2t+1-\frac{t^3}{60\sqrt{2}}+\frac{1}{60\sqrt{2}}+\int_1^t \frac{1}{20\sqrt{2}}(s^2+s+\frac{1}{3})\Delta s \\
&= 2t+1-\frac{t^3}{60\sqrt{2}}+\frac{1}{60\sqrt{2}}+\int_1^t f^{\Delta}(s)\Delta s \\
&= 2t+1-\frac{t^3}{60\sqrt{2}}+\frac{1}{60\sqrt{2}}+f(s)\Big|_{s=1}^{t} \\
&= 2t+1-\frac{t^3}{60\sqrt{2}}+\frac{1}{60\sqrt{2}}+(\frac{s^3}{60\sqrt{2}}-\frac{1}{60\sqrt{2}})\Big|_{s=1}^{t} \\
&= 2t+1-\frac{t^3}{60\sqrt{2}}+\frac{1}{60\sqrt{2}}+\frac{t^3}{60\sqrt{2}}-\frac{1}{60\sqrt{2}} \\
&= 2t+1=\phi^{\Delta}(t), \quad t\in[1,4].
\end{aligned}
$$

For this example the map T in (8) *is given by*

$$
T\phi(t)=1+\int_1^t\left(2x+1-\frac{x^3}{60\sqrt{2}}+\frac{1}{60\sqrt{2}}\right)\Delta x+\int_1^t\int_1^x \frac{s}{20\sqrt{2}}\left[\sqrt{\phi(s)}+\frac{1}{3\sqrt{\phi(s)}}+1\right]\Delta s\Delta x, \tag{20}
$$

for $t \in [1,4]_{\mathbb{T}}$ *where* $K(s,x,\sigma(s),\sigma(x))=\frac{s}{20}$, *and* $F(\phi(s))=\frac{1}{\sqrt{2}}(\sqrt{\phi(s)}+\frac{1}{3\sqrt{\phi(s)}}+1)$. *By the assumptions of theorem, let* $1\le|\phi(t)|<C$ *for some* $C>1$.

Then, by the definition of the map T in (8) *and the Cauchy-Schwarz inequality, it follows that*

$$
\begin{aligned}
|T\phi-T\psi|^2 &= \left|\int_1^t\int_1^x \frac{s}{20\sqrt{2}}\left[(\sqrt{\phi(s)}+\frac{1}{3\sqrt{\phi(s)}}+1)-(\sqrt{\psi(s)}+\frac{1}{3\sqrt{\psi(s)}}+1)\right]\Delta s\Delta x\right|^2 \\
&\le \left(\int_1^t\int_1^x \frac{s^2}{400}\Delta s\Delta x\right).\left(\int_1^t\int_1^x \frac{1}{2}\left|(\sqrt{\phi(s)}-\sqrt{\psi(s)}-\frac{\sqrt{\phi(s)}-\sqrt{\psi(s)}}{3\sqrt{\phi(s)}\sqrt{\psi(s)}})\right|^2\Delta s\Delta x\right).
\end{aligned} \tag{21}
$$

We observe that

$$
\begin{aligned}
\frac{1}{2}\left|(\sqrt{\phi(s)}-\sqrt{\psi(s)}-\frac{\sqrt{\phi(s)}-\sqrt{\psi(s)}}{3\sqrt{\phi(s)}\sqrt{\psi(s)}})\right|^2 &\le \frac{1}{2}\Big(\frac{3\sqrt{\phi(s)}\sqrt{\psi(s)}-1}{3\sqrt{\phi(s)}\sqrt{\psi(s)}}\Big)^2\left|\sqrt{\phi(s)}-\sqrt{\psi(s)}\right|^2 \\
&\le \frac{1}{2}\left|\sqrt{\phi(s)}-\sqrt{\psi(s)}\right|^2 \\
&\le \frac{1}{2}|\phi(s)-\psi(s)|^2
\end{aligned}
$$

since $|\sqrt{\phi(s)}-\sqrt{\psi(s)}|\le|\phi(s)-\psi(s)|$, *for* $\phi(s),\psi(s)\ge 1$. *The first integral in* (21) *is computed as*

$$
\int_1^t\int_1^x \frac{s^2}{400}\Delta s\Delta x=\frac{1}{400}\left(\frac{t^4}{12}-\frac{t^3}{3}+\frac{5}{12}t^2-\frac{t}{6}\right),
$$

whereupon,

$$
d(T\phi,T\psi)\le\frac{1}{400}\left(\frac{t^4}{12}-\frac{t^3}{3}+\frac{5}{12}t^2-\frac{t}{6}\right)\left(\int_1^t\int_1^x \frac{1}{2}|\phi(s)-\psi(s)|^2\Delta s\Delta x\right).
$$

Since $|x| \leq 4$, *we have*

$$d(T\phi, T\psi) \leq L\left(\int_1^t \int_1^x \frac{1}{2}|\phi(s) - \psi(s)|^2 \Delta s \Delta x\right),$$

where $L = \dfrac{1}{10}$. *Taking the supremum over* $t \in [1,4]_{\mathbb{T}}$ *together with the definition of the b-metric* (6), *one gets*

$$\begin{aligned} d(T\phi, T\psi) &\leq L\frac{1}{2}d(\phi,\psi)\int_1^t \int_1^x \Delta s \Delta x \\ &= L\frac{1}{2}d(\phi,\psi)h_2(t,1). \end{aligned}$$

By the Theorem 1.13 we estimate

$$d(T\phi, T\psi) \leq L\frac{(A-1)^2}{2}\frac{1}{2}d(\phi,\psi).$$

Since $L = \dfrac{1}{10}$ *and* $t, x \in [1,4]_{\mathbb{T}}$, *so that,* $|x| \leq 4$ *and* $|t| \leq 4$, *we get*

$$d(T\phi, T\psi) \leq L\frac{(A-1)^2}{2}\frac{1}{2}d(\phi,\psi) = \frac{9}{20}d(\phi,\psi)$$

because of the fact that $L\frac{(A-1)^2}{2} = \frac{9}{20}$. *Choosing the b-comparison function as* $\varphi_b(t) = \dfrac{1}{4}t$ *(see the Example 1.19) leads to*

$$d(T\phi, T\psi) \leq \varphi_b(d(\phi,\psi)) = \frac{1}{4}d(\phi,\psi) < d(\phi,\psi).$$

Therefore, by the Theorem 2.1, the map T defined in (20) *has a unique fixed point, that is, the integral equation* (19) *given in the example has a unique solution in* $C[1,4]_{\mathbb{T}}$.

4. Conclusion

The existence-uniqueness problem studied in this paper is solved by means of the Theorem 2.1 which gives conditions for the existence and uniqueness of solutions for a class of nonlinear Volterra integro-dynamic equations of the second kind on arbitrary time scales. The equation considered here contains delta derivative of first order. However, it is possible to extend this study to the initial value problems associated with integro-dynamic equations containing higher order delta derivatives which can be regarded as a direction for a future study.

References

[1] T. Abdeljawad, R. P. Agarwal, E. Karapnar,P.S. Kumari, Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space, Symmetry, 11 (2019) 686 doi:10.3390/sym11050686.

[2] T. Abdeljawad, B. Abdalla, Monotonicity results for delta and nabla Caputo and Reimann fractional differences via dual identities, Filomat, 31 (2017) 3671–3683.

[3] T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities, Discrete Dynamics in Nature and Society, 2013 (2013) Article Id. 406910.

[4] T. Abdeljawad, Meir-Keeler α-contractive fixed and common fixed point theorems, Fixed Point Theory and Applications, 2013 (2013) Article Id. 19.

[5] R. P. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: A survey, Mathematical Inequalities and Applications, 4 (2001) 535–557.

[6] H. Aydi, M. F. Bota, E. Karapinar, S. Moradi,A common fixed point for weak φ-contractions on b-metric spaces, Fixed Point Theory, **13(1)**, 337–346, 2012.

[7] I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, Functional Analysis, 30 (1989) 26–37.
[8] V. Berinde, Sequences of operators and fixed points in quasi-metric spaces, Mathematica, 41 (1996) 23–27.
[9] V. Berinde, Contracţii Generalizate şi Aplicaţii , Vol. 2, Editura Cub Press, Baia Mare, Romania, 1997.
[10] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2003.
[11] M. F. Bota, E. Karapinar, O. Mleşniţe, Ulam-Hyers Stability Results for Fixed Point Problems via $\alpha-\psi$-Contractive Mapping in (b)-Metric Space, Abtract and Applied Analysis, 2013 (2013) Article ID 825293.
[12] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis, 1 (1993), 5-11.
[13] S. Georgiev, Integral Equations on Time Scales, Atlantis Press, 2016.
[14] S. Georgiev, Fractional Dynamic Calculus and Fractional Dynamic Equations on Time Scales, Springer, 2018.
[15] S. Georgiev and İ M. Erhan, Nonlinear Integral Equations on Time Scales, Nova Science Publishers, 2019.
[16] D. Guo, Existence of solutions for n-th order integro-differential equations in Banach spaces, Computers & Mathematics with Application, 41 (2001) 597–606.
[17] E. Karapinar, B. Samet, Generalized $\alpha-\psi$-Contractive Type Mappings and Related Fixed Point Theorems with Applications, Abstract and Applied Analysis, 2012 (2012), Article ID 793486.
[18] M. Pacurar, A fixed point result for φ-contractions on b-metric spaces without boundedness assumption, Fasciculi Mathematici, 43 (2010) 127–137.
[19] D. K. Patel, T. Abdeljawad, D. Gopal, Common fixed points of generalized Meir-Keeler α-contractions, Fixed Point Theory and Applications, 2013 (2013), Article Id. 260.
[20] I. A. Rus, Generalized contractions and application, Cluj University Press, Cluj-Napoca, Romania, 2001.
[21] I. A. Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances, Fixed Point Theory, 9 (2008) 541-559.
[22] S. Şevgin, H. Şevli, Stability of a nonlinear Volterra integro-differential equation via a fixed point approach, Journal of Nonlinear Analysis and Applications, 9 (201) 200–207.
[23] W. Zhuang, Existence and uniqueness of solutions of nonlinear integro-differential equations of Volterra type in a Banach space, Applicable Analysis, 22 (198) 157–166.