Filomat 33:16 (2019), 5291–5330
https://doi.org/10.2298/FIL1916291S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Quasi Xgamma-Geometric Distribution with Application in Medicine

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Abstract. In this paper, a new probability distribution, which is synthesized based on the quasi xgamma [26] and geometric distributions, is proposed and studied. The proposed distribution so synthesized is basically a family of positively skewed probability distributions and possesses increasing and decreasing hazard rate properties depending on the values of the unknown parameters. Different important distributional and survival and/or reliability properties are also studied. A unique characterization of the distribution is presented based on reversed hazard rate. Seven different frequentist methods of estimating unknown parameters are proposed and the methods are justified with Monte-Carlo simulation study. Flexible data generation algorithm eases the utility of the proposed model in survival and/or reliability application which is accomplished by real data analyses and by comparing with other competitive life distributions.

1. Introduction

In recent years, a large list of probability distributions have been proposed and studied in statistical literature, the reason being simple owing the fact that in a data driven technological age it is logical to model the data sets, coming from diverse areas, with proper probabilistic interface and hence, development of newer and more flexible probability distributions are in their pick hour of progress. However, the process of developing newer probability distributions are sometimes driven by merely mathematical interest and hence finding applications with real world uncertainties compels the researchers to make a trade-off between the nature of the data and the characteristics of the proposed distributions. Moreover, in too many situations, the actual need and proposal do not apply to simplicity of inferential aspects. Life distributions, as termed in the survival and reliability contexts, play vital roles in modeling even non-survival data sets for an obvious reason of the range of the concerned random variable. Popular life distributions sometimes fail to compete with newly developed life distributions as the data description is more accurately captured and depicted by newer ones.
Recently, [29] introduced and studied a new life distribution namely, xgamma distribution which has a single parameter. The xgamma distribution, with the help of the simplicity of its density function and lucrative

2010 *Mathematics Subject Classification*. 62E10; 60K10, 60N05
Keywords. Life distributions, geometric distribution, maximum likelihood estimation, characterization.
Received: 10 January 2019; Revised: 13 April 2019; Accepted: 14 August 2019
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properties, serves as a potential life distribution (see [27] for more details). A two-parameter extension of xgamma distribution, called quasi-xgamma distribution ([26]) resembles with xgamma distribution with respect to several distributional and survival and/or reliability properties. Moreover, the xgamma distribution falls as a special case of quasi xgamma distribution.
Compounding two or more well established life distributions for the purpose of betterment in application is not new in statistical literature. A two-parameter lifetime distribution with decreasing failure rate by compounding exponential and geometric distributions, named as exponential geometric (EG) distribution, was introduced by [4]. The exponential Poisson (EP) and exponential logarithmic (EL) distributions were introduced and studied by [17] and [33], respectively. A class of distributions, named exponential power series (EPS) distributions, by compounding exponential and power series distributions, proposed by [7]. In a similar fashion, [30] and [19] introduced the Weibull-geometric (WG) and Weibull-Poisson (WP) distributions that extend the EG and EP distributions, respectively. A generalization of the exponential-Poisson distribution has been presented by [5]. The Weibull power series (WPS) class of distributions, as defined by [22], contains the EPS distributions as sub-models. [3] proposed the extended exponential-geometric (EEG) distribution as a generalization of the EG distribution and its different statistical and reliability properties have been studied.

1.1. Motivation

Keeping in mind the role of compounding techniques in improving any existing probability distribution, in this article we propose and study a generalized version of quasi xgamma, named as the quasi xgamma-geometric distribution, by some kind of compounding technique with geometric distribution. However, the compounding adopted is slightly different in sense of its actual idea of synthesis. In a very natural way, the quasi xgamma and the xgamma distributions come as special cases of the proposed distribution. There are following motivations in the current investigation.

(i) To obtain a generalized version of quasi xgamma distribution that includes different other sub-models useful for explaining typical types of uncertainties.
(ii) To obtain an improvement on the hazard rate function that may accommodate constant, decreasing, increasing, bathtub, upside down bathtub and reversed-J hazard rates.
(iii) To find specific characteristic, if any, of the proposed density function.
(iv) To identify better applicability of the proposed distribution and to establish competency over the other popular lifetime models.

The possible situation of encountering such a compounding form is described in the next section as a synthesis of the distribution. The rest of the article is organized as follows.
The synthesis of the proposed distribution, its probability density function, cumulative distribution function, survival function and hazard rate function are discussed in section 2. Section 3 deal with a series representation of the proposed distribution. In section 4 and in its dedicated subsections, we study some distributional and survival/reliability properties. A typical characterization is presented in section 5. Seven methods of estimating unknown parameters of the proposed distribution are studied in section 6 and in its subsections. Random data generation algorithm along with a Monte-Carlo simulation study is presented in section 7. In section 8, two real data sets are analyzed to illustrate the applicability and competence of the proposed model. Finally, section 9 concludes.

2. Synthesis of the distribution

If Y is a random variable following quasi xgamma (QXG) distribution with parameters α and θ (see [26]), then it has probability density function (pdf) as

$$f(y) = \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{\theta^2}{2}y^2\right)e^{-\theta y}, y > 0, \alpha, \theta > 0.$$

Let us denote it by $Y \sim$QXG(α,θ).
The corresponding cumulative distribution function (cdf) is given by

$$F(y) = 1 - \frac{\left(1+\alpha+\theta y+\frac{\theta^2}{2}y^2\right)}{(1+\alpha)}e^{-\theta y}, y>0, \alpha, \theta>0.$$

A new distribution, following a very similar way of [28], can be synthesized as follows.
Suppose that the life of a unit (be it mechanical or biological) fails due to the presence of M (an unknown number) initial defects of some kind. Let $Y_1, Y_2, \ldots, Y_M$ denote the lives of the initial defects, then the life of the unit, say X, can be expressed as

$$X = Min\{Y_1, Y_2, \ldots, Y_M\}.$$

Suppose that the lives of the initial defects, $Y_1, Y_2, \ldots, Y_M$, follow identically and independently distributed (i.i.d) QXG(α,θ) and the number of initial defects M follows a zero-truncated geometric distribution with parameter p, i.e., the probability mass function (pmf) of M is

$$\Pr(M=m) = p(m) = (1-p)p^{m-1}, 0<p<1, m=1,2,\ldots.$$

Assuming that the random variables $Y_i (i=1,2,\ldots,M)$ and M are independent, the conditional density of X given $M=m$ is

$$f(x|m) = \frac{m\theta\left(\alpha+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)^m}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)^{m-1} e^{-m\theta x}.$$

Then, the marginal density of X can be obtained as

$$\begin{aligned}
f(x) &= \sum_{m=1}^{\infty} f(x|m)p(m) \\
&= \sum_{m=1}^{\infty} \frac{m\theta\left(\alpha+\frac{\theta^2}{2}x^2\right)\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)^{m-1} e^{-m\theta x}}{(1+\alpha)^m} \times (1-p)p^{m-1} \\
&= \frac{\theta(1-p)e^{-\theta x}\left(\alpha+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)} \sum_{m=1}^{\infty} \frac{mp^{m-1}e^{-\theta x}}{(1+\alpha)^{m-1}}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)^{m-1} \\
&= \frac{\theta(1-p)e^{-\theta x}\left(\alpha+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\left[1-\frac{pe^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right]^{-2}.
\end{aligned}$$

We have the following definition for the new distribution obtained from the above synthesis.

Definition 2.1. *An absolutely continuous random variable X will be said to follow the quasi xgamma-geometric (QXGGc) distribution with parameters $\alpha(>0)$, $\theta(>0)$, and $p\in(0,1)$ if its pdf is of the form*

$$f(x) = K(\alpha,\theta,p)\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}\left[1-\frac{pe^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right]^{-2}, x>0, \quad (1)$$

where $K(\alpha,\theta,p) = \frac{\theta(1-p)}{(1+\alpha)}$.

We denote it by $X \sim QXGGc(\alpha,\theta,p)$.

Special cases:

(i) When $\alpha = \theta$ in (1), we obtain a new family of probability distributions, can be termed as xgamma-geometric (XGGc) distribution, with the following pdf:

$$f_1(x) = K_1(\theta, p)\left(1 + \frac{\theta}{2}x^2\right)e^{-\theta x}\left[1 - \frac{pe^{-\theta x}\left(1 + \theta + \theta x + \frac{\theta^2}{2}x^2\right)}{(1+\theta)}\right]^{-2}, x > 0,$$

where $K_1(\theta, p) = \frac{\theta^2(1-p)}{(1+\theta)}$.
We may denote it by $X \sim XGGc(\theta, p)$.

(ii) While $p \to 0$ in (1), the QXG model is obtained.

(iii) When $\alpha = \theta$ and $p \to 0$ in (1), we obtain xgamma (XG) distribution with parameter θ (see for more details, [29]).

The cdf of QXGGc(α, θ, p) is obtained as

$$\begin{aligned} F(x) &= K(\alpha, \theta, p)\int_0^x \left(\alpha + \frac{\theta^2}{2}z^2\right)e^{-\theta z}\left[1 - \frac{pe^{-\theta z}\left(1 + \alpha + \theta z + \frac{\theta^2}{2}z^2\right)}{(1+\alpha)}\right]^{-2} dz \\ &= \left[1 - \frac{e^{-\theta x}\left(1 + \alpha + \theta x + \frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right]\left[1 - \frac{pe^{-\theta x}\left(1 + \alpha + \theta x + \frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right]^{-1}, x > 0. \end{aligned} \tag{2}$$

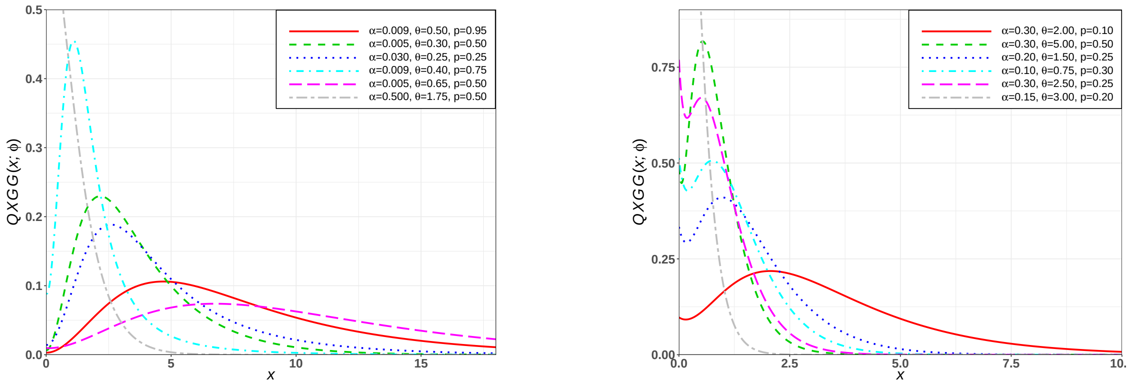
The survival function (or reliability function) (sf) of QXGGc model is derived as

$$S(x) = \frac{(1-p)}{(1+\alpha)}e^{-\theta x}\left(1 + \alpha + \theta x + \frac{\theta^2}{2}x^2\right)\left[1 - \frac{pe^{-\theta x}\left(1 + \alpha + \theta x + \frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right]^{-1}.$$

The hazard rate function (hrf) is obtained as

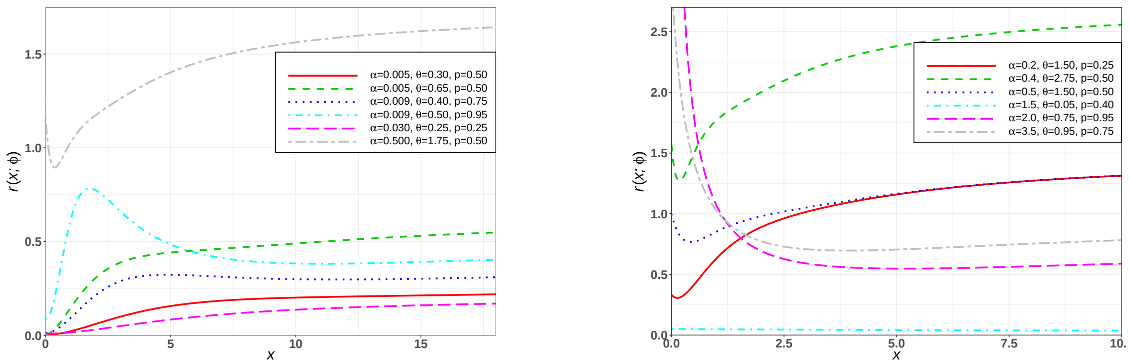
$$r(x) = \frac{f(x)}{S(x)} = \frac{\theta\left(\alpha + \frac{\theta^2}{2}x^2\right)}{\left(1 + \alpha + \theta x + \frac{\theta^2}{2}x^2\right)}\left[1 - \frac{pe^{-\theta x}\left(1 + \alpha + \theta x + \frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right]^{-1}. \tag{3}$$

The probability density curves of the QXGGc(α, θ, p) for different values of α, θ and p are shown in Figure 1. The hrf plots of the QXGGc distribution are displayed in Figure 2. Figure 1 reveals that the QXGGc density function can be reversed J-shape, concave down, symmetric, right skewed, unimodal or bimodal shapes. The hrf of the QXGGc distribution can be constant, decreasing, increasing, decreasing-increasing, upside down bathtub or bathtub failure rate shapes.



(a) (b)

Figure 1: QXGGc density plots for different values of α, θ and p.



(a) (b)

Figure 2: QXGGc hrf plots for different values of α, θ and p.

3. Series representation

In this section, we provide a useful series representation for the QXGGc pdf, cdf and sf. Using the Maclaurin series of $(1-x)^{-2} = \sum_{k=0}^{\infty}(k+1)x^k$, we can write

$$\left[1 - \frac{pe^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2x^2}{2}\right)}{\alpha+1}\right]^{-2} = \sum_{k=0}^{\infty}\frac{(k+1)p^k e^{-\theta k x}}{(\alpha+1)^k}\left[(1+\theta x)+\left(\alpha+\frac{\theta^2x^2}{2}\right)\right]^k,$$

and by using the binomial theorem, we can write

$$\left[(1+\theta x)+\left(\alpha+\frac{\theta^2x^2}{2}\right)\right]^k = \sum_{n=0}^{k}\binom{k}{n}(1+\theta x)^{k-n}\left(\alpha+\frac{\theta^2x^2}{2}\right)^n. \tag{4}$$

Substituting (4) in equation (1), we can write

$$f(x) = \theta(1-p)\sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k}{n}\frac{(k+1)p^k e^{-\theta(k+1)x}}{(\alpha+1)^{k+1}}(1+\theta x)^{k-n}\left(\alpha+\frac{\theta^2x^2}{2}\right)^{n+1}.$$

Similarly, apply the binomial theorem on the expressions $(1+\theta x)^{k-n}$ and $\left(\alpha+\frac{\theta^2x^2}{2}\right)^{n+1}$, respectively

$$(1+\theta x)^{k-n}=\sum_{j=0}^{k-n}\binom{k-n}{j}\theta^j x^j \tag{5}$$

and

$$\left(\alpha+\frac{\theta^2x^2}{2}\right)^{n+1}=\sum_{i=0}^{n+1}\binom{n+1}{i}\frac{\theta^{2i}\alpha^{n-i+1}}{2^i}x^{2i}. \tag{6}$$

Therefore, the pdf of the QXGGc in (1) can be expressed as

$$f(x)=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\,\Gamma(2i+j+1)\,g\left(x;2i+j+1,(k+1)\theta\right),$$

where

$$K_{k,n,i,j}(\alpha,\theta,p)=\binom{n+1}{i}\binom{k}{n}\binom{k-n}{j}\frac{(1-p)p^k\alpha^{n-i+1}}{2^i(\alpha+1)^{k+1}(k+1)^{2i+j}},$$

and $g(t;\alpha,\beta)=\frac{\beta^\alpha}{\Gamma(\alpha)}t^{\alpha-1}e^{-\beta t}$ is the gamma distribution.

The cdf of the QXGGc in (2) can be expressed as

$$F(x)=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\,\gamma\left(2i+j+1,(k+1)\theta x\right), \tag{7}$$

where $\gamma(\alpha,y)=\int_0^y e^{-t}t^{\alpha-1}dt$ is the lower incomplete gamma function.

The pdf of QXGGc can be expressed as a mixture of gamma densities.

Using the Maclaurin series of $(1-x)^{-1}=\sum_{k=0}^{\infty}x^k$, we can write

$$\left[1-\frac{pe^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2x^2}{2}\right)}{\alpha+1}\right]^{-1}=\sum_{k=0}^{\infty}\frac{p^ke^{-\theta kx}}{(\alpha+1)^k}\left[(1+\theta x)+\left(\alpha+\frac{\theta^2x^2}{2}\right)\right]^k.$$

Form equations (4), (5) and (6), sf can be written as

$$\begin{aligned}S(x)&=\frac{(1-p)}{(1+\alpha)}e^{-\theta x}\sum_{k=0}^{\infty}\frac{p^ke^{-\theta kx}}{(\alpha+1)^k}\left[(1+\theta x)+\left(\alpha+\frac{\theta^2x^2}{2}\right)\right]^{k+1}.\\&=(1-p)\sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\binom{k+1}{n}\frac{p^ke^{-\theta(k+1)x}}{(\alpha+1)^{k+1}}(1+\theta x)^{k-n+1}\left(\alpha+\frac{\theta^2x^2}{2}\right)^n.\\&=\sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\sum_{j=0}^{k-n+1}\sum_{i=0}^{n}K^*_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)g\left(x;2i+j+1,(k+1)\theta\right),\end{aligned}$$

where

$$K^*_{k,n,i,j}(\alpha,\theta,p)=\binom{k+1}{n}\binom{k-n+1}{j}\binom{n}{i}\frac{(1-p)p^k\alpha^{n-i}}{2^i\theta(\alpha+1)^{k+1}(k+1)^{2i+j+1}}.$$

4. Properties

In this section, we provide some statistical and reliability properties of the QXGGc distribution such as shape, quantile function, ordinary and incomplete moments, mean deviation about the mean and mean deviation about the median, moment generating function, Rényi entropy measure, mean residual life function and mean inactivity time.

4.1. Shapes of pdf and hrf

In this subsection, the behavior of the shapes of pdf and hrf of the random variable X follows the distribution in (1).
The pdf in (1) has the following limits to 0 and $+\infty$, respectively

$$\lim_{x\to 0} f(x) = \frac{\alpha\theta}{\alpha(1-p)-p+1}$$

and

$$\lim_{x\to +\infty} f(x) = 0.$$

The hrf in (3) has the following limits to 0 and $+\infty$, respectively

$$\lim_{x\to 0} r(x) = \frac{\alpha\theta}{(\alpha+1)(1-p)}$$

and

$$\lim_{x\to +\infty} r(x) = \theta.$$

4.2. Identifiablity of the distribution

In this subsection, we discuss about the identifiability of $QXGGc(\alpha, \theta, p)$.
Let $\phi_1 = (\alpha_1, \theta_1, p_1)$ and $\phi_2 = (\alpha_2, \theta_2, p_2)$ be two sets of parameters and $f_1(x; \phi_1)$ and $f_2(x; \phi_2)$ be the corresponding pdfs. Keeping in mind the definition of identifiability, we have,

$$\begin{aligned}
& f_1(x;\phi_1) = f_2(x;\phi_2) \\
& \iff r_1(x;\phi_1) = r_2(x;\phi_2) \text{ where } r_1(\cdot) \text{ and } r_1(\cdot) \text{ given in (3)} \\
& \iff \theta_1 = \theta_2 \text{ when } x \to \infty.
\end{aligned}$$

Hence the parameter θ is identifiable.

The parameters α and p are not identifiable, because we find $f_1(\alpha_1 = 2, \theta = 2, p_1 = 0.5) = f_2(\alpha_2 = 3, \theta = 2, p_2 = 0.4375) = 2.66667$ when $x \to 0$, but $\alpha_1 \neq \alpha_2$ and $p_1 \neq p_2$.

4.3. Quantile function

The quantile function (qf) of the QXGGc distribution can be obtained by solve $F(x) = \lambda$ for x, where $0 < \lambda < 1$. There is no closed form for the qf of QXGGc distribution. The λ^{th} quantile of the QXGGc distribution can be obtained by numerically solving the following equation for x

$$e^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)-\frac{(\alpha+1)(1-\lambda)}{1-p\lambda} = 0 \tag{8}$$

The median (M) of the QXGGc distribution can be calculated from (8) by substituting $\lambda = 0.5$ and solving the equation numerically.

4.4. Moments

The r^{th} raw moment of the QXGGc distribution is given by

$$\begin{aligned}
\mu'_r &= E(X^r) \\
&= \int_0^\infty x^r \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)\, g\left(x; 2i+j+1, (k+1)\theta\right) dx \\
&= \int_0^\infty \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)\, x^r\, g\left(x; 2i+j+1, (k+1)\theta\right) dx \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)\int_0^\infty x^r\, g\left(x; 2i+j+1, (k+1)\theta\right) dx \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\frac{\Gamma(2i+j+r+1)}{(k+1)^r\theta^r}. \qquad (9)
\end{aligned}$$

In particular,

$$\mu_X = \mu'_1 = E(X) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\frac{\Gamma(2i+j+2)}{(k+1)\theta},$$

$$\mu'_2 = E(X^2) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\frac{\Gamma(2i+j+3)}{(k+1)^2\theta^2}$$

and

$$Var(X) = \mu'_2 - \mu_X^2.$$

The mean is shown graphically in Figure 3, from this plot, it is noted that the mean is decreasing for increasing values of α, θ and p. The variance is shown graphically in Figure 4, from this plot, it is noted that the variance is decreasing for increasing values of θ and p, and the variance is increasing then decreasing for increasing values of α.
The central moments μ_r can be found from (9) as

$$\mu_r = E(X-\mu_X)^r = \sum_{m=0}^{r}\binom{r}{m}(-1)^m \mu_X^r \mu'_{r-m}.$$

Table 1 shows some results of numerical integration of the central mean, $\int_0^\infty xf(x)dx$, via the approximation by the summation formula above, μ_X, at truncated N terms instead of ∞, for different values of α, θ and p. These results are calculated by R, the R code can be found in Appendix Appendix A.

4.5. Incomplete moments

For any $t > 0$, the r^{th} incomplete moment is defined as

$$\mu_r(t) = E(X^r | x < t) = \int_0^t x^r f(x) dx.$$

Table 1: The numerical integration and the summation formula of the central mean for different values of α, θ and p at truncated N terms.

p	α	θ	N	Summation	Numerical integration
0.1	0.8	1.2	10	1.67744	
			50	1.67744	1.67744
			80	1.67744	
		6.0	10	0.33549	
			50	0.33549	0.33549
			80	0.33549	
	2.5	1.2	10	1.24126	
			50	1.24126	1.24126
			80	1.24126	
		6.0	10	0.24825	
			50	0.24825	0.24825
			80	0.24825	
0.4	0.8	1.2	10	1.38272	
			50	1.38273	1.38273
			80	1.38273	
		6.0	10	0.27654	
			50	0.27655	0.27655
			80	0.27655	
	2.5	1.2	10	1.00063	
			50	1.00064	1.00064
			80	1.00064	
		6.0	10	0.20013	
			50	0.20013	0.20013
			80	0.20013	
0.9	0.8	1.2	10	0.45420	
			50	0.48836	0.48851
			80	0.48851	
		6.0	10	0.09084	
			50	0.09767	0.09770
			80	0.09770	
	2.5	1.2	10	0.30650	
			50	0.32702	0.32711
			80	0.32711	
		6.0	10	0.06130	
			50	0.06540	0.06540
			80	0.06540	

The r^{th} incomplete momentof the QXGGc distribution is given by

$$\begin{aligned}
\mu_r(t) &= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)\int_0^t x^r\, g\left(x;2i+j+1,(k+1)\theta\right)dx \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)\int_0^t g\left(x;2i+j+r+1,(k+1)\theta\right)dx \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\frac{\gamma\left(2i+j+r+1,(k+1)\theta t\right)}{(k+1)^r\theta^r}. \qquad (10)
\end{aligned}$$

Also, we can write $F(x) = \mu_0(x)$ and $\mu'_r = \mu_r(\infty)$.

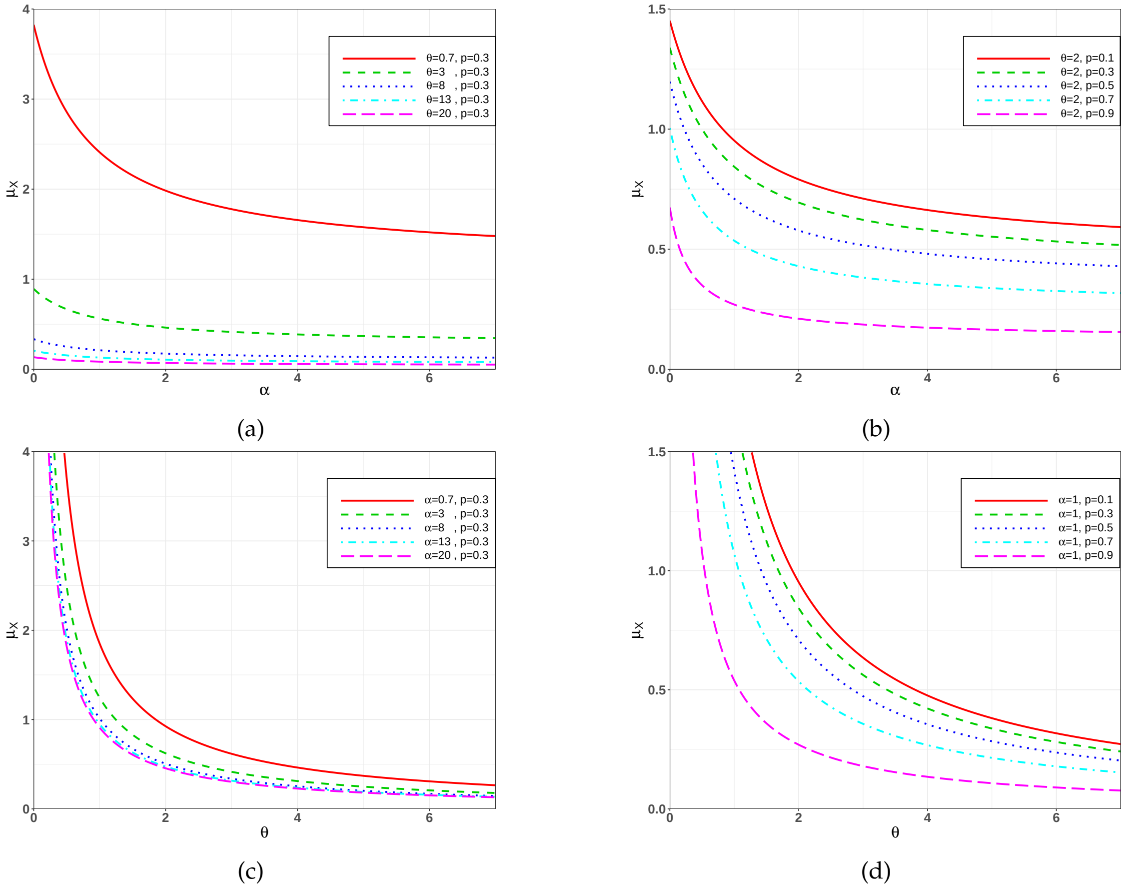


Figure 3: Plot of the mean of QXGGc for different values of α, θ and p.

4.6. *The mean deviations about the mean and the median*

Suppose that the median of the QXGGc distribution denoted by M, then the mean deviations about the mean and the median can be calculated as

$$\delta_{\mu_X} = \int_0^\infty |x - \mu_X| f(x)dx = 2F(\mu_X) - 2\mu_1(\mu_X)$$

and

$$\delta_M = \int_0^\infty |x - M| f(x)dx = \mu_X - 2\mu_1(M),$$

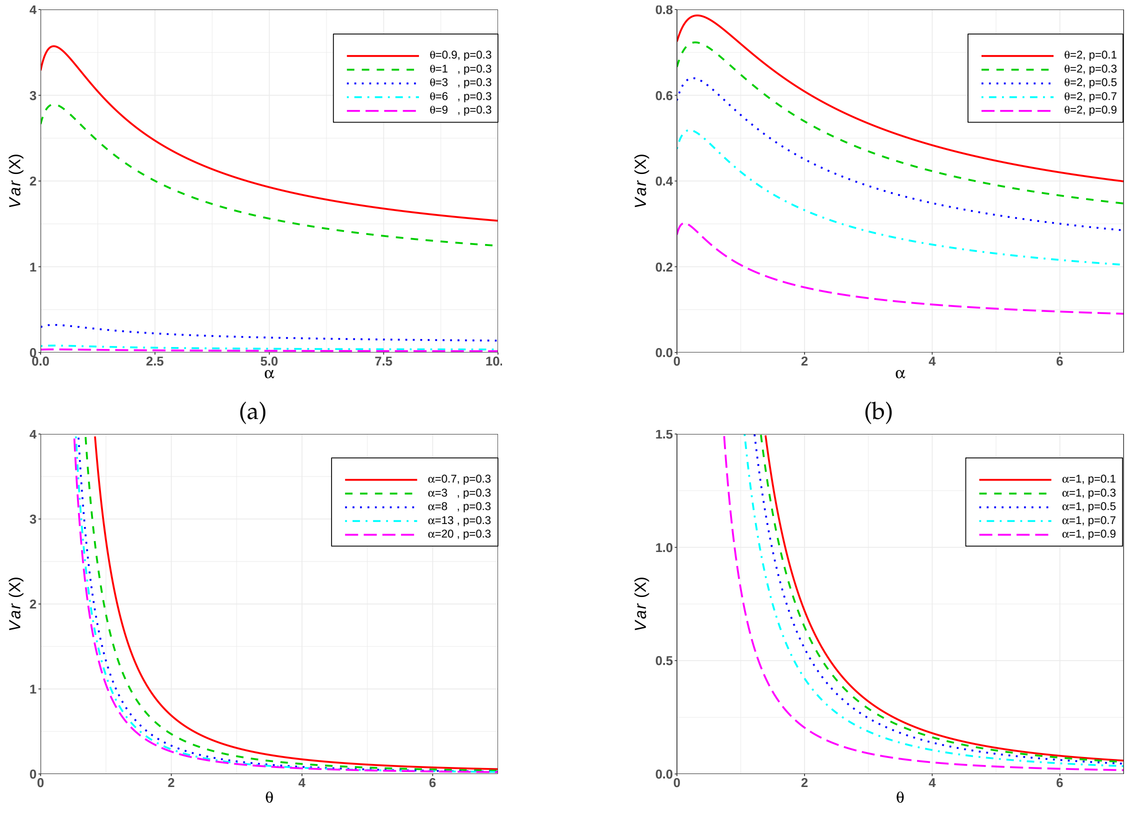
respectively, where $\mu_1(\mu_X)$ and $\mu_1(M)$ can be numerically computed from (10), and $F(\mu_X)$ can be numerically computed from (7).

4.7. Moment generating function

The moment generating function of the QXGGc distribution is given by

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)\int_0^{\infty} e^{tx}\, g\left(x;2i+j+1,(k+1)\theta\right)dx \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)\left(1-\frac{t}{(k+1)\theta}\right)^{-(2i+j+1)}, t<(k+1)\theta.
\end{aligned}
$$

It is to be noted that interchange of summation and integration is possible by Dominated Convergence Theorem.



(a) (b) (c) (d)

Figure 4: Plot of the variance of QXGGc for different values of α, θ and p.

4.8. Rényi entropy

The Rényi entropy of the random variable X is a measure of variation of the uncertainty and it is defined by

$$
H_\delta(x) = \frac{1}{1-\delta}\log\left[\int_0^{\infty} f^{\delta}(x)dx\right], \quad \delta>0, \quad \delta\neq 1.
$$

From equation (1), we have

$$f^{\delta}(x) = K_{\delta}(\alpha,\theta,p)\left(\alpha+\frac{\theta^2}{2}x^2\right)^{\delta} e^{-\theta\delta x}\left[1-\frac{pe^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right]^{-2\delta}, x>0,$$

where $K_{\delta}(\alpha,\theta,p) = \frac{\theta^{\delta}(1-p)^{\delta}}{(1+\alpha)^{\delta}}$. For $|a|<1$, the general binomial series holds

$$(1-a)^{-p} = \sum_{k=0}^{\infty}\frac{\Gamma(p+k)}{k!\Gamma(p)}a^k;\ p>0. \tag{11}$$

So, from (11), for $\delta>0$, we can write

$$\left[1-\frac{pe^{-\theta x}\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)}{(1+\alpha)}\right]^{-2\delta} = \sum_{k=0}^{\infty}\frac{\Gamma(2\delta+k)p^k e^{-\theta kx}}{k!\Gamma(2\delta)(1+\alpha)^k}\left[1+\theta x+\alpha+\frac{\theta^2}{2}x^2\right]^k.$$

Form equations (4), (5) and (6), $f^{\delta}(x)$ becomes

$$f^{\delta}(x) = (1-p)^{\delta}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+\delta}\sum_{j=0}^{k-n}S_{\delta,k,n,i,j}(\alpha,\theta,p)e^{-\theta(k+\delta)x}x^{2i+j},$$

where

$$S_{\delta,k,n,i,j}(\alpha,\theta,p) = \binom{n+\delta}{i}\binom{k}{n}\binom{k-n}{j}\frac{\Gamma(2\delta+k)p^k\theta^{2i+j+\delta}\alpha^{n-i+\delta}}{2^i k!\Gamma(2\delta)(1+\alpha)^{k+\delta}}.$$

Therefore, the Rényi entropy of the QXGGc distribution, is given by

$$H_{\delta}(x) = \frac{\delta}{1-\delta}\log(1-p)+\frac{1}{1-\delta}\log\left[\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+\delta}\sum_{j=0}^{k-n}S^{*}_{\delta,k,n,i,j}(\alpha,\theta,p)\right],$$

where

$$S^{*}_{\delta,k,n,i,j}(\alpha,\theta,p) = \binom{n+\delta}{i}\binom{k}{n}\binom{k-n}{j}\frac{\Gamma(2i+j+1)\Gamma(2\delta+k)p^k\theta^{\delta-1}\alpha^{n+\delta-i}}{2^i k!(\alpha+1)^{k+\delta}\Gamma(2\delta)(k+\delta)^{2i+j+1}}.$$

It is to be noted that interchange of summation and integration is possible by Dominated Convergence Theorem.

4.9. *Mean residual life and mean inactivity time*

The mean residual life (MRL) of the random variable X, is given by the general formula

$$\mu_T(t) = E[(X-t)|X>t] = \frac{1}{S(t)}\int_t^{\infty}S(u)du,\ t>0.$$

Then the integration $\int_t^{\infty}S(u)du$ can be written as

$$\begin{aligned}\int_t^{\infty}S(u)du &= \sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\sum_{j=0}^{k-n+1}\sum_{i=0}^{n}K^{*}_{k,n,i,j}(\alpha,\theta,p)\Gamma(2i+j+1)\int_t^{\infty}g\left(u;2i+j+1,(k+1)\theta\right)du\\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\sum_{j=0}^{k-n+1}\sum_{i=0}^{n}K^{*}_{k,n,i,j}(\alpha,\theta,p)\,\Gamma\left(2i+j+1,(k+1)\theta t\right),\end{aligned}$$

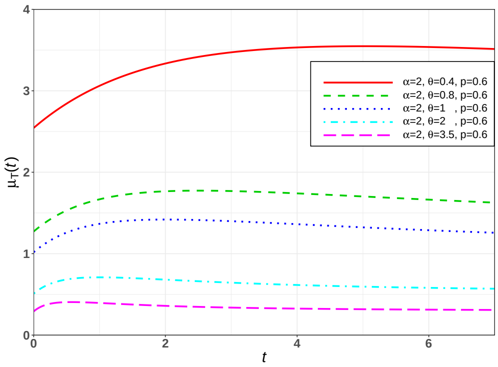
where $\Gamma(\alpha, y) = \int_y^\infty e^{-t} t^{\alpha-1} dt$ is the upper incomplete gamma function. So, the MRL of the QXGGc distribution is given by

$$\mu_T(t) = \frac{1}{S(t)} \sum_{k=0}^{\infty} \sum_{n=0}^{k+1} \sum_{j=0}^{k-n+1} \sum_{i=0}^{n} K^*_{k,n,i,j}(\alpha, \theta, p)\, \Gamma\left(2i + j + 1, (k+1)\theta t\right), \ t > 0.$$

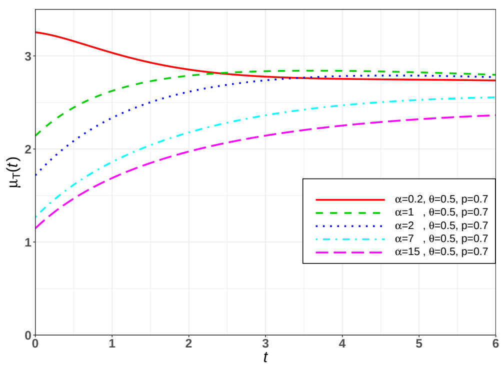
The mean inactivity time (MIT) function of the QXGGc distribution is given by

$$\begin{aligned} m_T(t) &= \frac{1}{F(t)} \int_0^t F(u) du, \ t > 0 \\ &= \frac{1}{F(t)} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha, \theta, p) \int_0^t \gamma\left(2i + j + 1, (k+1)\theta u\right) du \\ &= \frac{1}{F(t)} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha, \theta, p)\, t\, \gamma\left(2i + j + 1, (k+1)\theta t\right) \\ &- \frac{1}{F(t)} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha, \theta, p) \frac{1}{(k+1)\theta} \gamma\left(2i + j + 2, (k+1)\theta t\right). \end{aligned}$$

The MRL and MIT are shown graphically in Figure 5 and Figure 6, respectively, from these plots, it is observed that the MRL and MIT have different shapes based on the values of α, θ and p.



(a)



(b)

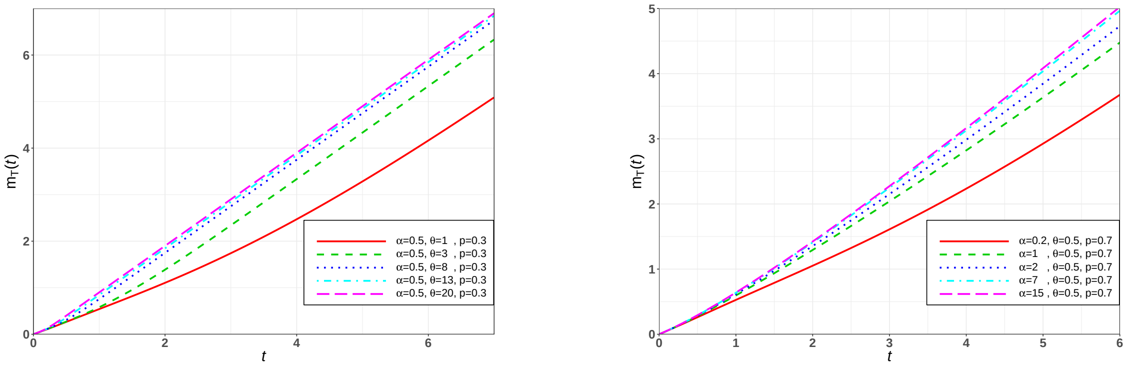
Figure 5: Plot of the MRL of QXGGc for different values of α, θ and p.

5. Characterization

The fitting of probability distributions to real world data is an important area of research. The characterizations give methods to identify a distribution using some basic properties. The characterizations presented in this section are how to identify that the distribution under consideration is QXGGc using the truncated first moment. We will need the following two Lemmas for the characterization of the proposed distribution.

Assumption A

Let X be an absolutely continuous random variable with cdf $F(x)$ and pdf $f(x)$. We assume $E(X)$ exists and $f(x)$ is differentiable. we assume further $\alpha = sup\{x|f(x) > 0\}$ and $\beta = \inf\{x|f(x) < 1\}$.



(a) (b)

Figure 6: Plot of the MIT of QXGGc for different values of α, θ and p.

Lemma 5.1. *If*

$$E(X|X \leq x) = g(x)\frac{f(x)}{F(x)},$$

where $g(x)$ is a continuous differentiable function in (α, β), then

$$f(x) = c\exp\left(\int \frac{x - g'(x)}{g(x)} dx\right),$$

and c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$

Proof. Let

$$g(x) = \frac{1}{f(x)} \int_\alpha^x u f(u) du,$$

thus

$$\int_\alpha^x u f(u) du = f(x) g(x).$$

Differentiating both sides of the above equation with respect to x, we obtain

$$x f(x) = f'(x) g(x) + f(x) g'(x).$$

On simplification, we get

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}.$$

On integrating both sides of the above equation, we obtain

$$f(x) = c\exp\left(\int \frac{x - g'(x)}{g(x)} dx\right),$$

and c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$. □

Lemma 5.2. *Under the assumption A, if*

$$E(X|X \geq x) = h(x)\frac{f(x)}{1 - F(x)},$$

where h(x) is a continuous differentiable function in (α, β), *then*

$$f(x) = c\exp\left(\int \frac{x + h'(x)}{h(x)}dx\right),$$

and c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$.

Proof. Let

$$h(x) = \frac{1}{f(x)}\int_x^\infty uf(u)du,$$

thus

$$\int_x^\infty uf(u)du = f(x)h(x).$$

Differentiating both sides of the above equation, we obtain

$$-xf(x) = f'(x)h(x) + f(x)h'(x).$$

On simplification, we obtain

$$\frac{f'(x)}{f(x)} = -\frac{x + h'(x)}{h(x)}.$$

On integrating both sides of the above equation, we obtain

$$f(x) = c\exp\left(-\int \frac{x + h'(x)}{h(x)}dx\right),$$

and c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$. □

Theorem 5.3. *Suppose that the random variable X satisfies the conditions of the assumption A with* $\alpha = 0$ *and* $\beta = \infty$. *Then*

$$E(X|X \leq x) = g(x)\tau(x),$$

where $\tau(x) = \frac{f(x)}{F(x)}$ *and*

$$g(x) = \frac{\sum_{k=0}^\infty \sum_{n=0}^k \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha, \theta, p)\, \Gamma(x, 2i + j + 2, \theta(k + 1))}{\sum_{k=0}^\infty \sum_{n=0}^k \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha, \theta, p)\, g(x, 2i + j + 1, \theta(k + 1))},$$

where $\Gamma(x, \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x u^{\alpha-1}e^{-\beta u}du$, *if and only if*

$$f(x) = \sum_{k=0}^\infty \sum_{n=0}^k \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha, \theta, p)\, g(x, 2i + j + 1, \theta(k + 1)).$$

Proof. Suppose that

$$f(x)=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;g(x,2i+j+1,\theta(k+1)),$$

then

$$\begin{aligned}g(x)f(x) &= \int_x^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;u\;g(u,2i+j+1,\theta(k+1))du\\ &= \int_0^{x}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;\frac{u^{2i+j+1}e^{-\theta(k+1)u}}{\Gamma(2i+j+1)}du\\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;\Gamma(x,2i+j+2,\theta(k+1)),\end{aligned}$$

where

$$\Gamma(x,\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^x u^{\alpha-1}e^{-\beta u}du.$$

Thus

$$g(x)=\frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;\Gamma(x,2i+j+2,\theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;g(x,2i+j+1,\theta(k+1))}.$$

Suppose that,

$$g(x)=\frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;\Gamma(x,2i+j+2,\theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;g(x,2i+j+1,\theta(k+1))},$$

then

$$\begin{aligned}g\prime(x) = x-&\left(\frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;\Gamma(x,2i+j+2,\theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;g(x,2i+j+1,\theta(k+1))}\right)\\ \times&\left(\frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=1}^{n+1}\sum_{j=1}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\frac{\theta(k+1)(2i+j-\theta(k+1)x)}{2i+j}g(x,2i+j,\theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;g(x,2i+j+1,\theta(k+1))}\right).\\ = x-g(x)&\left(\frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=1}^{n+1}\sum_{j=1}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\frac{\theta(k+1)(2i+j-\theta(k+1)x)}{2i+j}\;g(x,2i+j,\theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;g(x,2i+j+1,\theta(k+1))}\right).\end{aligned}$$

Thus

$$\frac{x-g'(x)}{g(x)}=\frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=1}^{n+1}\sum_{j=1}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\frac{\theta(k+1)(2i+j-\theta(k+1)x)}{2i+j}\;g(x,2i+j,\theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;g(x,2i+j+1,\theta(k+1))}.$$

By Lemma 1

$$\frac{f'(x)}{f(x)}=\frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=1}^{n+1}\sum_{j=1}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\frac{\theta(k+1)(2i+j-\theta(k+1)x)}{2i+j}\;g(x,2i+j,\theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n}K_{k,n,i,j}(\alpha,\theta,p)\;g(x,2i+j+1,\theta(k+1))}.$$

On integrating the above equation with respect to x, we obtain

$$f(x) = c\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1)).$$

Using the condition $\int_0^\infty f(x)dx = 1$, we obtain

$$f(x) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1)).$$

□

Theorem 5.4. *Suppose that the random variable X satisfies the conditions of the assumption A with $\alpha = 0$ and $\beta = \infty$. Then*

$$E(X|X \geq x) = h(x)\tau(x),$$

where $\tau(x) = \frac{f(x)}{F(x)}$ *and*

$$h(x) = \frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ \gamma(x, 2i+j+2, \theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1))},$$

where $\gamma(x,\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_x^\infty u^{\alpha-1}e^{-\beta u}du$, *if and only if*

$$f(x) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1)).$$

Proof. If the pdf of the random variable X is given by

$$f(x) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1)),$$

then

$$\begin{aligned}
f(x)h(x) &= \int_x^\infty \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ u\ g(u, 2i+j+1, \theta(k+1))du. \\
&= \int_x^\infty \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ \frac{u^{2i+j+1}e^{-\theta(k+1)u}}{\Gamma(2i+j+1)}du. \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ \gamma(x, 2i+j+2, \theta(k+1)).
\end{aligned}$$

Thus

$$h(x) = \frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ \gamma(x, 2i+j+2, \theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1))}.$$

Suppose that

$$h(x) = \frac{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ \gamma(x, 2i+j+2, \theta(k+1))}{\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{i=0}^{n+1}\sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1))},$$

then

$$h'(x) = -x - \left(\frac{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ \gamma(x, 2i+j+2, \theta(k+1))}{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1))} \right)$$
$$\times \left(\frac{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=1}^{n+1} \sum_{j=1}^{k-n} K_{k,n,i,j}(\alpha,\theta,p) \frac{\theta(k+1)(2i+j-\theta(k+1)x)}{2i+j} g(x, 2i+j, \theta(k+1))}{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1))} \right)$$
$$= -x - h(x) \times \left(\frac{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=1}^{n+1} \sum_{j=1}^{k-n} K_{k,n,i,j}(\alpha,\theta,p) \frac{\theta(k+1)(2i+j-\theta(k+1)x)}{2i+j}\ g(x, 2i+j, \theta(k+1))}{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1))} \right).$$

Thus

$$-\frac{x + h'(x))}{h(x)} = \frac{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=1}^{n+1} \sum_{j=1}^{k-n} K_{k,n,i,j}(\alpha,\theta,p) \frac{\theta(k+1)(2i+j-\theta(k+1)x)}{2i+j}\ g(x, 2i+j, \theta(k+1))}{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1))}.$$

By Lemma 5.2

$$\frac{f'(x)}{f(x)} = \frac{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=1}^{n+1} \sum_{j=1}^{k-n} K_{k,n,i,j}(\alpha,\theta,p) \frac{\theta(k+1)(2i+j-\theta(k+1)x)}{2i+j}\ g(x, 2i+j, \theta(k+1))}{\sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1))}.$$

On integrating the above equation with respect to x, we obtain

$$f(x) = c \sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1)).$$

Using the condition $\int_0^\infty f(x)dx = 1$, we obtain

$$f(x) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{i=0}^{n+1} \sum_{j=0}^{k-n} K_{k,n,i,j}(\alpha,\theta,p)\ g(x, 2i+j+1, \theta(k+1)).$$

□

6. Estimation of parameters

In this section, we estimate of the QXGGc parameters using seven frequentist approaches. These methods are: the maximum likelihood, least squares, weighted least-squares, maximum product of spacing, Cramér–von Mises estimation, Anderson–Darling and Right-tail Anderson–Darling estimators.

6.1. Maximum likelihood estimation

In this section, we consider the Maximum likelihood method to estimate the QXGG parameters from complete samples. Let $x_1, \ldots, x_n$ be a random sample of size n from the QXGG distribution with parameter vector $\varphi = (\alpha, \theta, p)^\intercal$. Then, the log-likelihood function for φ reduces to

$$\begin{aligned} \ell &= n\log\theta + n\log(1-p) - n\log(1+\alpha) + \sum_{i=1}^{n} \log\left(\alpha + \frac{\theta^2}{2}x_i^2\right) \\ &\quad -\theta\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} \log\left[1 - \frac{pe^{-\theta x_i}\left(1+\alpha+\theta x_i + \frac{\theta^2}{2}x_i^2\right)}{(1+\alpha)}\right]. \end{aligned}$$

The Maximum likelihood estimators (MLEs) of the unknown parameters α, θ and p of the QXGGc distribution can be obtained by maximizing the last equation. This can also be done by using different programs namely `R` (`optim` function), `SAS` (`PROC NLMIXED`) or by solving the nonlinear likelihood equations obtained by differentiating ℓ.

The components of the score vector, $\mathbf{U}(\varphi) = \frac{\partial \ell}{\partial \varphi} = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial p}\right)^{\intercal}$, are given by

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n}{1+\alpha} + \sum_{i=1}^{n} \frac{1}{\alpha + \frac{\theta^2}{2} x_i^2} - \frac{2}{1+\alpha} \sum_{i=1}^{n} \frac{p e^{-\theta x_i}\left(\theta x_i + \frac{\theta^2}{2} x_i^2\right)}{\alpha + 1 - s_i} = 0, \tag{12}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{\theta x_i^2}{\alpha + \frac{\theta^2}{2} x_i^2} - \sum_{i=1}^{n} x_i + 2 \sum_{i=1}^{n} \frac{p e^{-\theta x_i}\left(x_i + \theta x_i^2\right) - s_i}{1 + \alpha - s_i} = 0 \tag{13}$$

and

$$\frac{\partial \ell}{\partial p} = \frac{n}{p-1} + \frac{2}{p} \sum_{i=1}^{n} \frac{s_i}{\alpha + 1 - s_i} = 0, \tag{14}$$

where $s_i = p e^{-\theta x_i}\left(1 + \alpha + \theta x_i + \frac{\theta^2}{2} x_i^2\right)$.

Now, we will study the existence and uniqueness of the MLE estimates when the other parameter is known (or given).

Theorem 6.1. *Let us suppose that the parameters θ and p are known. Then, the of equation (12) has only one solution for $\alpha > 0$.*

Proof. Let $x_{(1)} = \min(x_1, x_2, ..., x_n)$ and $x_{(n)} = \max(x_1, x_2, ..., x_n)$, so $e^{-\theta x_{(n)}} \leq e^{-\theta x_i} \leq e^{-\theta x_{(1)}}$ for all $i = 1, 2, ..., n$ and

$$p e^{-\theta x_{(n)}}\left(1 + \alpha + \theta x_{(1)} + \frac{\theta^2}{2} x_{(1)}^2\right) \leq s_i \leq p e^{-\theta x_{(1)}}\left(1 + \alpha + \theta x_{(n)} + \frac{\theta^2}{2} x_{(n)}^2\right),$$

for all $i = 1, 2, ..., n$, so

$$-p e^{-\theta x_{(1)}}\left(1 + \alpha + \theta x_{(n)} + \frac{\theta^2}{2} x_{(n)}^2\right) \leq -s_i \leq -p e^{-\theta x_{(n)}}\left(1 + \alpha + \theta x_{(1)} + \frac{\theta^2}{2} x_{(1)}^2\right),$$

for all $i = 1, 2, ..., n$,

$$\frac{1}{\alpha + 1 - p e^{-\theta x_{(n)}}\left(1 + \alpha + \theta x_{(1)} + \frac{\theta^2}{2} x_{(1)}^2\right)} \leq \frac{1}{\alpha + 1 - s_i} \leq \frac{1}{\alpha + 1 - p e^{-\theta x_{(1)}}\left(1 + \alpha + \theta x_{(n)} + \frac{\theta^2}{2} x_{(n)}^2\right)}$$

for all $i = 1, 2, ..., n$, and

$$p e^{-\theta x_{(n)}}\left(\theta x_{(1)} + \frac{\theta^2}{2} x_{(1)}^2\right) \leq p e^{-\theta x_i}\left(\theta x_i + \frac{\theta^2}{2} x_i^2\right) \leq p e^{-\theta x_{(1)}}\left(\theta x_{(n)} + \frac{\theta^2}{2} x_{(n)}^2\right),$$

for all $i = 1, 2, ..., n$, this implies

$$\frac{p e^{-\theta x_{(n)}}\left(\theta x_{(1)} + \frac{\theta^2}{2} x_{(1)}^2\right)}{\alpha + 1 - p e^{-\theta x_{(n)}}\left(1 + \alpha + \theta x_{(1)} + \frac{\theta^2}{2} x_{(1)}^2\right)} \leq \frac{p e^{-\theta x_i}\left(\theta x_i + \frac{\theta^2}{2} x_i^2\right)}{\alpha + 1 - s_i} \leq \frac{p e^{-\theta x_{(1)}}\left(\theta x_{(n)} + \frac{\theta^2}{2} x_{(n)}^2\right)}{\alpha + 1 - p e^{-\theta x_{(1)}}\left(1 + \alpha + \theta x_{(n)} + \frac{\theta^2}{2} x_{(n)}^2\right)}.$$

for all $i = 1, 2, ..., n$, and

$$-\frac{2n}{1+\alpha}\frac{pe^{-\theta x_{(1)}}\left(\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)}{\alpha+1-pe^{-\theta x_{(1)}}\left(1+\alpha+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)} \leq -\frac{2}{1+\alpha}\sum_{i=1}^{n}\frac{pe^{-\theta x_i}\left(\theta x_i+\frac{\theta^2}{2}x_i^2\right)}{\alpha+1-s_i} \leq$$
$$-\frac{2n}{1+\alpha}\frac{pe^{-\theta x_{(n)}}\left(\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)}{\alpha+1-pe^{-\theta x_{(n)}}\left(1+\alpha+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)}.$$

Therefore, the equation (12) can be approximate as follows

$$-\frac{n}{1+\alpha}+\frac{n}{\alpha+\frac{\theta^2}{2}x_{(1)}^2}-\left(\frac{2n}{1+\alpha}\right)\frac{pe^{-\theta x_{(1)}}\left(\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)}{\alpha+1-pe^{-\theta x_{(1)}}\left(1+\alpha+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)} \leq$$
$$-\frac{n}{1+\alpha}+\sum_{i=1}^{n}\frac{1}{\alpha+\frac{\theta^2}{2}x_i^2}-\frac{2}{1+\alpha}\sum_{i=1}^{n}\frac{pe^{-\theta x_i}\left(\theta x_i+\frac{\theta^2}{2}x_i^2\right)}{\alpha+1-s_i} \leq$$
$$-\frac{n}{1+\alpha}+\frac{n}{\alpha+\frac{\theta^2}{2}x_{(n)}^2}-\left(\frac{2n}{1+\alpha}\right)\frac{pe^{-\theta x_{(n)}}\left(\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)}{\alpha+1-pe^{-\theta x_{(n)}}\left(1+\alpha+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)}.$$

The left hand of the inequality has only one solution for α at

$$\alpha=\frac{e^{\theta x_{(1)}}\left(4-2\theta^2x_{(1)}^2\right)-p\left(\theta\left(x_{(n)}\left(\theta^2x_{(1)}^2+2\right)\left(\theta x_{(n)}^2+2\right)-2\theta x_{(1)}^2\right)+4\right)}{2p\left(\theta\left(2x_{(n)}\left(\theta x_{(n)}+2\right)-\theta x_{(1)}^2\right)+2\right)+2e^{\theta x_{(1)}}\left(\theta^2x_{(1)}^2-2\right)},$$

similarly the right hand of the inequality has only one solution for α at

$$\alpha=\frac{e^{\theta x_{(n)}}\left(4-2\theta^2x_{(n)}^2\right)-p\left(\theta^2x_{(n)}^2\left(\theta x_{(1)}\left(\theta x_{(1)}+2\right)-2\right)+2\left(\theta x_{(1)}\left(\theta x_{(1)}+2\right)+2\right)\right)}{-2\theta^2px_{(n)}^2+2e^{\theta x_{(n)}}\left(\theta^2x_{(n)}^2-2\right)+4p\left(\theta x_{(1)}+1\right)^2}.$$

Therefore, the equation (12) has a unique solution. □

Theorem 6.2. *If the parameters α and p are known, then the equation (13) has at least one root on the interval $(0,\infty)$.*

Proof. It is easy to verify that $\lim_{\theta\to 0}\frac{\partial\ell}{\partial\theta}=+\infty$ and $\lim_{\theta\to+\infty}\frac{\partial\ell}{\partial\theta}=-\sum_{i=1}^{n}x_i$. So, there exists at least one solution on the interval $(0,\infty)$. This completes the proof. □

Theorem 6.3. *Let us suppose that the parameters α and θ are known. Then, the of equation (14) has a unique solution for $0<p<1$.*

Proof. Let $x_{(1)}=\min(x_1,x_2,...,x_n)$ and $x_{(n)}=\max(x_1,x_2,...,x_n)$ for all $i=1,2,...,n$, so $e^{-\theta x_{(n)}}\leq e^{-\theta x_i}\leq e^{-\theta x_{(1)}}$ for all $i=1,2,...,n$ and

$$pe^{-\theta x_{(n)}}\left(1+\alpha+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)\leq s_i\leq pe^{-\theta x_{(1)}}\left(1+\alpha+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right),$$

for all $i=1,2,...,n$, this implies

$$\frac{1}{(\alpha+1)-pe^{-\theta x_{(n)}}\left(1+\alpha+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)}\leq\frac{1}{(\alpha+1)-s_i}\leq\frac{1}{(\alpha+1)-pe^{-\theta x_{(1)}}\left(1+\alpha+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)}$$

for all $i = 1, 2, ..., n$, and then

$$\frac{pe^{-\theta x_{(n)}}\left(1+\alpha+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)}{(\alpha+1)-pe^{-\theta x_{(n)}}\left(1+\alpha+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)} \leq \frac{s_i}{(\alpha+1)-s_i} \leq \frac{pe^{-\theta x_{(1)}}\left(1+\alpha+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)}{(\alpha+1)-pe^{-\theta x_{(1)}}\left(1+\alpha+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)},$$

for all $i = 1, 2, ..., n$. Therefore, the equation (14) can be approximate as follows

$$\frac{n}{p-1}+\frac{2n}{p}\frac{pe^{-\theta x_{(n)}}\left(1+\alpha+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)}{(\alpha+1)-pe^{-\theta x_{(n)}}\left(1+\alpha+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2\right)} \leq \frac{n}{p-1}+\frac{2}{p}\sum_{i=1}^{n}\frac{s_i}{\alpha+1-s_i} \leq$$
$$\frac{n}{p-1}+\frac{2n}{p}\frac{pe^{-\theta x_{(1)}}\left(1+\alpha+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)}{(\alpha+1)-pe^{-\theta x_{(1)}}\left(1+\alpha+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2\right)}.$$

The left hand of the inequality has only one solution for p at

$$p = 2-\frac{(\alpha+1)e^{\theta x_{(n)}}}{(\alpha+1)+\theta x_{(1)}+\frac{\theta^2}{2}x_{(1)}^2},$$

similarly the right hand of the inequality has only one solution for p at

$$p = 2-\frac{(\alpha+1)e^{\theta x_{(1)}}}{(\alpha+1)+\theta x_{(n)}+\frac{\theta^2}{2}x_{(n)}^2}.$$

Therefore, the equation (14) has a unique solution. □

Newton-Rapshon as a numerical method is required to find the solution of the nonlinear system.

6.2. Ordinary and weighted least-square estimators

Let $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ be the order statistics of the random sample of size n from $F(\mathbf{x}; \alpha, \theta, p)$. The ordinary least square estimators (OLSEs) ([32]) $\widehat{\alpha}_{LSE}$, $\widehat{\theta}_{LSE}$ and $\widehat{p}_{LSE}$ can be obtained by minimizing

$$V(\alpha,\theta,p) = \sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\theta,p\right)-\frac{i}{n+1}\right]^2,$$

with respect to α, θ and p. Or equivalently, the OLSEs follow by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\theta,p\right)-\frac{i}{n+1}\right]\Delta_s\left(x_{(i)}|\alpha,\theta,p\right)=0,\ \ s=1,2,3,$$

where

$$\begin{aligned}\Delta_1\left(x_{(i)}|\alpha,\theta,p\right) &= \frac{\partial}{\partial\alpha}F\left(x_{(i)}|\alpha,\theta,p\right),\ \Delta_2\left(x_{(i)}|\alpha,\theta,p\right)=\frac{\partial}{\partial\theta}F\left(x_{(i)}|\alpha,\theta,p\right)\\ \text{and } \Delta_3\left(x_{(i)}|\alpha,\theta,p\right) &= \frac{\partial}{\partial p}F\left(x_{(i)}|\alpha,\theta,p\right).\end{aligned} \tag{15}$$

Note that the solution of Δ_s for $s = 1, 2, 3$ can be obtained numerically.

The weighted least-squares estimators (WLSEs) ([32]), $\widehat{\alpha}_{WLSE}$, $\widehat{\theta}_{WLSE}$ and $\widehat{p}_{WLSE}$, can be obtained by minimizing the following equation

$$W(\alpha,\theta,p) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)}|\alpha,\theta,p\right)-\frac{i}{n+1}\right]^2.$$

Further, the WLSEs can also be derived by solving the non-linear equations

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)}|\alpha,\theta,p\right)-\frac{i}{n+1}\right]\Delta_s\left(x_{(i)}|\alpha,\theta,p\right)=0,\ \ s=1,2,3,$$

where $\Delta_1(\cdot|\alpha,\theta,p)$, $\Delta_2(\cdot|\alpha,\theta,p)$ and $\Delta_3(\cdot|\alpha,\theta,p)$ are provided in (15).

6.3. *Method of maximum product of spacings*

The maximum product of spacings (MPS) method ([8, 9] and [24]), as an approximation to the Kullback-Leibler information measure, is a good alternative to the MLE method.

Let $D_i(\alpha,\theta,p)=F\left(x_{(i)}|\alpha,\theta,p\right)-F\left(x_{(i-1)}|\alpha,\theta,p\right)$, for $i=1,2,\dots,n+1$, be the uniform spacings of a random sample from the QXGGc distribution, where $F\left(x_{(0)}|\alpha,\theta,p\right)=0$, $F\left(x_{(n+1)}|\alpha,\theta,p\right)=1$ and $\sum_{i=1}^{n+1}D_i(\alpha,\theta,p)=1$. The maximum product of spacings estimators (MPSEs) for $\widehat{\alpha}_{MPSE}$, $\widehat{\theta}_{MPSE}$ and $\widehat{p}_{MPSE}$ can be obtained by maximizing the geometric mean of the spacings

$$G(\alpha,\theta,p)=\left[\prod_{i=1}^{n+1}D_i(\alpha,\theta,p)\right]^{\frac{1}{n+1}}$$

with respect to α,θ and p, or, equivalently, by maximizing the logarithm of the geometric mean of sample spacings

$$H(\alpha,\theta,p)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\theta,p).$$

The MPSEs of the QXGGc parameters can be obtained by solving the nonlinear equations defined by

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\theta,p)}\left[\Delta_s(x_{(i)}|\alpha,\theta,p)-\Delta_s(x_{(i-1)}|\alpha,\theta,p)\right]=0,\quad s=1,2,3,$$

where $\Delta_1(\cdot|\alpha,\theta,p)$, $\Delta_2(\cdot|\alpha,\theta,p)$ and $\Delta_3(\cdot|\alpha,\theta,p)$ are defined in (15).

6.4. *The Cramer-von Mises minimum distance estimators*

The Cramer-von Mises estimators (CVMEs) as a type of minimum distance estimators have less bias than the other minimum distance estimators ([21]). The CVMEs are obtained based on the difference between the estimates of the cdf and the empirical distribution function ([20]). The CVMEs of the QXGGc parameters are obtained by minimizing

$$C(\alpha,\theta,p)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\theta,p\right)-\frac{2i-1}{2n}\right]^2,$$

with respect to α,θ and p. Further, the CVMEs follow by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\theta,p\right)-\frac{2i-1}{2n}\right]\Delta_s\left(x_{(i)}|\alpha,\theta,p\right)=0,\quad s=1,2,3,$$

where $\Delta_1(\cdot|\alpha,\theta,p)$, $\Delta_2(\cdot|\alpha,\theta,p)$ and $\Delta_3(\cdot|\alpha,\theta,p)$ are provided in (15).

6.5. The Anderson-Darling and right-tail Anderson-Darling estimators

The Anderson-Darling statistic or Anderson-Darling estimator is another type of minimum distance estimators. The Anderson-Darling estimators (ADEs) of the QXGGc parameters are obtained by minimizing

$$A(\alpha,\theta,p) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{(i)}|\alpha,\theta,p\right) + \log S\left(x_{(i)}|\alpha,\theta,p\right)\right],$$

with respect to α, θ and p. These ADEs can also be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s\left(x_{(i)}|\alpha,\theta,p\right)}{F\left(x_{(i)}|\alpha,\theta,p\right)} - \frac{\Delta_j\left(x_{(n+1-i)}|\alpha,\theta,p\right)}{S\left(x_{(n+1-i)}|\alpha,\theta,p\right)}\right] = 0, \quad s = 1,2,3.$$

The right-tail Anderson-Darling estimators (RADEs) of the QXGGc parameters are obtained by minimizing

$$R(\alpha,\theta,p) = \frac{n}{2} - 2\sum_{i=1}^{n} F\left(x_{i:n}|\alpha,\theta,p\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{n+1-i:n}|\alpha,\theta,p\right),$$

with respect to α, θ and p. The RADEs can also be obtained by solving the non-linear equations

$$-2\sum_{i=1}^{n}\Delta_s\left(x_{i:n}|\alpha,\theta,p\right) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_s\left(x_{n+1-i:n}|\alpha,\theta,p\right)}{S\left(x_{n+1-i:n}|\alpha,\theta,p\right)} = 0, \quad s = 1,2,3.$$

where $\Delta_1(\cdot|\alpha,\theta,p)$, $\Delta_2(\cdot|\alpha,\theta,p)$ and $\Delta_3(\cdot|\alpha,\theta,p)$ are defined in Equation (15).

7. Data generation and simulation study

We present below two different algorithms for generating random sample from QXGGc(α,θ,p).
Algorithm 1:

1. Generate $U \sim uniform(0,1)$.
2. Put $\lambda = U$ in equation (8).
3. Solve the new equation numerically for x.

Algorithm 2:
Also, we can generate a random data from the QXGGc(α,θ,p) distribution using the following simulation algorithm:

1. Generate $M \sim$ zero-truncated geometric (p).
2. Generate $U_i \sim uniform(0,1), i = 1,2,\ldots,M$.
3. Generate $V_i \sim exponential(\theta), i = 1,2,\ldots,M$.
4. Generate $W_i \sim gamma(3,\theta), i = 1,2,\ldots,M$.
5. If $U_i \le \alpha/(1+\alpha)$, then set $Y_i = V_i$, otherwise, set $Y_i = W_i, i = 1,2,\ldots,M$.
6. Set $X = min(Y_1, Y_2, \ldots, Y_M)$, then X is the required sample.

Now, we present a simulation study to compare the performance of the different estimators of the unknown parameters for the QXGGc distribution. To investigate the behavior of the MLEs in the previous section, and by R software (version 3.4.4), we generate 10,000 samples of the QXGGc distribution, where $n = \{50, 100, 350, 600, 800\}$, and by choosing $\alpha = (0.02, 0.10, 30, 65, 100)$, $\theta = (0.75, 7, 8, 20.20, 39.50)$ and $p = (0.10, 0.30, 0.50, 0.60, 0.70)$, for each parameters combination and each sample, we evaluate: the average of absolute value of biases $(|Bias(\widehat{\boldsymbol{\phi}})|)$, $|Bias(\widehat{\boldsymbol{\phi}})| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}|$, the mean square error of the estimates (MSEs), $MSEs = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi})^2$, and the mean relative estimates (MREs), $MREs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\phi}} - \boldsymbol{\phi}|/\boldsymbol{\phi}$, where $\boldsymbol{\phi} = (\alpha,\theta,p)'$.

Tables 2, 3, 4, 5 and 6 display $|Bias(\widehat{\boldsymbol{\phi}})|$, *MSEs* and *MREs* of the WLSE, OLSE, MLE, MPS, CVME and ADE. Furthermore, these tables show the rank of each of the estimators among all the estimators in each row, which is the superscript indicators, and the $\sum$ *Ranks*, which is the partial partial sum of the ranks for each column in a certain sample size. Table 7 shows the the partial and overall rank of the estimators. From tables 2, 3, 4, 5, 6 and 7, we observe that:

- Estimator methods MLE and MPS are the only estimators show the property of consistency i.e., the *MSEs* and *MREs* decrease as sample size increase, for all parameter combinations.
- Estimator method CVME shows the property of consistency for all parameter combinations, except when $\alpha = 30$, for the parameter α.
- Estimator method ADE shows the property of consistency for all parameter combinations, except when $\alpha = 30, 100$, for the parameter α.
- Estimator method RADE shows the property of consistency for all parameter combinations, except when $\alpha = 30, 65, 100$, for the parameter α.
- Estimator method WLSE shows the property of consistency for all parameter combinations, except when $\alpha = 30, 65, 100$, for the parameter α.
- Estimator method OLSE shows the property of consistency for all parameter combinations, except when $\alpha = 30$, for the parameter α.
- Form Table 7, and for the parameter combinations, we can conclude that the MPS and MLE estimator methods outperform all the other estimator methods (overall scores of 44 and 46.5, respectively). Therefore, depends on our study, we can consider the MPS and MLE estimator methods are the best.

Table 2: Simulation results for $\boldsymbol{\phi} = (\alpha = 0.02, \theta = 0.75, p = 0.3)'$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPS	CVME	ADE	RADE
50	BIAS	$\hat{\alpha}$	0.03436[3]	0.04031[6]	0.02999[1]	0.10282[7]	0.03669[4]	0.03071[2]	0.03809[5]
		$\hat{\theta}$	0.15155[5]	0.16675[6]	0.11786[1]	0.18642[7]	0.15025[4]	0.13842[3]	0.13002[2]
		$\hat{p}$	0.30275[4]	0.32125[6]	0.26583[1]	0.32178[7]	0.30836[5]	0.29314[3]	0.28343[2]
	MSE	$\hat{\alpha}$	0.00291[3]	0.00418[6]	0.00210[1]	21.89015[7]	0.00349[4]	0.00227[2]	0.00386[5]
		$\hat{\theta}$	0.04427[5]	0.05252[6]	0.02637[1]	0.07273[7]	0.04261[4]	0.03669[3]	0.03305[2]
		$\hat{p}$	0.11727[4]	0.13110[6]	0.08874[1]	0.13901[7]	0.11860[5]	0.10843[3]	0.10143[2]
	MRE	$\hat{\alpha}$	1.71819[3]	2.01562[6]	1.49938[1]	5.14109[7]	1.83453[4]	1.53531[2]	1.90472[5]
		$\hat{\theta}$	0.20207[5]	0.22233[6]	0.15715[1]	0.24856[7]	0.20033[4]	0.18456[3]	0.17336[2]
		$\hat{p}$	1.00916[4]	1.07083[6]	0.88611[1]	1.07260[7]	1.02788[5]	0.97714[3]	0.94477[2]
	$\sum$ Ranks		36[4]	54[6]	9[1]	63[7]	39[5]	24[2]	27[3]
100	BIAS	$\hat{\alpha}$	0.02520[4]	0.03076[6]	0.02259[1]	0.02382[3]	0.02891[5]	0.02376[2]	0.03135[7]
		$\hat{\theta}$	0.10807[4]	0.12590[6]	0.09351[1]	0.12766[7]	0.11443[5]	0.10153[3]	0.09937[2]
		$\hat{p}$	0.24968[4]	0.27310[7]	0.22776[1]	0.26043[5]	0.26258[6]	0.23960[2]	0.24272[3]
	MSE	$\hat{\alpha}$	0.00130[4]	0.00207[6]	0.00095[1]	0.00120[3]	0.00176[5]	0.00114[2]	0.00216[7]
		$\hat{\theta}$	0.02096[4]	0.03071[6]	0.01495[1]	0.03549[7]	0.02430[5]	0.01855[3]	0.01662[2]
		$\hat{p}$	0.08091[4]	0.09783[7]	0.06705[1]	0.09345[6]	0.08862[5]	0.07526[3]	0.07499[2]
	MRE	$\hat{\alpha}$	1.25979[4]	1.53807[6]	1.12974[1]	1.19075[3]	1.44548[5]	1.18777[2]	1.56759[7]
		$\hat{\theta}$	0.14409[4]	0.16787[6]	0.12468[1]	0.17021[7]	0.15257[5]	0.13537[3]	0.13250[2]
		$\hat{p}$	0.83228[4]	0.91034[7]	0.75919[1]	0.86812[5]	0.87528[6]	0.79866[2]	0.80905[3]
	$\sum$ Ranks		36[4]	57[7]	9[1]	46[5]	47[6]	22[2]	35[3]
350	BIAS	$\hat{\alpha}$	0.01511[4]	0.01916[6]	0.01303[2]	0.01254[1]	0.01878[5]	0.01480[3]	0.02196[7]
		$\hat{\theta}$	0.06303[5]	0.07477[7]	0.05641[1]	0.06213[4]	0.07281[6]	0.06203[3]	0.06003[2]
		$\hat{p}$	0.16460[4]	0.18995[7]	0.15348[1]	0.15536[2]	0.18837[6]	0.16368[3]	0.16610[5]
	MSE	$\hat{\alpha}$	0.00038[4]	0.00063[6]	0.00028[1.5]	0.00028[1.5]	0.00061[5]	0.00037[3]	0.00085[7]
		$\hat{\theta}$	0.00595[4]	0.00862[7]	0.00480[1]	0.00608[5]	0.00804[6]	0.00574[3]	0.00539[2]
		$\hat{p}$	0.03793[4]	0.04914[7]	0.03366[1]	0.03594[2]	0.04799[6]	0.03768[3]	0.03834[5]
	MRE	$\hat{\alpha}$	0.75533[4]	0.95780[6]	0.65125[2]	0.62703[1]	0.93900[5]	0.74014[3]	1.09807[7]
		$\hat{\theta}$	0.08404[5]	0.09970[7]	0.07522[1]	0.08284[4]	0.09708[6]	0.08270[3]	0.08004[2]
		$\hat{p}$	0.54865[4]	0.63318[7]	0.51160[1]	0.51786[2]	0.62791[6]	0.54560[3]	0.55368[5]
	$\sum$ Ranks		38[4]	60[7]	11.5[1]	22.5[2]	51[6]	27[3]	42[5]
600	BIAS	$\hat{\alpha}$	0.01204[4]	0.01613[6]	0.00991[2]	0.00965[1]	0.01574[5]	0.01180[3]	0.01867[7]
		$\hat{\theta}$	0.05061[5]	0.05981[7]	0.04599[1]	0.04723[2]	0.05964[6]	0.05040[4]	0.04806[3]
		$\hat{p}$	0.13487[3]	0.15687[6]	0.12518[2]	0.11943[1]	0.15780[7]	0.13545[4]	0.13666[5]
	MSE	$\hat{\alpha}$	0.00024[4]	0.00041[6]	0.00016[1.5]	0.00016[1.5]	0.00040[5]	0.00023[3]	0.00058[7]
		$\hat{\theta}$	0.00387[5]	0.00545[7]	0.00325[1]	0.00349[2]	0.00531[6]	0.00382[4]	0.00353[3]
		$\hat{p}$	0.02679[3]	0.03524[6]	0.02351[2]	0.02294[1]	0.03529[7]	0.02696[4]	0.02762[5]
	MRE	$\hat{\alpha}$	0.60212[4]	0.80667[6]	0.49545[2]	0.48249[1]	0.78689[5]	0.59022[3]	0.93347[7]
		$\hat{\theta}$	0.06748[5]	0.07975[7]	0.06131[1]	0.06297[2]	0.07951[6]	0.06720[4]	0.06408[3]
		$\hat{p}$	0.44958[3]	0.52291[6]	0.41725[2]	0.39809[1]	0.52602[7]	0.45150[4]	0.45553[5]
	$\sum$ Ranks		36[4]	57[7]	14.5[2]	12.5[1]	54[6]	33[3]	45[5]
800	BIAS	$\hat{\alpha}$	0.01045[3]	0.01418[5]	0.00866[2]	0.00833[1]	0.01426[6]	0.01058[4]	0.01669[7]
		$\hat{\theta}$	0.04458[5]	0.05360[7]	0.03988[1]	0.04012[2]	0.05242[6]	0.04425[4]	0.04224[3]
		$\hat{p}$	0.11922[5]	0.14161[7]	0.10861[2]	0.10226[1]	0.14033[6]	0.11874[4]	0.11828[3]
	MSE	$\hat{\alpha}$	0.00018[3.5]	0.00032[5.5]	0.00012[1.5]	0.00012[1.5]	0.00032[5.5]	0.00018[3.5]	0.00046[7]
		$\hat{\theta}$	0.00302[5]	0.00433[7]	0.00248[1]	0.00255[2]	0.00412[6]	0.00296[4]	0.00275[3]
		$\hat{p}$	0.02153[5]	0.02951[7]	0.01833[2]	0.01778[1]	0.02877[6]	0.02137[3]	0.02141[4]
	MRE	$\hat{\alpha}$	0.52228[3]	0.70884[5]	0.43286[2]	0.41630[1]	0.71300[6]	0.52912[4]	0.83453[7]
		$\hat{\theta}$	0.05945[5]	0.07147[7]	0.05318[1]	0.05349[2]	0.06989[6]	0.05900[4]	0.05633[3]
		$\hat{p}$	0.39741[5]	0.47203[7]	0.36204[2]	0.34086[1]	0.46776[6]	0.39581[4]	0.39428[3]
	$\sum$ Ranks		39.5[4]	57.5[7]	14.5[2]	12.5[1]	53.5[6]	34.5[3]	40[5]

Table 3: Simulation results for $\boldsymbol{\phi} = (\alpha = 0.1, \theta = 7, p = 0.6)'$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPS	CVME	ADE	RADE
50	BIAS	$\hat{\alpha}$	0.10527[2]	3.90475[6]	0.10859[3]	1.59240[5]	11.39283[7]	0.09933[1]	0.22862[4]
		$\hat{\theta}$	2.15840[4]	2.26317[7]	1.73786[1]	2.23498[6]	2.22565[5]	2.05692[3]	1.99534[2]
		$\hat{p}$	0.33148[5]	0.35433[6]	0.30913[1]	0.30929[2]	0.35986[7]	0.31866[3]	0.31893[4]
	MSE	$\hat{\alpha}$	0.02808[2]	66973.27738[6]	0.03696[3]	2769.73461[5]	617906.53968[7]	0.02585[1]	59.22590[4]
		$\hat{\theta}$	7.05821[4]	7.51135[6]	4.83105[1]	8.92959[7]	7.07057[5]	6.57283[3]	6.25447[2]
		$\hat{p}$	0.14744[5]	0.16307[6]	0.13876[2]	0.13115[1]	0.16896[7]	0.14124[3]	0.14233[4]
	MRE	$\hat{\alpha}$	1.05267[2]	39.04752[6]	1.08594[3]	15.92397[5]	113.92826[7]	0.99327[1]	2.28624[4]
		$\hat{\theta}$	0.30834[4]	0.32331[7]	0.24827[1]	0.31928[6]	0.31795[5]	0.29385[3]	0.28505[2]
		$\hat{p}$	0.55247[5]	0.59055[6]	0.51522[1]	0.51549[2]	0.59977[7]	0.53110[3]	0.53155[4]
	$\sum$ Ranks		33[4]	56[6]	16[1]	39[5]	57[7]	21[2]	30[3]
100	BIAS	$\hat{\alpha}$	0.07608[2]	0.08549[6]	0.07716[3]	0.40139[7]	0.08297[5]	0.07303[1]	0.08163[4]
		$\hat{\theta}$	1.62181[4]	1.91524[7]	1.33069[1]	1.62885[5]	1.85462[6]	1.54574[3]	1.47733[2]
		$\hat{p}$	0.26487[5]	0.29747[6]	0.24826[2]	0.22747[1]	0.30337[7]	0.25401[4]	0.25217[3]
	MSE	$\hat{\alpha}$	0.01285[2]	0.01908[6]	0.01463[3]	541.10536[7]	0.01515[4]	0.01169[1]	0.01569[5]
		$\hat{\theta}$	4.18640[4]	5.52546[6]	2.82475[1]	5.73999[7]	5.08204[5]	3.82151[3]	3.52666[2]
		$\hat{p}$	0.10602[5]	0.12572[6]	0.09897[3]	0.08127[1]	0.13107[7]	0.09951[4]	0.09892[2]
	MRE	$\hat{\alpha}$	0.76082[2]	0.85486[6]	0.77157[3]	4.01392[7]	0.82972[5]	0.73028[1]	0.81633[4]
		$\hat{\theta}$	0.23169[4]	0.27361[7]	0.19010[1]	0.23269[5]	0.26495[6]	0.22082[3]	0.21105[2]
		$\hat{p}$	0.44145[5]	0.49578[6]	0.41377[2]	0.37912[1]	0.50561[7]	0.42334[4]	0.42029[3]
	$\sum$ Ranks		33[4]	56[7]	19[1]	41[5]	52[6]	24[2]	27[3]
350	BIAS	$\hat{\alpha}$	0.03818[3]	0.04572[7]	0.03558[2]	0.03350[1]	0.04517[6]	0.03837[4]	0.04311[5]
		$\hat{\theta}$	0.86631[5]	1.09438[7]	0.73961[2]	0.57996[1]	1.09166[6]	0.82953[4]	0.80322[3]
		$\hat{p}$	0.14653[5]	0.17963[6]	0.12806[2]	0.10718[1]	0.18173[7]	0.14301[4]	0.14117[3]
	MSE	$\hat{\alpha}$	0.00277[3]	0.00396[7]	0.00238[2]	0.00219[1]	0.00380[6]	0.00282[4]	0.00367[5]
		$\hat{\theta}$	1.17431[5]	1.89761[7]	0.87150[1]	1.00656[3]	1.89594[6]	1.07345[4]	1.00412[2]
		$\hat{p}$	0.03696[5]	0.05321[6]	0.02871[2]	0.02368[1]	0.05498[7]	0.03548[4]	0.03505[3]
	MRE	$\hat{\alpha}$	0.38182[3]	0.45716[7]	0.35580[2]	0.33498[1]	0.45172[6]	0.38370[4]	0.43110[5]
		$\hat{\theta}$	0.12376[5]	0.15634[7]	0.10566[2]	0.08285[1]	0.15595[6]	0.11850[4]	0.11475[3]
		$\hat{p}$	0.24421[5]	0.29939[6]	0.21343[2]	0.17863[1]	0.30289[7]	0.23834[4]	0.23528[3]
	$\sum$ Ranks		39[5]	60[7]	17[2]	11[1]	57[6]	36[4]	32[3]
600	BIAS	$\hat{\alpha}$	0.02824[3]	0.03445[7]	0.02655[2]	0.02367[1]	0.03360[6]	0.02881[4]	0.03213[5]
		$\hat{\theta}$	0.65476[5]	0.84735[7]	0.55274[2]	0.29889[1]	0.83918[6]	0.64561[4]	0.60952[3]
		$\hat{p}$	0.10884[5]	0.13692[7]	0.09547[2]	0.06681[1]	0.13646[6]	0.10868[4]	0.10591[3]
	MSE	$\hat{\alpha}$	0.00142[3]	0.00218[7]	0.00127[2]	0.00106[1]	0.00207[6]	0.00150[4]	0.00193[5]
		$\hat{\theta}$	0.67710[5]	1.12902[7]	0.48471[2]	0.39267[1]	1.10510[6]	0.65410[4]	0.59019[3]
		$\hat{p}$	0.02028[5]	0.03118[6]	0.01563[2]	0.01089[1]	0.03150[7]	0.01986[4]	0.01961[3]
	MRE	$\hat{\alpha}$	0.28240[3]	0.34452[7]	0.26552[2]	0.23673[1]	0.33596[6]	0.28815[4]	0.32128[5]
		$\hat{\theta}$	0.09354[5]	0.12105[7]	0.07896[2]	0.04270[1]	0.11988[6]	0.09223[4]	0.08707[3]
		$\hat{p}$	0.18140[5]	0.22821[7]	0.15911[2]	0.11135[1]	0.22744[6]	0.18113[4]	0.17652[3]
	$\sum$ Ranks		39[5]	62[7]	18[2]	9[1]	55[6]	36[4]	33[3]
800	BIAS	$\hat{\alpha}$	0.02477[4]	0.02949[6]	0.02285[2]	0.01974[1]	0.02950[7]	0.02360[3]	0.02701[5]
		$\hat{\theta}$	0.56638[5]	0.73247[7]	0.48275[2]	0.20106[1]	0.72638[6]	0.55376[4]	0.51783[3]
		$\hat{p}$	0.09395[5]	0.11908[7]	0.08202[2]	0.05102[1]	0.11840[6]	0.09058[4]	0.08840[3]
	MSE	$\hat{\alpha}$	0.00106[4]	0.00157[7]	0.00092[2]	0.00071[1]	0.00154[6]	0.00097[3]	0.00129[5]
		$\hat{\theta}$	0.50097[5]	0.83850[7]	0.36965[2]	0.23563[1]	0.83098[6]	0.48212[4]	0.42251[3]
		$\hat{p}$	0.01480[5]	0.02382[7]	0.01146[2]	0.00671[1]	0.02364[6]	0.01378[4]	0.01313[3]
	MRE	$\hat{\alpha}$	0.24768[4]	0.29494[6]	0.22852[2]	0.19736[1]	0.29499[7]	0.23599[3]	0.27012[5]
		$\hat{\theta}$	0.08091[5]	0.10464[7]	0.06896[2]	0.02872[1]	0.10377[6]	0.07911[4]	0.07398[3]
		$\hat{p}$	0.15659[5]	0.19847[7]	0.13670[2]	0.08503[1]	0.19733[6]	0.15097[4]	0.14733[3]
	$\sum$ Ranks		42[5]	61[7]	18[2]	9[1]	56[6]	33[3.5]	33[3.5]

Table 4: Simulation results for $\boldsymbol{\phi} = (\alpha = 30, \theta = 20.2, p = 0.5)'$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPS	CVME	ADE	RADE
50	BIAS	$\hat{\alpha}$	$45313.64938^{\{7\}}$	$0.43586^{\{3\}}$	$4.83192^{\{4\}}$	$0.23324^{\{1\}}$	$0.41939^{\{2\}}$	$18829.94899^{\{6\}}$	$15964.75954^{\{5\}}$
		$\hat{\theta}$	$6.99589^{\{5\}}$	$8.09820^{\{7\}}$	$6.14694^{\{2\}}$	$3.87117^{\{1\}}$	$7.80262^{\{6\}}$	$6.70426^{\{4\}}$	$6.23173^{\{3\}}$
		$\hat{p}$	$0.24028^{\{5\}}$	$0.26351^{\{6\}}$	$0.22389^{\{2\}}$	$0.17123^{\{1\}}$	$0.26465^{\{7\}}$	$0.23477^{\{4\}}$	$0.22485^{\{3\}}$
	MSE	$\hat{\alpha}$	$2.41054\times10^{10\{7\}}$	$0.27877^{\{3\}}$	$99493.96456^{\{4\}}$	$0.15906^{\{1\}}$	$0.26421^{\{2\}}$	$1.27598\times10^{10\{5\}}$	$1.46374\times10^{10\{6\}}$
		$\hat{\theta}$	$72.86557^{\{5\}}$	$97.15129^{\{7\}}$	$60.56764^{\{3\}}$	$44.21256^{\{1\}}$	$92.51742^{\{6\}}$	$68.67589^{\{4\}}$	$59.25793^{\{2\}}$
		$\hat{p}$	$0.08210^{\{5\}}$	$0.09485^{\{6\}}$	$0.07611^{\{3\}}$	$0.04737^{\{1\}}$	$0.09747^{\{7\}}$	$0.07934^{\{4\}}$	$0.07462^{\{2\}}$
	MRE	$\hat{\alpha}$	$1510.45498^{\{7\}}$	$0.01453^{\{3\}}$	$0.16106^{\{4\}}$	$0.00777^{\{1\}}$	$0.01398^{\{2\}}$	$627.66497^{\{6\}}$	$532.15865^{\{5\}}$
		$\hat{\theta}$	$0.34633^{\{5\}}$	$0.40090^{\{7\}}$	$0.30430^{\{2\}}$	$0.19164^{\{1\}}$	$0.38627^{\{6\}}$	$0.33189^{\{4\}}$	$0.30850^{\{3\}}$
		$\hat{p}$	$0.48056^{\{5\}}$	$0.52702^{\{6\}}$	$0.44778^{\{2\}}$	$0.34247^{\{1\}}$	$0.52929^{\{7\}}$	$0.46955^{\{4\}}$	$0.44969^{\{3\}}$
	$\sum$ Ranks		$51^{\{7\}}$	$48^{\{6\}}$	$26^{\{2\}}$	$9^{\{1\}}$	$45^{\{5\}}$	$41^{\{4\}}$	$32^{\{3\}}$
100	BIAS	$\hat{\alpha}$	$50208.96830^{\{7\}}$	$785.67107^{\{3\}}$	$0.26928^{\{2\}}$	$0.11378^{\{1\}}$	$1125.62633^{\{4\}}$	$10403.24306^{\{5\}}$	$11326.00266^{\{6\}}$
		$\hat{\theta}$	$5.45655^{\{5\}}$	$6.04027^{\{6\}}$	$4.41676^{\{2\}}$	$1.89348^{\{1\}}$	$6.11648^{\{7\}}$	$5.05480^{\{4\}}$	$4.62872^{\{3\}}$
		$\hat{p}$	$0.18377^{\{5\}}$	$0.20085^{\{6\}}$	$0.16545^{\{3\}}$	$0.10928^{\{1\}}$	$0.21028^{\{7\}}$	$0.17816^{\{4\}}$	$0.16489^{\{2\}}$
	MSE	$\hat{\alpha}$	$1.51930\times10^{10\{7\}}$	$8.07537\times10^{8\{3\}}$	$0.11847^{\{2\}}$	$0.06345^{\{1\}}$	$1.27477\times10^{9\{4\}}$	$8.51999\times10^{9\{5\}}$	$1.42741\times10^{10\{6\}}$
		$\hat{\theta}$	$45.25379^{\{5\}}$	$55.49562^{\{6\}}$	$31.91002^{\{2\}}$	$17.63927^{\{1\}}$	$57.71834^{\{7\}}$	$39.67589^{\{4\}}$	$33.67889^{\{3\}}$
		$\hat{p}$	$0.05258^{\{5\}}$	$0.06074^{\{6\}}$	$0.04481^{\{3\}}$	$0.02239^{\{1\}}$	$0.06687^{\{7\}}$	$0.04985^{\{4\}}$	$0.04366^{\{2\}}$
	MRE	$\hat{\alpha}$	$1673.63228^{\{7\}}$	$26.18904^{\{3\}}$	$0.00898^{\{2\}}$	$0.00379^{\{1\}}$	$37.52088^{\{4\}}$	$346.77477^{\{5\}}$	$377.53342^{\{6\}}$
		$\hat{\theta}$	$0.27013^{\{5\}}$	$0.29902^{\{6\}}$	$0.21865^{\{2\}}$	$0.09374^{\{1\}}$	$0.30280^{\{7\}}$	$0.25024^{\{4\}}$	$0.22914^{\{3\}}$
		$\hat{p}$	$0.36753^{\{5\}}$	$0.40171^{\{6\}}$	$0.33089^{\{3\}}$	$0.21856^{\{1\}}$	$0.42056^{\{7\}}$	$0.35631^{\{4\}}$	$0.32977^{\{2\}}$
	$\sum$ Ranks		$51^{\{6\}}$	$45^{\{5\}}$	$21^{\{2\}}$	$9^{\{1\}}$	$54^{\{7\}}$	$39^{\{4\}}$	$33^{\{3\}}$
350	BIAS	$\hat{\alpha}$	$97157.73883^{\{7\}}$	$0.18115^{\{3\}}$	$0.14295^{\{2\}}$	$0.00499^{\{1\}}$	$0.18345^{\{4\}}$	$71.52307^{\{5\}}$	$226.23567^{\{6\}}$
		$\hat{\theta}$	$3.58184^{\{7\}}$	$3.35448^{\{5\}}$	$2.34983^{\{2\}}$	$0.08349^{\{1\}}$	$3.39905^{\{6\}}$	$2.72720^{\{4\}}$	$2.58415^{\{3\}}$
		$\hat{p}$	$0.10601^{\{5\}}$	$0.11216^{\{6\}}$	$0.08610^{\{2\}}$	$0.03837^{\{1\}}$	$0.11632^{\{7\}}$	$0.09558^{\{4\}}$	$0.09227^{\{3\}}$
	MSE	$\hat{\alpha}$	$3.01468\times10^{10\{7\}}$	$0.05204^{\{3\}}$	$0.03352^{\{2\}}$	$0.00206^{\{1\}}$	$0.05318^{\{4\}}$	$2.54485\times10^{7\{5\}}$	$1.99627\times10^{10\{6\}}$
		$\hat{\theta}$	$19.58152^{\{7\}}$	$17.89431^{\{5\}}$	$8.93256^{\{2\}}$	$0.57495^{\{1\}}$	$18.31580^{\{6\}}$	$11.90795^{\{4\}}$	$10.57245^{\{3\}}$
		$\hat{p}$	$0.01835^{\{5\}}$	$0.02044^{\{6\}}$	$0.01236^{\{2\}}$	$0.00259^{\{1\}}$	$0.02209^{\{7\}}$	$0.01509^{\{4\}}$	$0.01392^{\{3\}}$
	MRE	$\hat{\alpha}$	$3238.59129^{\{7\}}$	$0.00604^{\{3\}}$	$0.00477^{\{2\}}$	$0.00017^{\{1\}}$	$0.00611^{\{4\}}$	$2.38410^{\{5\}}$	$7.54119^{\{6\}}$
		$\hat{\theta}$	$0.17732^{\{7\}}$	$0.16606^{\{5\}}$	$0.11633^{\{2\}}$	$0.00413^{\{1\}}$	$0.16827^{\{6\}}$	$0.13501^{\{4\}}$	$0.12793^{\{3\}}$
		$\hat{p}$	$0.21202^{\{5\}}$	$0.22431^{\{6\}}$	$0.17221^{\{2\}}$	$0.07673^{\{1\}}$	$0.23264^{\{7\}}$	$0.19116^{\{4\}}$	$0.18455^{\{3\}}$
	$\sum$ Ranks		$57^{\{7\}}$	$42^{\{5\}}$	$18^{\{2\}}$	$9^{\{1\}}$	$51^{\{6\}}$	$39^{\{4\}}$	$36^{\{3\}}$
600	BIAS	$\hat{\alpha}$	$133799.27748^{\{7\}}$	$119.90040^{\{3\}}$	$0.10662^{\{2\}}$	$0.00058^{\{1\}}$	$299.86576^{\{4\}}$	$457.22755^{\{6\}}$	$301.14773^{\{5\}}$
		$\hat{\theta}$	$3.02759^{\{7\}}$	$2.57190^{\{5\}}$	$1.74324^{\{2\}}$	$0.00980^{\{1\}}$	$2.58836^{\{6\}}$	$2.10353^{\{4\}}$	$1.99239^{\{3\}}$
		$\hat{p}$	$0.08339^{\{5\}}$	$0.08683^{\{6\}}$	$0.06339^{\{2\}}$	$0.02921^{\{1\}}$	$0.08711^{\{7\}}$	$0.07322^{\{4\}}$	$0.07073^{\{3\}}$
	MSE	$\hat{\alpha}$	$5.54733\times10^{10\{7\}}$	$7.17145\times10^{7\{3\}}$	$0.01889^{\{2\}}$	$0.00015^{\{1\}}$	$4.49179\times10^{8\{6\}}$	$2.62513\times10^{8\{5\}}$	$2.36423\times10^{10\{4\}}$
		$\hat{\theta}$	$13.63681^{\{7\}}$	$10.52490^{\{5\}}$	$4.95697^{\{2\}}$	$0.04166^{\{1\}}$	$10.58099^{\{6\}}$	$6.97005^{\{4\}}$	$6.27288^{\{3\}}$
		$\hat{p}$	$0.01118^{\{5\}}$	$0.01217^{\{6\}}$	$0.00658^{\{2\}}$	$0.00137^{\{1\}}$	$0.01240^{\{7\}}$	$0.00850^{\{4\}}$	$0.00798^{\{3\}}$
	MRE	$\hat{\alpha}$	$4459.97592^{\{7\}}$	$3.99668^{\{3\}}$	$0.00355^{\{2\}}$	$0.00002^{\{1\}}$	$9.99553^{\{4\}}$	$15.24092^{\{6\}}$	$10.03826^{\{5\}}$
		$\hat{\theta}$	$0.14988^{\{7\}}$	$0.12732^{\{5\}}$	$0.08630^{\{2\}}$	$0.00048^{\{1\}}$	$0.12814^{\{6\}}$	$0.10414^{\{4\}}$	$0.09863^{\{3\}}$
		$\hat{p}$	$0.16679^{\{5\}}$	$0.17367^{\{6\}}$	$0.12678^{\{2\}}$	$0.05842^{\{1\}}$	$0.17421^{\{7\}}$	$0.14645^{\{4\}}$	$0.14146^{\{3\}}$
	$\sum$ Ranks		$57^{\{7\}}$	$42^{\{5\}}$	$18^{\{2\}}$	$9^{\{1\}}$	$53^{\{6\}}$	$41^{\{4\}}$	$32^{\{3\}}$
800	BIAS	$\hat{\alpha}$	$148216.40320^{\{7\}}$	$0.11990^{\{4\}}$	$0.09331^{\{2\}}$	$0.00034^{\{1\}}$	$0.11835^{\{3\}}$	$2458.22833^{\{6\}}$	$156.09153^{\{5\}}$
		$\hat{\theta}$	$2.91804^{\{7\}}$	$2.22518^{\{6\}}$	$1.52682^{\{2\}}$	$0.00552^{\{1\}}$	$2.19502^{\{5\}}$	$1.84438^{\{4\}}$	$1.69365^{\{3\}}$
		$\hat{p}$	$0.07668^{\{7\}}$	$0.07465^{\{5\}}$	$0.05469^{\{2\}}$	$0.02465^{\{1\}}$	$0.07469^{\{6\}}$	$0.06350^{\{4\}}$	$0.06027^{\{3\}}$
	MSE	$\hat{\alpha}$	$7.24071\times10^{10\{7\}}$	$0.02272^{\{4\}}$	$0.01420^{\{2\}}$	$0.00008^{\{1\}}$	$0.02219^{\{3\}}$	$1.48351\times10^{9\{6\}}$	$1.21614\times10^{8\{5\}}$
		$\hat{\theta}$	$12.10730^{\{7\}}$	$7.83549^{\{6\}}$	$3.74245^{\{2\}}$	$0.02100^{\{1\}}$	$7.64873^{\{5\}}$	$5.37238^{\{4\}}$	$4.56717^{\{3\}}$
		$\hat{p}$	$0.00918^{\{7\}}$	$0.00894^{\{5\}}$	$0.00486^{\{2\}}$	$0.00097^{\{1\}}$	$0.00896^{\{6\}}$	$0.00638^{\{4\}}$	$0.00589^{\{3\}}$
	MRE	$\hat{\alpha}$	$4940.54677^{\{7\}}$	$0.00400^{\{4\}}$	$0.00311^{\{2\}}$	$0.00001^{\{1\}}$	$0.00394^{\{3\}}$	$81.94094^{\{6\}}$	$5.20305^{\{5\}}$
		$\hat{\theta}$	$0.14446^{\{7\}}$	$0.11016^{\{6\}}$	$0.07559^{\{2\}}$	$0.00027^{\{1\}}$	$0.10866^{\{5\}}$	$0.09131^{\{4\}}$	$0.08384^{\{3\}}$
		$\hat{p}$	$0.15337^{\{7\}}$	$0.14930^{\{5\}}$	$0.10938^{\{2\}}$	$0.04930^{\{1\}}$	$0.14937^{\{6\}}$	$0.12699^{\{4\}}$	$0.12055^{\{3\}}$
	$\sum$ Ranks		$63^{\{7\}}$	$45^{\{6\}}$	$18^{\{2\}}$	$9^{\{1\}}$	$42^{\{4.5\}}$	$42^{\{4.5\}}$	$33^{\{3\}}$

Table 5: Simulation results for $\boldsymbol{\phi} = (\alpha = 100, \theta = 39.5, p = 0.1)'$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPS	CVME	ADE	RADE
50	BIAS	$\hat{\alpha}$	96520.69023[7]	0.08508[6]	0.07281[4]	0.05149[1]	0.07116[2]	0.07181[3]	0.07300[5]
		$\hat{\theta}$	7.59159[5]	9.05567[7]	6.29992[2]	4.12546[1]	7.70995[6]	7.30793[4]	7.01237[3]
		$\hat{p}$	0.20261[6]	0.23488[7]	0.16129[1]	0.18479[3]	0.19351[4]	0.19617[5]	0.18464[2]
	MSE	$\hat{\alpha}$	8.33879×10^{10}[7]	0.01328[6]	0.00985[4]	0.01275[5]	0.00949[2]	0.00953[3]	0.00938[1]
		$\hat{\theta}$	93.52205[5]	137.58828[7]	65.74495[1]	70.60265[2]	101.20135[6]	89.62244[4]	80.28634[3]
		$\hat{p}$	0.07035[6]	0.09601[7]	0.04391[1]	0.06344[3]	0.06702[4]	0.06738[5]	0.05828[2]
	MRE	$\hat{\alpha}$	965.20690[7]	0.00085[6]	0.00073[4.5]	0.00051[1]	0.00071[2]	0.00072[3]	0.00073[4.5]
		$\hat{\theta}$	0.19219[5]	0.22926[7]	0.15949[2]	0.10444[1]	0.19519[6]	0.18501[4]	0.17753[3]
		$\hat{p}$	2.02611[6]	2.34883[7]	1.61291[1]	1.84792[3]	1.93514[4]	1.96169[5]	1.84636[2]
	$\sum$ Ranks		54[6]	60[7]	20.5[2]	20[1]	36[4.5]	36[4.5]	25.5[3]
100	BIAS	$\hat{\alpha}$	223059.53424[7]	0.06059[4]	0.05604[3]	0.01653[1]	0.05448[2]	170.67134[5]	1263.39077[6]
		$\hat{\theta}$	5.36189[5]	6.35788[7]	4.72463[2]	1.33592[1]	5.75947[6]	5.35318[4]	5.10499[3]
		$\hat{p}$	0.16005[5]	0.18132[7]	0.13889[2]	0.11588[1]	0.16309[6]	0.16002[4]	0.15142[3]
	MSE	$\hat{\alpha}$	2.34717×10^{11}[7]	0.00659[4]	0.00574[3]	0.00336[1]	0.00543[2]	1.45553×10^{8}[5]	1.74937×10^{9}[6]
		$\hat{\theta}$	47.68192[4]	67.83288[7]	37.06550[2]	19.01506[1]	56.30716[6]	47.68988[5]	42.79370[3]
		$\hat{p}$	0.04206[4]	0.05511[7]	0.03018[2]	0.02489[1]	0.04497[6]	0.04215[5]	0.03700[3]
	MRE	$\hat{\alpha}$	2230.59534[7]	0.00061[4]	0.00056[3]	0.00017[1]	0.00054[2]	1.70671[5]	12.63391[6]
		$\hat{\theta}$	0.13574[5]	0.16096[7]	0.11961[2]	0.03382[1]	0.14581[6]	0.13552[4]	0.12924[3]
		$\hat{p}$	1.60051[5]	1.81318[7]	1.38889[2]	1.15883[1]	1.63086[6]	1.60023[4]	1.51422[3]
	$\sum$ Ranks		49[6]	54[7]	21[2]	9[1]	42[5]	41[4]	36[3]
350	BIAS	$\hat{\alpha}$	428051.54819[7]	0.03540[4]	0.03425[3]	0.00033[1]	0.03374[2]	1258.22531[5]	5502.71743[6]
		$\hat{\theta}$	3.38709[5]	3.59035[7]	2.83370[2]	0.03134[1]	3.40906[6]	3.09999[4]	3.01386[3]
		$\hat{p}$	0.10673[4]	0.11774[7]	0.09761[2]	0.06124[1]	0.11305[6]	0.10775[5]	0.10348[3]
	MSE	$\hat{\alpha}$	6.27722×10^{11}[7]	0.00215[4]	0.00209[3]	0.00003[1]	0.00194[2]	1.93994×10^{9}[5]	1.16169×10^{10}[6]
		$\hat{\theta}$	18.74097[5]	21.48638[7]	12.93938[2]	0.23912[1]	19.13282[6]	15.57838[4]	14.37513[3]
		$\hat{p}$	0.01667[4]	0.02122[7]	0.01380[2]	0.00548[1]	0.01922[6]	0.01709[5]	0.01554[3]
	MRE	$\hat{\alpha}$	4280.51548[7]	0.00035[4]	0.00034[2.5]	0.00000[1]	0.00034[2.5]	12.58225[5]	55.02717[6]
		$\hat{\theta}$	0.08575[5]	0.09089[7]	0.07174[2]	0.00079[1]	0.08631[6]	0.07848[4]	0.07630[3]
		$\hat{p}$	1.06733[4]	1.17737[7]	0.97611[2]	0.61239[1]	1.13053[6]	1.07754[5]	1.03482[3]
	$\sum$ Ranks		48[6]	54[7]	20.5[2]	9[1]	42.5[5]	42[4]	36[3]
600	BIAS	$\hat{\alpha}$	473925.65175[7]	0.02898[4]	0.02851[3]	0.00012[1]	0.02821[2]	2502.38251[5]	9796.83869[6]
		$\hat{\theta}$	3.10624[7]	2.90239[6]	2.28344[2]	0.00999[1]	2.83399[5]	2.47664[4]	2.42723[3]
		$\hat{p}$	0.09189[5]	0.09914[7]	0.08225[2]	0.04851[1]	0.09654[6]	0.08916[4]	0.08771[3]
	MSE	$\hat{\alpha}$	7.83338×10^{11}[7]	0.00140[4]	0.00136[3]	0.00001[1]	0.00131[2]	5.42295×10^{9}[5]	2.62655×10^{10}[6]
		$\hat{\theta}$	14.84295[7]	13.75145[6]	8.15081[2]	0.05679[1]	13.00521[5]	9.73673[4]	9.30240[3]
		$\hat{p}$	0.01188[5]	0.01460[7]	0.00957[2]	0.00351[1]	0.01360[6]	0.01133[4]	0.01110[3]
	MRE	$\hat{\alpha}$	4739.25652[7]	0.00029[3.5]	0.00029[3.5]	0.00000[1]	0.00028[2]	25.02383[5]	97.96839[6]
		$\hat{\theta}$	0.07864[7]	0.07348[6]	0.05781[2]	0.00025[1]	0.07175[5]	0.06270[4]	0.06145[3]
		$\hat{p}$	0.91889[5]	0.99142[7]	0.82251[2]	0.48512[1]	0.96536[6]	0.89159[4]	0.87709[3]
	$\sum$ Ranks		57[7]	50.5[6]	21.5[2]	9[1]	39[4.5]	39[4.5]	36[3]
800	BIAS	$\hat{\alpha}$	510035.02543[7]	0.02509[3]	0.02552[4]	0.00005[1]	0.02467[2]	6406.12800[5]	8221.68252[6]
		$\hat{\theta}$	3.05455[7]	2.49836[6]	2.01298[2]	0.00423[1]	2.47002[5]	2.18949[4]	2.09741[3]
		$\hat{p}$	0.08671[5]	0.08781[6]	0.07489[2]	0.04193[1]	0.08843[7]	0.07972[4]	0.07679[3]
	MSE	$\hat{\alpha}$	8.97626×10^{11}[7]	0.00103[3]	0.00109[4]	0.00000[1]	0.00100[2]	1.64356×10^{10}[5]	2.11638×10^{10}[6]
		$\hat{\theta}$	14.06801[7]	9.96421[6]	6.36157[2]	0.00899[1]	9.76417[5]	7.58842[4]	6.85928[3]
		$\hat{p}$	0.01021[5]	0.01127[7]	0.00791[2]	0.00271[1]	0.01119[6]	0.00919[4]	0.00847[3]
	MRE	$\hat{\alpha}$	5100.35025[7]	0.00025[2.5]	0.00026[4]	0.00000[1]	0.00025[2.5]	64.06128[5]	82.21683[6]
		$\hat{\theta}$	0.07733[7]	0.06325[6]	0.05096[2]	0.00011[1]	0.06253[5]	0.05543[4]	0.05310[3]
		$\hat{p}$	0.86711[5]	0.87811[6]	0.74892[2]	0.41931[1]	0.88432[7]	0.79716[4]	0.76790[3]
	$\sum$ Ranks		57[7]	45.5[6]	24[2]	9[1]	41.5[5]	39[4]	36[3]

Table 6: Simulation results for $\boldsymbol{\phi} = (\alpha = 65, \theta = 8, p = 0.7)'$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPS	CVME	ADE	RADE
50	BIAS	$\hat{\alpha}$	7508.79237[7]	0.02013[5]	0.01790[4]	0.01550[1]	0.02040[6]	0.01671[3]	0.01616[2]
		$\hat{\theta}$	3.96405[5]	4.57972[6]	3.20460[2]	2.76893[1]	4.66884[7]	3.64458[4]	3.30625[3]
		$\hat{p}$	0.19481[5]	0.21593[6]	0.17079[3]	0.14405[1]	0.23404[7]	0.18316[4]	0.16960[2]
	MSE	$\hat{\alpha}$	5.09381×10^{10}[7]	0.00062[5]	0.00058[4]	0.00045[3]	0.00066[6]	0.00044[2]	0.00041[1]
		$\hat{\theta}$	25.85010[5]	32.86599[6]	19.27062[3]	14.64428[1]	35.44173[7]	22.21309[4]	18.50493[2]
		$\hat{p}$	0.06767[5]	0.07793[6]	0.05667[3]	0.03616[1]	0.09418[7]	0.05988[4]	0.05255[2]
	MRE	$\hat{\alpha}$	115.51988[7]	0.00031[5.5]	0.00028[4]	0.00024[1]	0.00031[5.5]	0.00026[3]	0.00025[2]
		$\hat{\theta}$	0.49551[5]	0.57247[6]	0.40058[2]	0.34612[1]	0.58361[7]	0.45557[4]	0.41328[3]
		$\hat{p}$	0.27830[5]	0.30847[6]	0.24399[3]	0.20578[1]	0.33435[7]	0.26166[4]	0.24229[2]
	$\sum$ Ranks		51[5]	51.5[6]	28[3]	11[1]	59.5[7]	32[4]	19[2]
100	BIAS	$\hat{\alpha}$	2168.94781[7]	0.01476[5]	0.01263[4]	0.01018[1]	0.01495[6]	0.01170[3]	0.01161[2]
		$\hat{\theta}$	2.81212[5]	3.35911[6]	2.18264[2]	1.77211[1]	3.41555[7]	2.58932[4]	2.38959[3]
		$\hat{p}$	0.13675[5]	0.15958[6]	0.11411[2]	0.09783[1]	0.16636[7]	0.12725[4]	0.11963[3]
	MSE	$\hat{\alpha}$	1.23304×10^{9}[7]	0.00034[5]	0.00028[4]	0.00023[3]	0.00036[6]	0.00022[1.5]	0.00022[1.5]
		$\hat{\theta}$	13.85198[5]	18.11117[6]	8.34929[2]	7.02914[1]	19.55340[7]	11.24827[4]	9.53781[3]
		$\hat{p}$	0.03539[5]	0.04352[6]	0.02426[2]	0.01697[1]	0.04964[7]	0.02961[4]	0.02605[3]
	MRE	$\hat{\alpha}$	33.36843[7]	0.00023[5.5]	0.00019[4]	0.00016[1]	0.00023[5.5]	0.00018[2.5]	0.00018[2.5]
		$\hat{\theta}$	0.35151[5]	0.41989[6]	0.27283[2]	0.22151[1]	0.42694[7]	0.32366[4]	0.29870[3]
		$\hat{p}$	0.19536[5]	0.22797[6]	0.16302[2]	0.13975[1]	0.23766[7]	0.18178[4]	0.17089[3]
	$\sum$ Ranks		51[5]	51.5[6]	24[2.5]	11[1]	59.5[7]	31[4]	24[2.5]
350	BIAS	$\hat{\alpha}$	53749.07590[7]	0.00773[5]	0.00684[4]	0.00258[1]	0.00787[6]	0.00626[3]	0.00598[2]
		$\hat{\theta}$	1.73147[5]	1.77105[6]	1.13046[2]	0.44473[1]	1.80482[7]	1.38051[4]	1.23401[3]
		$\hat{p}$	0.07536[5]	0.08217[6]	0.05616[2]	0.03692[1]	0.08513[7]	0.06629[4]	0.06054[3]
	MSE	$\hat{\alpha}$	1.320479×10^{10}[7]	0.00009[5]	0.00008[4]	0.00004[1]	0.00010[6]	0.00006[2.5]	0.00006[2.5]
		$\hat{\theta}$	5.04900[6]	4.90594[5]	2.06105[2]	1.14392[1]	5.24121[7]	3.05046[4]	2.43607[3]
		$\hat{p}$	0.00974[5]	0.01095[6]	0.00521[2]	0.00303[1]	0.01221[7]	0.00724[4]	0.00607[3]
	MRE	$\hat{\alpha}$	826.90886[7]	0.00012[5.5]	0.00011[4]	0.00004[1]	0.00012[5.5]	0.00010[3]	0.00009[2]
		$\hat{\theta}$	0.21643[5]	0.22138[6]	0.14131[2]	0.05559[1]	0.22560[7]	0.17256[4]	0.15425[3]
		$\hat{p}$	0.10766[5]	0.11739[6]	0.08023[2]	0.05275[1]	0.12162[7]	0.09470[4]	0.08648[3]
	$\sum$ Ranks		52[6]	50.5[5]	24[2]	9[1]	59.5[7]	32.5[4]	24.5[3]
600	BIAS	$\hat{\alpha}$	92964.14919[7]	0.00577[5]	0.00530[4]	0.00089[1]	0.00593[6]	0.00469[3]	0.00458[2]
		$\hat{\theta}$	1.46220[7]	1.32030[5]	0.84937[2]	0.15330[1]	1.35698[6]	1.03802[4]	0.94863[3]
		$\hat{p}$	0.05874[5]	0.06206[6]	0.04236[2]	0.02217[1]	0.06326[7]	0.05014[4]	0.04640[3]
	MSE	$\hat{\alpha}$	2.763871×10^{10}[7]	0.00005[4.5]	0.00005[4.5]	0.00001[1]	0.00006[6]	0.00003[2.5]	0.00003[2.5]
		$\hat{\theta}$	3.45090[7]	2.80265[5]	1.14970[2]	0.33455[1]	2.93418[6]	1.72709[4]	1.43692[3]
		$\hat{p}$	0.00576[5]	0.00620[6]	0.00290[2]	0.00109[1]	0.00654[7]	0.00405[4]	0.00350[3]
	MRE	$\hat{\alpha}$	1430.21768[7]	0.00009[5.5]	0.00008[4]	0.00001[1]	0.00009[5.5]	0.00007[2.5]	0.00007[2.5]
		$\hat{\theta}$	0.18277[7]	0.16504[5]	0.10617[2]	0.01916[1]	0.16962[6]	0.12975[4]	0.11858[3]
		$\hat{p}$	0.08391[5]	0.08866[6]	0.06051[2]	0.03167[1]	0.09038[7]	0.07162[4]	0.06629[3]
	$\sum$ Ranks		57[7]	48[5]	24.5[2]	9[1]	56.5[6]	32[4]	25[3]
800	BIAS	$\hat{\alpha}$	111180.84109[7]	0.00505[4]	0.00466[3]	0.00044[1]	0.00511[5]	0.00402[2]	86.69058[6]
		$\hat{\theta}$	1.37954[7]	1.15712[5]	0.73508[2]	0.07530[1]	1.16876[6]	0.88724[4]	0.84238[3]
		$\hat{p}$	0.05277[5]	0.05405[6]	0.03688[2]	0.01725[1]	0.05468[7]	0.04310[4]	0.04042[3]
	MSE	$\hat{\alpha}$	3.81645×10^{10}[7]	0.00004[4]	0.00004[4]	0.00001[1]	0.00004[4]	0.00002[2]	3.75638×10^{7}[6]
		$\hat{\theta}$	3.00556[7]	2.11549[5]	0.86248[2]	0.15964[1]	2.17174[6]	1.24305[4]	1.12133[3]
		$\hat{p}$	0.00458[5]	0.00467[6]	0.00218[2]	0.00065[1]	0.00481[7]	0.00296[4]	0.00261[3]
	MRE	$\hat{\alpha}$	1710.47448[7]	0.00008[4.5]	0.00007[3]	0.00001[1]	0.00008[4.5]	0.00006[2]	1.33370[6]
		$\hat{\theta}$	0.17244[7]	0.14464[5]	0.09188[2]	0.00941[1]	0.14609[6]	0.11091[4]	0.10530[3]
		$\hat{p}$	0.07538[5]	0.07721[6]	0.05269[2]	0.02465[1]	0.07811[7]	0.06157[4]	0.05775[3]
	$\sum$ Ranks		57[7]	45.5[5]	22[2]	9[1]	52.5[6]	30[3]	36[4]

Table 7: Partial and overall ranks of all the methods of estimation for various combination of ϕ

ϕ'	n	WLSE	OLSE	MLE	MPS	CVME	ADE	RADE
	50	4	6	1	7	5	2	3
	100	4	7	1	5	6	2	3
$(\alpha = 0.02, \theta = 0.75, p = 0.3)$	350	4	7	1	2	6	3	5
	600	4	7	2	1	6	3	5
	800	4	7	2	1	6	3	5
	50	4	6	1	5	7	2	3
	100	4	7	1	5	6	2	3
$(\alpha = 0.1, \theta = 7, p = 0.6)$	350	5	7	2	1	6	4	3
	600	5	7	2	1	6	4	3
	800	5	7	2	1	6	3.5	3.5
	50	7	6	2	1	5	4	3
	100	6	5	2	1	7	4	3
$(\alpha = 30, \theta = 20.2, p = 0.5)$	350	7	5	2	1	6	4	3
	600	7	5	2	1	6	4	3
	800	7	6	2	1	4.5	4.5	3
	50	6	7	2	1	4.5	4.5	3
	100	6	7	2	1	5	4	3
$(\alpha = 100, \theta = 39.5, p = 0.1)$	350	6	7	2	1	5	4	3
	600	7	6	2	1	4.5	4.5	3
	800	7	6	2	1	5	4	3
	50	5	6	3	1	7	4	2
	100	5	6	2.5	1	7	4	2.5
$(\alpha = 65, \theta = 8, p = 0.7)$	350	6	5	2	1	7	4	3
	600	7	5	2	1	6	4	3
	800	7	5	2	1	6	3	4
$\sum$ Ranks		139	155	46.5	44	145.5	89	81
Overall Rank		5	7	2	1	6	4	3

8. Applications

In this section, we illustrate the importance of the QXGGc distribution in modeling skewed data. The first data set consists of 128 observations of remission times (in months) of bladder cancer patients ([18]). This data set is positively skewed (skewness: 3.286), with mean remission time of 9.366 months, standard deviation of 10.508 months and is unimodal (presented in Table 8). This data set was previously studied by [11] and [2].
The second data set refers the survival times of 33 patients, in weeks, suffering from acute Myelogeneous Leukemia ([13]). This data set is positively skewed (skewness: 1.165), with mean survival time of 40.88 weeks, standard deviation of 46.70 weeks and is unimodal (presented in Table 9). These data were previously studied by [1] and [23].

Table 8: Remission times (in months) of a random sample of 128 bladder cancer patients

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98
6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28
9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25
8.37	12.02	2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69				

Table 9: Survival times of patients suffering from acute Myelogeneous Leukemia

65	156	100	134	16	108	121	4	39	143	56	26	22	1
1	5	65	56	65	17	7	16	22	3	4	2	3	8
4	3	30	4	43									

In order to compare the fits of the new distributions with other some competitive models given in Table 10, we consider some measures of goodness-of-fit including the maximized log-likelihood under the model ($-\widehat{\ell}$), Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion ($HQIC$) and Kolmogorov Smirnov ($K-S$) statistics with its bootstrapped p-value (PV). We bootstrapped the p-value of $K-S$ by using the bootstrap approach as considered in [31], and described in detail by [6] and [25]. The smaller the values of these statistics, better the fit.

Table 10: The fitted competitive models

Distribution	Author(s)
Transmuted generalized exponential (TGEx)	[15]
Kumaraswamy exponential (KEx)	[12]
Transmuted two-parameter Lindley (TTLi)	[16]
Beta exponential (BEx)	[14]
Lindley Weibull (LiW)	[10]
Quasi xgamma Poisson (QXGPo)	[28]

The pdf's of these models (for $x > 0$) are given by

TGEx: $f(x) = \frac{\alpha a}{\exp(ax)}\left[1-\exp(-ax)\right]^{\alpha-1}\left\{1+\lambda-2\lambda\left[1-\exp(-ax)\right]^{\alpha}\right\}.$

KEx: $f(x) = ab\lambda\exp(-\lambda x)\left[1-\exp(-\lambda x)\right]^{a-1}\left\{1-\left[1-\exp(-\lambda x)\right]^{a}\right\}^{b-1}.$

TTLi: $f(x) = \frac{a^2}{\alpha+a}(1+\alpha x)\exp(-ax)\left[1-\lambda+2\lambda\frac{\alpha+a+\alpha ax}{\alpha+a}\exp(-ax)\right].$

BEx: $f(x) = \frac{\lambda}{B(a,b)}\exp(-b\lambda x)\left[1-\exp(-\lambda x)\right]^{a-1}.$

LiW: $f(x) = \frac{\beta\theta^2}{\theta+1}\left[\alpha^{\beta}x^{\beta-1}+\alpha^{2\beta}x^{2\beta-1}\right]\exp\left[-\theta\left(\alpha x\right)^{\beta}\right].$

QXGPo: $f(x) = \frac{\lambda\theta(1+\alpha)^{-1}}{\left[\exp(\lambda)-1\right]}\left(\alpha+\frac{\theta^2}{2}x^2\right)\exp\left[\left(1+\alpha+\theta x+\frac{\theta^2}{2}x^2\right)\frac{\lambda\exp(-\theta x)}{1+\alpha}\right]\exp\left(-\theta x\right).$

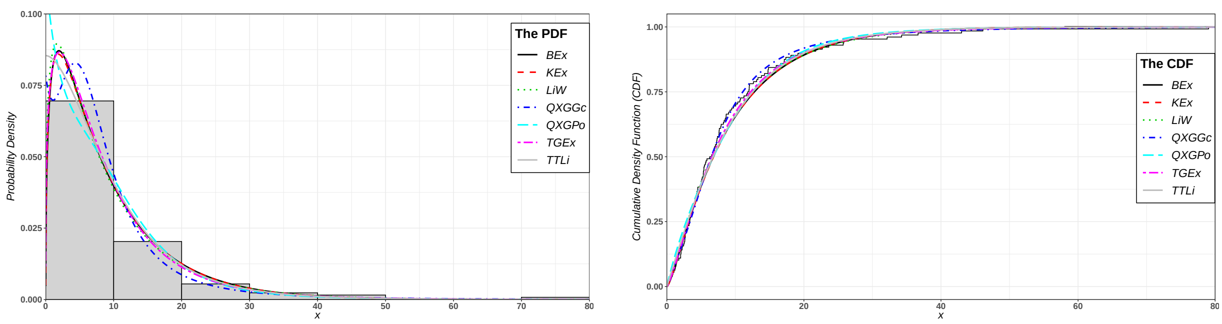
The parameters of the above densities are all positive real numbers except for the TGEx and TTLi distributions for which $|\lambda| \leq 1$. The numerical values of various measures, the parameter estimates and their corresponding standard errors (SEs) (in parentheses) are given in Tables 12 and 14, respectively.
The fitted pdfs of the fitting distributions as well as the empirical histogram are displayed in Figures 7 and 9, respectively. The corresponding probability (PP) plots are displayed in Figures 8 and 10. In Tables 12 and 14, we compare the fits of the QXGGc model with the distributions. The QXGGc distribution has the lowest values for all goodness-of-fit statistics among all fitted models.
From Tables 12 and 14, we conclude that the QXGGc distribution gives the smallest AIC, BIC, $HQIC$, $K-S$ values and the largest PV based on the KS statistic. Hence, the QXGGc distribution again provides the best fit for both data sets.
Now as we have aforementioned in Section 6, we use the different methods of estimation with $\hat{\ell}$, W, A, KS (and the corresponding p-value) as measures of goodness-of-fit for data sets I and II, Tables 11 and 13 display these results, respectively. From Tables 11 and 13 and based on goodness-of-fit measures, we recommend to use either MPS or MLE methods to estimate the parameters of the QXGGc distribution for data set I and data set II, which is expected from Table 7 in the simulation study.
Figure 11 displays the hrf plots of the QXGGc distribution for both data sets. It is seen that, the hrf is upside down bathtub for the first data set, whereas it is decreasing for the second data set.
Figure 12 displays the TTT plots of the QXGGc distribution for both data sets. The scaled TTT plot for the cancer data is concave then convex which indicates an upside down bathtub hazard rate, and the scaled TTT plot for the leukemia data is convex which indicates an decreasing hazard rate. Therefore, our QXGGc distribution is a suitable for modeling these data sets.

Table 11: The parameter estimates under various methods and the goodness-of-fit statistics for data set I.

Method	$\hat{\alpha}$	$\hat{\theta}$	$\hat{p}$	$-\hat{\ell}$	W	A	$K-S$	Bootstrapped PV
WLSE	0.0369	0.0905	0.9615	410.5629	0.0587	0.3303	0.0448	0.6423
OLSE	0.0475	0.1039	0.9463	410.7789	0.0600	0.3418	0.0469	0.2570
MLE	0.0285	0.0828	0.9708	410.4927	0.0560	0.3134	0.0460	0.6843
MPE	0.0197	0.0597	0.9863	410.6991	0.0634	0.3528	0.0467	0.6869
CVME	0.0472	0.1075	0.9431	410.7973	0.0588	0.3362	0.0438	0.5596
ADE	0.0346	0.0878	0.9645	410.5352	0.0582	0.3267	0.0451	0.6342
RADE	0.0361	0.0751	0.9729	410.8998	0.0635	0.3543	0.0579	0.1563

Table 12: Goodness-of-fit statistics and estimates for data set I

Model	$-\widehat{\ell}$	AIC	BIC	$HQIC$	$K-S$ (Bootstrapped PV)	Estimates (SEs)	
QXGGc	410.493	826.985	835.542	830.462	0.0460	α	0.0286(0.0200)
					(0.6843)	θ	0.0828(0.0336)
						p	0.9707(0.0295)
TGEx	410.947	827.893	836.449	831.370	0.0521	α	1.3025(0.1489)
					(0.2771)	a	0.0878(0.0183)
						λ	0.6980(0.2300)
KEx	412.464	830.927	839.483	834.403	0.0681	a	1.4502(0.2514)
					(0.7842)	b	0.2887(0.0373)
						λ	0.4045(0.0331)
TTLi	412.942	831.884	840.440	835.360	0.0637	α	0.1171(0.0294)
					(0.1177)	a	0.1580(0.1672)
						λ	0.7125(0.2069)
BEx	412.344	830.688	839.244	834.164	0.0665	a	1.4483(0.3276)
					(0.4285)	b	0.1794(0.1759)
						λ	0.6449(0.6076)
LiW	411.518	829.036	837.5919	832.512	0.0548	α	48.471(82.575)
					(0.3586)	β	0.7369(0.0500)
						θ	0.0237(0.0265)
QXGPo	415.015	836.030	844.586	839.506	0.0797	α	0.2902(0.1112)
					(0.0108)	θ	0.1227(0.0361)
						λ	4.0266(1.4030)



(a) (b)

Figure 7: Pdfs and cdfs of the fitted models for data set I.

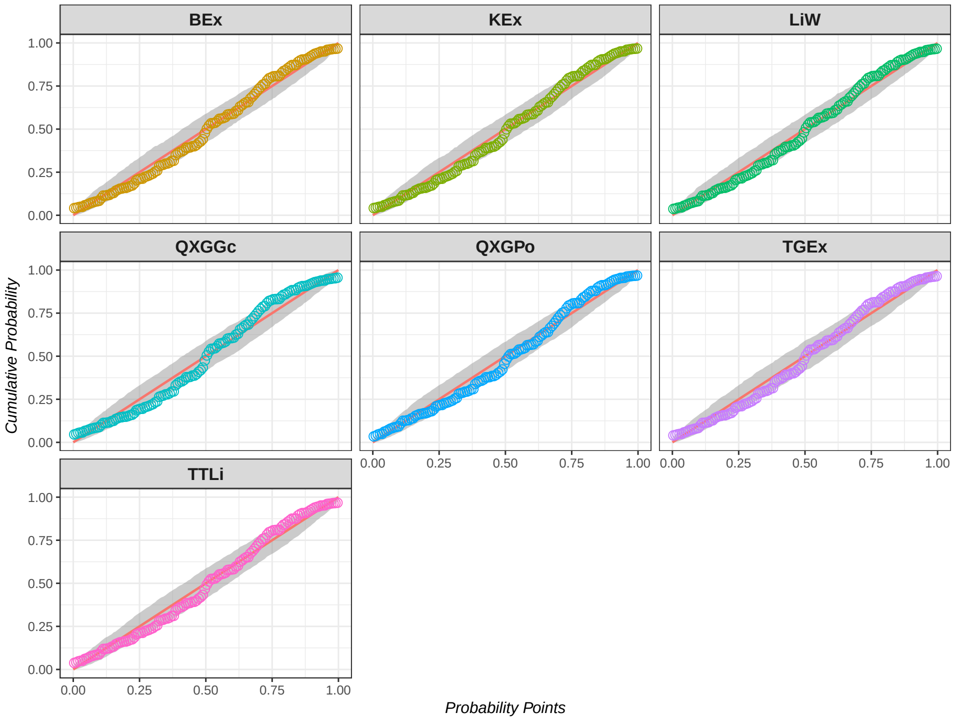


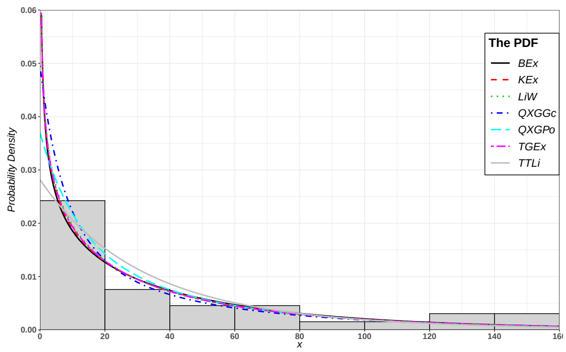
Figure 8: PP plots of the fitted models for data set I.

Table 13: The parameter estimates under various methods and the goodness-of-fit statistics for data set II.

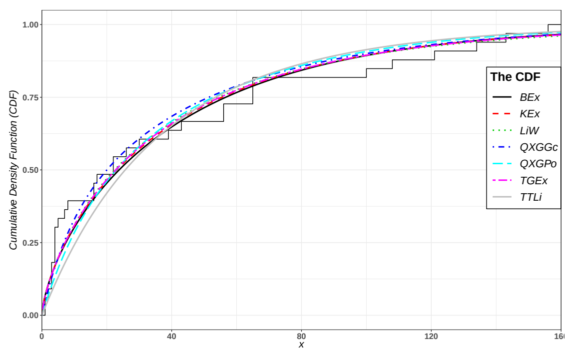
Method	$\hat{\alpha}$	$\hat{\theta}$	$\hat{p}$	$-\hat{\ell}$	W	A	$K-S$	PV
WLSE	0.3485	0.0126	0.9404	153.2798	0.0959	0.6261	0.1251	0.0896
OLSE	0.2694	0.0078	0.9701	153.9561	0.1075	0.6891	0.1237	0.0860
MLE	0.7892	0.0259	0.7698	152.6232	0.0830	0.5639	0.1382	0.1026
MPE	0.6151	0.0204	0.8516	152.7590	0.0872	0.5817	0.1314	0.1002
CVME	0.2838	0.0096	0.9613	153.6155	0.1016	0.6566	0.1248	0.0811
ADE	0.3340	0.0122	0.9446	153.3105	0.0965	0.6290	0.1237	0.0883
RADE	0.4995	0.0176	0.8804	152.9433	0.0891	0.5924	0.1405	0.1189

Table 14: Goodness-of-fit statistics and estimates for data set II

Model	$-\widehat{\ell}$	AIC	BIC	$HQIC$	$K-S$ (Bootstrapped PV)	Estimates (SEs)	
QXGG	152.623	311.246	315.735	312.756	0.1381	α	0.7892(0.9103)
					(0.1038)	θ	0.0259(0.0102)
						p	0.7698(0.2418)
TGEx	153.557	313.115	317.604	314.626	0.1359	α	0.7213(0.1633)
					(0.1402)	a	0.0172(0.0061)
						λ	0.2277(0.5029)
KEx	153.578	313.156	317.645	314.667	0.1363	a	0.7086(0.1476)
					(0.1284)	b	1.7469(2.2126)
						λ	0.0100(0.0144)
TTLi	154.856	315.713	320.202	317.224	0.2029	α	0.0010(0.0092)
					(0.0178)	a	0.0215(0.0096)
						λ	0.3708(0.3613)
BEx	153.651	313.301	317.790	314.812	0.1382	a	0.6745(0.1527)
					(0.2319)	b	0.8381(1.7002)
						λ	0.0227(0.0494)
LiW	153.602	313.203	317.693	314.714	0.1368	α	0.0103(0.0149)
					(0.1417)	β	0.7523(0.1115)
						θ	2.7340(2.6493)
QXGPo	153.388	312.775	317.264	314.285	0.1725	α	1.5958(1.3949)
					(0.0398)	θ	0.0295(0.0097)
						λ	1.6381(1.4001)



(a)



(b)

Figure 9: Pdfs and cdfs of the fitted models for data set II.

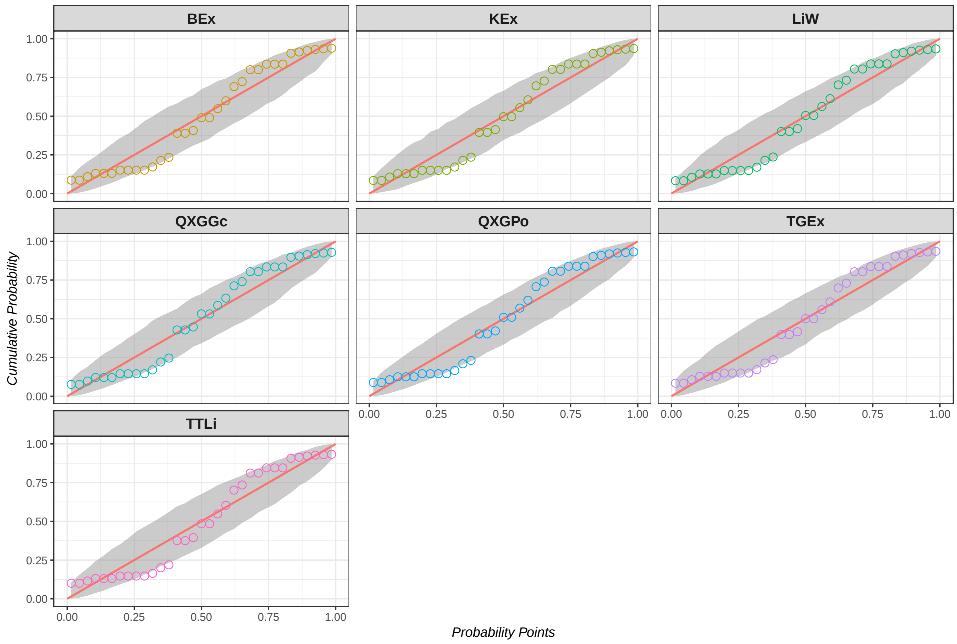
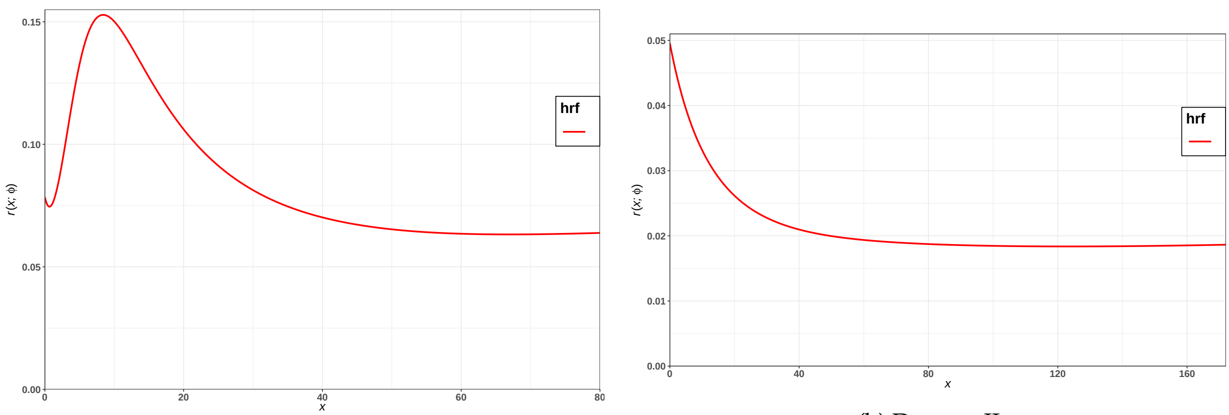


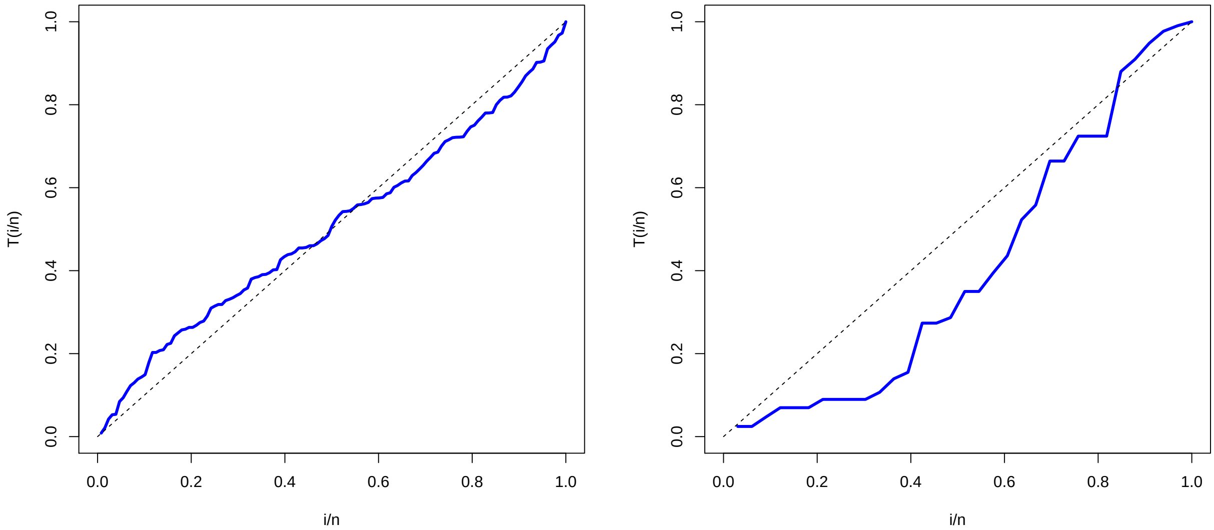
Figure 10: PP plots of the fitted models for data set II.



(a) Data set I

(b) Data set II

Figure 11: The hrf plots of the QXGGc distribution for both data sets.



(a) Data set I (b) Data set II

Figure 12: The TTT plots of the QXGGc distribution for both data sets.

9. Concluding remarks

In this article, a new distribution is synthesized, proposed and studied based on the quasi xgamma and geometric distributions. The QXGGc distribution, as named the proposed distribution, is basically a family of positively skewed probability distributions and possesses increasing and decreasing hazard rate properties depending on the values of the unknown parameters. The MRL function of QXGGc increases and decreases depending on the parameters as well. The QXGGc distribution is uniquely characterized utilizing its reversed hazard rate function. However, identifiability of the proposed probability distribution is not been investigated in this article and will be an important study.
Seven different frequentist methods of estimating unknown parameters of QXGGc model make it flexible in view of parametric inferential procedure and the same is justified with Monte-Carlo simulation study. Flexible data generation algorithm eases the utility of the proposed model in survival and/or reliability application which is accomplished by real data analyses and by comparing with other potential life distributions.
We believe that the QXGGc distribution can be utilized as useful generalization of the quasi xgamma distribution and can be well applied for data modeling in the field of survival and reliability analyses and other possible areas to explain uncertainties.

Acknowledgment

The authors would like place their sincere thanks to the anonymous referees and the editor-in-chief for their valuables comments and suggestions that have improved the article in its revised version.

References

[1] T. H. M. Abouelmagd, M. S. Hamed, A. Z. Afify, H. Al-Mofleh, Z. Iqbal, The Burr X Frchet distribution with its properties and applications, Journal of Applied Probability and Statistics 13 (2018a) 23–51.

[2] T. H. M. Abouelmagd, S. Al-mualim, A. Z. Afify, A. Munir, H. Al-Mofleh, The odd Lindley Burr XII distribution with applications, Pakistan Journal of Statistics 34 (2018b) 15–32.
[3] K. Adamidis, T. Dimitrakopoulou, S. Loukas, On an extension of the exponential-geometric distribution, Statistics and Probability Letters 73 (2005) 259–269.
[4] K. Adamidis, S. Loukas, A lifetime distribution with decreasing failure rate, Statistics and Probability Letters 39 (1998) 35–42.
[5] W. Barreto-Souza, F. Cribari-Neto, A generalization of the exponential-Poisson distribution, Statistics and Probability Letters 79 (2009) 2493–2500.
[6] R. Castro, Goodness-of-fit (GoF) Tests. http://www.win.tue.nl/ rmcastro/AppStat2013/files/lectures23.pdf, 2013.
[7] M. Chahkandi, M. Ganjali, On some lifetime distributions with decreasing failure rate, Computational Statistics and Data Analysis 53 (2009) 4433–4440.
[8] R. Cheng, N. Amin, Maximum product of spacings estimation with application to the lognormal distribution (Mathematical Report 79-1), Cardiff: University of Wales IST, 1979.
[9] R. Cheng, N. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, Journal of the Royal Statistical Society Series B (Methodological) 45 (1983) 394–403.
[10] G. M. Cordeiro, A. Z. Afify, H. M. Yousof, S. Cakmakyapan, G. Ozel, The Lindley Weibull distribution: properties and applications, Anais da Academia Brasileira de Ciências 90 (2018) 2579–2598.
[11] G. M. Cordeiro, A. Z. Afify, H. M. Yousof, R. R. Pescim, G. R. Aryal, The exponentiated Weibull-H family of distributions: theory and applications, Mediterranean Journal of Mathematics 14 (2017) 1–22.
[12] G. M. Cordeiro, E. M. Ortega, S. Nadarajah, The Kumaraswamy Weibull distribution with application to failure data, Journal of the Franklin Institute 347 (2010) 1399–1429.
[13] P. Feigl, M. Zelen, Estimation of exponential probabilities with concomitant information, Biometrics 21 (1965) 826–838.
[14] M. C. Jones, Families of distributions arising from distributions of order statistics, Test 13 (2004) 1–43.
[15] M. S. Khan, R. King, I. L. Hudson, Transmuted generalized exponential distribution: A generalization of the exponential distribution with applications to survival data, Communications in Statistics-Simulation and Computation 46 (2017) 4377–4398.
[16] S. A. Kemaloglu, M. Yilmaz, Transmuted two-parameter Lindley distribution, Communications in Statistics-Theory and Methods 46 (2017) 11866–11879.
[17] C. KuÅ, A new lifetime distribution, Computational Statistics and Data Analysis 51 (2007) 4497–4509.
[18] E. T. Lee, J. W. Wang, Statistical Methods for Survival Data Analysis, (3rd edition), Wiley, Hoboken, 2003.
[19] W. Lu, D. Shi, A new compounding life distribution: the WeibullPoisson distribution, Journal of Applied Statistics 39 (2012) 21–38.
[20] A. Luceño, Fitting the generalized pareto distribution to data using maximum goodness-of-fit estimators, Computational Statistics and Data Analysis 51 (2006) 904–917.
[21] P. Macdonald, Comment on "An Estimation Procedure for Mixtures of Distributions" by Choi and Bulgren, Journal of the Royal Statistical Society Series B (Methodological) 33 (1971) 326–329.
[22] A. L. Morais, W. Barreto-Souza, A compound class of Weibull and power series distributions, Computational Statistics and Data Analysis 55 (2011) 1410–1425.
[23] M. Nassar, A. Z. Afify, S. Dey, D. Kumar, A new extension of Weibull distribution: properties and different methods of estimation, Journal of Computational and Applied Mathematics 336 (2018) 439457
[24] B. Ranneby, The maximum spacing method: an estimation method related to the maximum likelihood method, Scandinavian Journal of Statistics 11 (1984) 93–112.
[25] M. M. Ristic, B. V. Popovic, K. Zografos, N. Balakrishnan, Discrimination among bivariate beta-generated distributions, Statistics 52 (2018) 303–320.
[26] S. Sen, N. Chandra, The quasi xgamma distribution with application in bladder cancer data, Journal of Data Science 15 (2017) 61–76.
[27] S. Sen, N. Chandra, S. S. Maiti, Survival estimation in xgamma distribution under progressively type-II right censored scheme, Model Assisted Statistics and Applications 13 (2018a) 107–121.
[28] S. Sen, M. Ç. Korkmaz, H. M. Yousof, The quasi xgamma-Poisson distribution: properties and application, iSTATiSTiK: Journal of the Turkish Statistical Association 11 (2018b) 65–76.
[29] S. Sen, S. S. Maiti, N. Chandra, The xgamma distribution: statistical properties and application, Journal of Modern Applied Statistical Method 15 (2016) 774–788.
[30] R. B. Silva, W. Barreto-Souza, G. M. Cordeiro, A new distribution with decreasing, increasing and upside-down bathtub failure rate, Computational Statistics and Data Analysis 54 (2010) 935–944.
[31] W. Stute, W. Gonzales-Manteiga, M. P. Quindimil, Bootstrap based goodness-of-fit-tests, Metrika 40 (1993) 243–256.
[32] J. Swain, S. Venkatraman, J. Wilson, Least squares estimation of distribution function in Johnsons translation system, Journal of Statistical Computation and Simulation 29 (1988) 271–297.
[33] R. Tahmasbi, S. Rezaei, A two-parameter lifetime distribution with decreasing failure rate, Computational Statistics and Data Analysis 52 (2008) 3889–3901.

Appendix A. R code

```
#The function QXGGc multiply by X#
XQXGG = function(x,theta)
              {
              alpha  = theta[1]
              theta1 = theta[2]
              p      = theta[3]
              x1     = alpha+((theta1^2/2))*x^2
              x2     = exp(-theta1*x)
              x3     = (1-((p*x2*(1+theta1*x+x1))/(1+alpha)))^(-2)
              const  = theta1*(1-p)/(1+alpha)
              const*x*x1*x2*x3
              }

#Numerical integration of the Mean of X#
MEAN = function(theta)
        {
        tt1 = seq(1,1,l=length(theta[,1]))
        for(i in 1:length(theta[,1])){
        tt1[i]= integrate(XQXGG,0,Inf,theta=theta[i,])$val
             }
        tt1
               }

#Mean of X by the Summation in equation (9)#
mom=function(r,M,theta1){
             alpha=theta1[1]
             theta=theta1[2]
             p    =theta1[3]
            sum=0
for(k in 0:M){
 for(n in 0:k){
   for(i in 0:(n+1)){
     for(j in 0:(k-n)){
         t1=choose(n+1,i)*choose(k,n)*choose(k-n,j)
         t2=(p^k)*(alpha^(n-i+1))/((2^i)*((alpha+1)^(k+1)))
         t3=gamma(2*i+j+r+1)/((theta^r)*(k+1)^(2*i+j+r))
         sum=sum+(t1*t2*t3)
                 }
               }
          }
          }
          sum*(1-p)
                    }
############################################
#Numerical integration at N truncated terms#
theta1=rbind(c(0.8,1.2,0.1),
             c(0.8,6  ,0.1),
             c(2.5,1.2,0.1),
             c(2.5,6  ,0.1),
```

```
            c(0.8,1.2,0.4),
            c(0.8,6  ,0.4),
            c(2.5,1.2,0.4),
            c(2.5,6  ,0.4),
            c(0.8,1.2,0.9),
            c(0.8,6  ,0.9),
            c(2.5,1.2,0.9),
            c(2.5,6  ,0.9))
round(cbind(MEAN(theta1)),5)

#Mean of X by the Summation#
for(i in 1:12){
print(rbind(round(mom(1,10,theta1[i,]),5),
      round(mom(1,50,theta1[i,]),5),
      round(mom(1,80,theta1[i,]),5)))
}
```