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Inverse Boundary-Value Problem for the Equation of Longitudinal Wave Propagation with Non-self-adjoint Boundary Conditions

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Abstract. We study the inverse coefficient problem for the equation of longitudinal wave propagation with non-self-adjoint boundary conditions. The main purpose of this paper is to prove the existence and uniqueness of the classical solutions of an inverse boundary-value problem. To investigate the solvability of the inverse problem, we carried out a transformation from the original problem to some equivalent auxiliary problem with trivial boundary conditions. Applying the Fourier method and contraction mappings principle, the solvability of the appropriate auxiliary inverse problem is proved. Furthermore, using the equivalency, the existence and uniqueness of the classical solution of the original problem are shown.

1. Introduction

In this paper we study the unique solvability of the nonlocal inverse boundary-value problem for the longitudinal wave propagation equation [8]

$$u_{tt}(x,t) - u_{ttxx}(x,t) - u_{xx}(x,t) = q(t)u(x,t) + f(x,t) \ \ (x,t) \in D_T, \tag{1}$$

with the nonlocal initial conditions

$$\begin{aligned} u(x,0) &= \int_0^T P_1(t)u(x,t)dt + \varphi(x), \\ u_t(x,0) &= \int_0^T P_2(t)u(x,t)dt + \psi(x), \ 0 \le x \le 1, \end{aligned} \tag{2}$$

non-self-adjoint boundary conditions

$$u(0,t) = \beta u(1,t), \ \ u_x(0,t) = u_x(1,t), \ 0 \le t \le T, \tag{3}$$

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and the overdetermination condition

$$u\left(\frac{1}{2}, t\right) = h(t), \ 0 \le t \le T, \tag{4}$$

where $D_T := \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ is a rectangular domain, $\beta \neq \pm 1$ is given number, $f(x,t)$, $P_1(t)$, $P_2(t)$, $\varphi(x)$, $\psi(x)$, $h(t)$ are given functions, and $u(x,t)$, $q(t)$ are unknown functions.

The problem of finding a pair $\{u(x,t), q(t)\}$ in (1)-(4) will be called an inverse problem.

Most generally, the inverse coefficient problems arises in many different areas of mathematical modelings, such as mineral exploration, biology, medicine, seismology, etc. Problems for the solvability of inverse problems for various types of partial differential equations were discussed in papers and monographs by Belov [2, 3], Kozhanov [16], Ivanchov [13], Prilepko [24], Pyatkov [25], Kabanikhin [14], and many others [5, 6, 18, 26, 27].

In some instances, the equation of propagation of longitudinal waves can arise in the theory of long waves, plasma physics, problems of hydrodynamics and so on [1, 7, 9–11, 19, 30]. The investigation of the equation of propagation of longitudinal waves is of interest both for specialists in mathematical modeling and for mathematicians. In fundamental science, the equation for the propagation of longitudinal waves is considered as a Sobolev type equation and many works have been devoted to it [4, 28, 29]. But, the inverse problems for the equation of longitudinal wave propagation equation have been studied relatively little, and one can get acquainted with them in the articles [20, 21, 23].

It will be noted that the statement of the problem and the proof technique used in this article are different from those of the above-mentioned articles, and the conditions in the theorems significantly differ from those therein.

We introduce the following set of functions

$$\tilde{C}^{(2,2)}(D_T) = \{u(x,t) | u(x,t) \in C^2(D_T), u_{txx}(x,t), u_{ttx}(x,t), u_{ttxx}(x,t) \in C(D_T)\}.$$

Definition 1.1. *The pair $\{u(x,t), q(t)\}$ is said to be a classical solution of the problem (1)-(4), if the functions $u(x,t) \in \tilde{C}^{(2,2)}(D_T)$ and $q(t) \in C[0,T]$ satisfy Equation (1) in D_T, the condition (2) on $[0,1]$, and the statements (3)-(4) on the interval $[0,T]$.*

In order to investigate the problem (1) - (4), we first consider the following auxiliary problem

$$y''(t) = q(t)y(t), \ 0 \le t \le T, \tag{5}$$

$$y(0) = \int_0^T P_1(t)y(t)dt, \ y'(0) = \int_0^T P_2(t)y(t)dt, \tag{6}$$

where $P_1(t), P_2(t), q(t) \in C[0,T]$ are given functions, and $y = y(t)$ is unknown function. By a solution of the problem (5), (6), we understand a function $y(t) \in C^2[0,T]$ satisfying the conditions (5) and (6) in the usual sense.

Lemma 1.2. [21] *Suppose that $P_i(t) \in C[0,T]$ $(i = 1,2)$, $q(t) \in C[0,T]$, $\|q(t)\|_{C[0,T]} \le R = const$, and the condition*

$$\left(T \|P_2(t)\|_{C[0,T]} + \|P_1(t)\|_{C[0,T]} + \frac{T}{2}R\right)T < 1$$

hold. Then the problem (5), (6) has a unique trivial solution.

Now along with the inverse boundary-value problem (1) - (4), we consider the following auxiliary inverse boundary-value problem. It is required to determine a pair $\{u(x,t), q(t)\}$ of functions $u(x,t) \in \tilde{C}^{(2,2)}(D_T)$ and $q(t) \in C[0,T]$ from relations (1)-(3), and

$$h''(t) - u_{ttxx}\left(\frac{1}{2}, t\right) - u_{xx}\left(\frac{1}{2}, t\right) = q(t)h(t) + f\left(\frac{1}{2}, t\right), \ 0 \le t \le T. \tag{7}$$

Theorem 1.3. *Assume that $\varphi(x), \psi(x) \in C[0,1]$, $P_1(t), P_2(t) \in C[0,T]$, $h(t) \in C^2[0,T]$, $h(t) \neq 0$, $0 \leq t \leq T$, $f(x,t) \in C(D_T)$, and the compatibility conditions*

$$h(0) = \int_0^T P_1(t)h(t)dt + \varphi\left(\frac{1}{2}\right), \quad h'(0) = \int_0^T P_2(t)h(t)dt + \psi\left(\frac{1}{2}\right) \tag{8}$$

hold. Then the following assertions are valid:

(i) *each classical solution $\{u(x,t), q(t)\}$ of the problem (1)-(4) is a solution of problem (1)-(3), (7), and*

(ii) *each solution $\{u(x,t), q(t)\}$ of the problem (1)-(3), (7), satisfying the condition*

$$\left(T \|P_2(t)\|_{C[0,T]} + \|P_1(t)\|_{C[0,T]} + \frac{T}{2} \left\|q(t)\right\|_{C[0,T]}\right) T < 1 \tag{9}$$

is a classical solution of problem (1)-(4).

Proof. Let $\{u(x,t), q(t)\}$ be a classical solution of the problem (1)-(4). Taking into consideration $h(t) \in C^2[0,T]$ and differentiating (4), we get

$$u_t\left(\frac{1}{2}, t\right) = h'(t), \; u_{tt}\left(\frac{1}{2}, t\right) = h''(t), \; 0 \leq t \leq T. \tag{10}$$

Setting $x = \frac{1}{2}$ in Equation (1), we find

$$u_{tt}\left(\frac{1}{2}, t\right) - u_{ttxx}\left(\frac{1}{2}, t\right) - u_{xx}\left(\frac{1}{2}, t\right) = q(t)u\left(\frac{1}{2}, t\right) + f\left(\frac{1}{2}, t\right), \; 0 \leq t \leq T. \tag{11}$$

By (4) and (10), we conclude that the relation (7) is fulfilled.

Now, assume that $\{u(x,t), q(t)\}$ is a solution of problem (1)-(3), (7) and the condition (9) is fulfilled. Then from (7), taking into account (11), we have

$$\frac{d^2}{dt^2}\left(u\left(\frac{1}{2}, t\right) - h(t)\right) = q(t)\left(u\left(\frac{1}{2}, t\right) - h(t)\right), \; 0 \leq t \leq T. \tag{12}$$

Furthermore, from (2) and (8), we obtain

$$\begin{aligned} u\left(\tfrac{1}{2}, 0\right) - h(0) - \int_0^T P_1(t)(u\left(\tfrac{1}{2}, t\right) - h(0))dt &= \varphi\left(\tfrac{1}{2}\right) - \left(h(0) - \int_0^T P_1(t)h(t)dt\right) = 0, \\ u_t\left(\tfrac{1}{2}, 0\right) - h'(0) - \int_0^T P_2(t)(u\left(\tfrac{1}{2}, t\right) - h(0))dt &= \psi\left(\tfrac{1}{2}\right) - \left(h'(0) - \int_0^T P_2(t)h(t)dt\right) = 0. \end{aligned} \tag{13}$$

Since by Lemma 1.2, problem (12), (13) has only a trivial solution, it follows that

$$u\left(\frac{1}{2}, t\right) - h(t) = 0, \; 0 \leq t \leq T,$$

i.e. the condition (4) holds. □

2. Auxiliary facts and notations

Now, in order to investigate the problem (1)-(3), (7), we cite some known facts.

Consider the following spectral problem [12, 15]

$$X''(x) + \lambda X(x) = 0, \; 0 \leq x \leq 1, \tag{14}$$

$$X(0) = \beta X(1),\ X'(0) = X'(1),\ \beta \neq \pm 1. \tag{15}$$

It is seen that the boundary-value problem (14), (15) is not self-adjoint. But the problem

$$Y''(x) + \lambda Y(x) = 0,\ 0 \leq x \leq 1, \tag{16}$$

$$Y(0) = Y(1),\ Y'(1) = \beta Y'(0), \tag{17}$$

will be a conjugated problem.

Denote the system of eigen and adjoint functions of the problem (14), (15) in the following way [15],

$$X_0(x) = ax + b, ..., X_{2k-1}(x) = (ax + b)\cos\lambda_k x, X_{2k}(x) = \sin\lambda_k x, ..., \tag{18}$$

where

$$\lambda_k = 2k\pi, k = 0, 1, 2, ...,\ a = (1-\beta)/(1+\beta) \neq 0,\ b = \beta/(1+\beta). \tag{19}$$

We choose the system of eigen and adjoint functions of the conjugated problem as follows [15]

$$Y_0(x) = 2, ..., Y_{2k-1}(x) = 4\cos\lambda_k x, Y_{2k}(x) = 4(1 - b - ax)\sin\lambda_k x, ... \tag{20}$$

It is directly verified that the biorthogonality conditions

$$(X_i, Y_j) = \int_0^1 X_i(x) Y_j(x) dx = \delta_{ij}$$

are fulfilled. Here, δ_{ij} is Kronecker's symbol.

The following theorem is valid:

Theorem 2.1. [22] *The system of functions (18) forms a Riesz basis in the space $L_2(0, 1)$ and the estimates*

$$r \left\| g(x) \right\|_{L_2(0,1)} \leq \sum_{k=0}^{\infty} g_k^2 \leq R \left\| g(x) \right\|_{L_2(0,1)}, \tag{21}$$

where

$$g_k = (g(x), Y_k(x)) = \int_0^1 g(x) Y_k(x) dx,\ k = 0, 1, ...,$$

$$r = \left\{ \frac{1}{3}\left(\left(a + \frac{3}{2}b\right)^2 + \frac{3}{4}b^2 \right) + \frac{1}{2}\left(1 + \left\| (ax+b)^2 \right\|_{c[0,1]}\right) \right\}^{-1},$$

$$R = 8\left(1 + \left\| (1 - b - ax)^2 \right\|_{C[0,1]}\right),$$

are valid for any function $g(x) \in L_2(0, 1)$.

Under the assumptions

$$g(x) \in C^{2i-1}[0, 1],\ g^{(2i)}(x) \in L_2(0, 1),$$

$$g^{(2s)}(0) = \beta g^{(2s)}(1),\ g^{(2s+1)}(0) = g^{(2s+1)}(1),\ s = \overline{0, i-1},\ i \geq 1,$$

we establish the validity of the inequalities

$$\left(\sum_{k=1}^{\infty} (\lambda_k^{2i} g_{2k-1})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{2} \left\| g^{(2i)}(x) \right\|_{L_2(0,1)}, \tag{22}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i}g_{2k})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\left\|g^{(2i)}(x)(1-b-ax)-2aig^{(2i-1)}(x)\right\|_{L_2(0,1)}. \tag{23}$$

Further, under the suppositions

$$g(x)\in C^{2i}[0,1],\ g^{(2i+1)}(x)\in L_2(0,1),$$

$$g^{(2s)}(0)=\beta g^{(2s)}(1),\ g^{(2s-1)}(0)=g^{(2s-1)}(1),\ s=\overline{0,i},$$

we prove the validity of the estimates

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i+1}g_{2k-1})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\left\|g^{(2i+1)}(x)\right\|_{L_2(0,1)}, \tag{24}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^{2i+1}g_{2k})^2\right)^{\frac{1}{2}} \le 2\sqrt{2}\left\|g^{(2i+1)}(x)(1-b-ax)-a(2i+1)g^{(2i)}(x)\right\|_{L_2(0,1)}. \tag{25}$$

Now, denote by $B_{2,T}^3$ [17] a set of all the functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x),$$

considered in D_T, where each of the functions $u_k(t)$ $(k=0,1,2...)$ is continuous on $[0,T]$ and

$$J(u)\equiv\|u_0(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}<+\infty.$$

The norm in this set is defined as follows

$$\|u(x,t)\|_{B_{2,T}^3}=J(u).$$

That is, we denote by E_T^3 the space $B_{2,T}^3\times C[0,T]$ of vector-functions $z(x,t)=\{u(x,t),q(t)\}$ with the norm

$$\|z\|_{E_T^3}=\|u(x,t)\|_{B_{2,T}^3}+\left\|q(t)\right\|_{C[0T]}.$$

It is known that $B_{2,T}^3$ and E_T^3 are the Banach spaces.

3. Classical Solvability of Inverse Boundary-Value Problem

The system (18) is Riesz basis in $L_2(0,1)$ and the systems (18), and (20) are biorthogonal in $L_2(0,1)$. We'll seek the first component $u(x,t)$ of classical solution $\{u(x,t),q(t)\}$ of the problem (1)-(3), (7) in the form

$$u(x,t)=u_0(t)X_0(x)+\sum_{k=1}^{\infty}u_{2k-1}(t)X_{2k-1}(x)+\sum_{k=1}^{\infty}u_{2k}(t)X_{2k}(x), \tag{26}$$

where

$$u_0(t)=\int_0^1 u(x,t)Y_0(x)dx,\ u_{2k-1}(t)=\int_0^1 u(x,t)Y_{2k-1}(x)dx,\ u_{2k}(t)=\int_0^1 u(x,t)Y_{2k}(x)dx. \tag{27}$$

Here, $X_k(x)$ $(k = 0, 1, 2, ...)$ and $Y_k(x)$ $(k = 0, 1, 2, ...)$ are expressed by (18) and (19), respectively. Then by separation of variables, from (1) and (2), we have

$$u_0''(t) = q(t)u_0(t) + f_0(t),\ 0 \le t \le T, \tag{28}$$

$$(1+\lambda_k^2)u_{2k-1}''(t) + \lambda_k^2 u_{2k-1}(t) = q(t)u_{2k-1}(t) + f_{2k-1}(t),\ k = 1, 2, ...,\ 0 \le t \le T, \tag{29}$$

$$(1+\lambda_k^2)u_{2k}''(t) + \lambda_k^2 u_{2k}(t) = q(t)u_{2k}(t) + f_{2k}(t) - 2a\lambda_k(u_{2k-1}''(t) + u_{2k-1}(t)),\ k = 1, 2, ...,\ 0 \le t \le T, \tag{30}$$

$$u_k(0) = \varphi_k + \int_0^T P_1(t)u_k(t)dt,\ u_k'(0) = \psi_k + \int_0^T P_2(t)u_k(t)dt,\ k = 0, 1, 2, ..., \tag{31}$$

where

$$f_k(t) = \int_0^1 f(x,t)Y_k(x)dx,\ \varphi_k = \int_0^1 \varphi(x)Y_k(x)dx,\ \psi_k = \int_0^1 \psi(x)Y_k(x)dx,\ k = 0, 1, ...$$

Solving the problem (28)-(31), we get

$$u_0(t) = \left(\varphi_0 + \int_0^T P_1(t)u_0(t)dt\right) + t\left(\psi_0 + \int_0^T P_2(t)u_0(t)dt\right) + \int_0^t (t-\tau)F_0(\tau; u, q)d\tau,\ 0 \le t \le T, \tag{32}$$

$$u_{2k-1}(t) = \left(\varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt\right)\cos\beta_k t + \frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt\right) sin\beta_k t$$
$$+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_{2k-1}(\tau; u, q)\sin\beta_k(t-\tau)d\tau,\ k = 1, 2, ...,\ 0 \le t \le T, \tag{33}$$

$$u_{2k}(t) = \left(\varphi_{2k} + \int_0^T P_1(t)u_{2k}(t)dt\right)\cos\beta_k t + \frac{1}{\beta_k}\left(\psi_{2k} + \int_0^T P_2(t)u_{2k}(t)dt\right)\sin\beta_k t$$
$$+\frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_{2k}(\tau; u, q)\sin\beta_k(t-\tau)d\tau$$
$$-\frac{a\lambda_k(1-\beta_k^2)}{\beta_k(1+\lambda_k^2)}\left[t\left(\varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt\right)\sin\beta_k t\right.$$
$$\left.+\left(\frac{1}{\beta_k}\sin\beta_k t - t\cos\beta_k t\right)\frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt\right)\right]$$
$$-\frac{2a\lambda_k(1-\beta_k^2)}{\beta_k^2(1+\lambda_k^2)^2}\int_0^t\left(\int_0^\tau F_{2k-1}(\xi; u, q)\sin\beta_k(t-\xi)d\xi\right)\sin\beta_k(t-\tau)d\tau$$
$$-\frac{2a\lambda_k}{\beta_k(1+\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau; u, q)\sin\lambda_k(t-\tau)d\tau,\ k = 1, 2, ...,\ 0 \le t \le T, \tag{34}$$

where

$$F_k(t;u,q) = f_k(t) + q(t)u_k(t)\ (k = 0,1,2,...)$$

and

$$\beta_k = \frac{\lambda_k}{\sqrt{1+\lambda_k^2}}.$$

After substitution of the expressions $u_0(t)$, $u_{2k-1}(t)$, $u_{2k}(t)$ described by (32), (33), and (34) in (26), respectively, we obtain

$$u(x,t) = \left\{ \left(\varphi_0 + \int_0^T P_1(t)u_0(t)dt \right) + t\left(\psi_0 + \int_0^T P_2(t)u_0(t)dt \right) \right.$$

$$\left. + \int_0^t (t-\tau)F_0(\tau;u,q)d\tau \right\} X_0(x) + \sum_{k=1}^{\infty} \left\{ \left(\varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt \right)\cos\beta_k t \right.$$

$$+ \frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt \right) sin\beta_k t$$

$$\left. + \frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_{2k-1}(\tau;u,q)\sin\beta_k(t-\tau)d\tau \right\} X_{2k-1}(x)$$

$$+ \sum_{k=1}^{\infty} \left\{ \left(\varphi_{2k} + \int_0^T P_1(t)u_{2k}(t)dt \right)\cos\beta_k t + \frac{1}{\beta_k}\left(\psi_{2k} + \int_0^T P_2(t)u_{2k}(t)dt \right)\sin\beta_k t \right.$$

$$+ \frac{1}{\beta_k(1+\lambda_k^2)}\int_0^t F_{2k}(\tau;u,q)\sin\beta_k(t-\tau)d\tau - \frac{a\lambda_k(1-\beta_k^2)}{\beta_k(1+\lambda_k^2)}$$

$$\times \left[t\left(\varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt \right)\sin\beta_k t + \left(\frac{1}{\beta_k}\sin\beta_k t \right.\right.$$

$$\left.\left. - \ t\cos\beta_k t \right)\frac{1}{\beta_k}\left(\psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt \right) \right]$$

$$- \frac{2a\lambda_k(1-\beta_k^2)}{\beta_k^2(1+\lambda_k^2)^2}\int_0^t \left(\int_0^\tau F_{2k-1}(\xi;u,q)\sin\beta_k(t-\xi)d\xi \right)\sin\beta_k(t-\tau)d\tau$$

$$\left. - \frac{2a\lambda_k}{\beta_k(1+\lambda_k^2)^2}\int_0^t F_{2k-1}(\tau;u,q)\sin\lambda_k(t-\tau)d\tau \right\} X_{2k}(x). \tag{35}$$

Now, in order to find the equation for the component $q(t)$ of $\{u(x,t), q(t)\}$, from (7) by virtue of (26), we have

$$q(t) = [h(t)]^{-1}\left\{h''(t) - f\left(\frac{1}{2}, t\right) - \left(\frac{1}{2}a + b\right)\sum_{k=1}^{\infty}(-1)^k\lambda_k^2(u_{2k-1}(t) + u''_{2k-1}(t))\right\}. \quad (36)$$

By (33), from (29) we get

$$\lambda_k^2(u_{2k-1}(t) + u''_{2k-1}(t)) = -u''_{2k-1}(t) + F_{2k-1}(t;u,q)$$

$$= \beta_k^2 u_{2k-1}(t) + \left(1 - \frac{1}{1+\lambda_k^2}\right)F_{2k-1}(t;u,q) = \beta_k^2 u_{2k-1}(t) + \beta_k^2 F_{2k-1}(t;u,q)$$

$$= \beta_k^2\left(\varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt\right)\cos\beta_k t + \beta_k\left(\psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt\right)sin\beta_k t$$

$$+\frac{\beta_k}{1+\lambda_k^2}\int_0^t F_{2k-1}(\tau;u,q)\sin\lambda_k(t-\tau)d\tau + \beta_k^2 F_{2k-1}(t;u,q). \quad (37)$$

Then from (36), taking into account (37), we have

$$q(t) = [h(t)]^{-1}\left\{h''(t) - f\left(\frac{1}{2}, t\right) - \left(\frac{1}{2}a + b\right)\sum_{k=1}^{\infty}(-1)^k\right.$$

$$\times\left(\beta_k^2\left(\varphi_{2k-1} + \int_0^T P_1(t)u_{2k-1}(t)dt\right)\cos\beta_k t + \beta_k\left(\psi_{2k-1} + \int_0^T P_2(t)u_{2k-1}(t)dt\right)sin\beta_k t\right.$$

$$\left.\left.+\frac{\beta_k}{1+\lambda_k^2}\int_0^t F_{2k-1}(\tau;u,q)\sin\lambda_k(t-\tau)d\tau + \beta_k^2 F_{2k-1}(t;u,q)\right)\right\}. \quad (38)$$

Thus, the problem of finding the solution of (1) - (3), (7) reduces to finding a solution of system (35) and (38) with respect to functions $u(x,t)$ and $q(t)$.

Based on the definition of the solution of problem (1) - (3), (7), the following lemma was proved.

Lemma 3.1. *If $\{u(x,t), q(t)\}$ is any solution of (1)-(3), (7), then the functions are defined by formulas (27) satisfy system (32), (33), and (34).*

From Lemma 3.1 it follows

Corollary 3.2. *Suppose that system (35) and (38) has a unique solution. Then the problem (1)- (3), (7), can't have more than one solution.*

In other words, if the problem (1) - (3), (7) has a solution, then it is unique.

Next, in the space E_T^2 consider the operator

$$\Phi(u,q) = \{\Phi_1(u,q), \Phi_2(u,q)\},$$

where

$$\Phi_1(u,q) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty}\tilde{u}_k(t)X_k(x),\ \Phi_2(u,q) = \tilde{q}(t),$$

and the functions $\tilde{u}_0(t), \tilde{u}_{2k-1}(t), \tilde{u}_{2k}(t)$, and $\tilde{q}(t)$ are equal to the right-hand sides of (32), (33), (34), and (38), respectively.

It is easy to see that

$$1/\sqrt{2} < \beta_k < 1, \ 0 < 1 - \beta_k^2 < \frac{1}{2}.$$

Taking into account these relations, we have

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq \left|\varphi_0\right| + T\left|\psi_0\right| + T(\|P_1(t)\|_{C[0,T]} + T\,\|P_2(t)\|_{C[0,T]})\,\|u_0(t)\|_{C[0,T]}$$

$$+T\sqrt{T}\left(\int_0^T \left|f_0(\tau)\right|^2 d\tau\right)^{\frac{1}{2}} + T^2\left\|q(t)\right\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}, \tag{39}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{2k-1}\right|)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{3}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_{2k-1}\right|)^2\right)^{\frac{1}{2}} + T\sqrt{6}(\|P_1(t)\|_{C[0,T]}$$

$$+\sqrt{2}\,\|P_2(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + 2\sqrt{3T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\left|f_{2k-1}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}}$$

$$+2\sqrt{3}T\left\|q(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{2k-1}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{40}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{14}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{2k}\right|)^2\right)^{\frac{1}{2}} + 2\sqrt{7}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_{2k}\right|)^2\right)^{\frac{1}{2}}$$

$$+T\sqrt{14}(\|P_1(t)\|_{C[0,T]} + \sqrt{2}\,\|P_2(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{7T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\left|f_{2k}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}} + 2\sqrt{7}T\left\|q(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{7}aT\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{2k-1}\right|)^2\right)^{\frac{1}{2}} + \sqrt{14}a(\sqrt{2}+T)\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_{2k-1}\right|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{14}aT(T\sqrt{2}\,\|P_1(t)\|_{C[0,T]} + (\sqrt{2}+T)\,\|P_2(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{2k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{14T}a(T+\sqrt{2})\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\left|f_{2k-1}(\tau)\right|)^2 d\tau\right)^{\frac{1}{2}}$$

$$+2\sqrt{14}a(T+\sqrt{2})T^2\left\|q(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}. \tag{41}$$

$$\left\|q(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}$$

$$\times\left\{\left\|h''(t)-f\left(\frac{1}{2},t\right)\right\|_{C[0,T]}+\left(\frac{1}{2}a+b\right)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{2k-1}\right|)^2\right)^{\frac{1}{2}}\right.\right.$$

$$+\left(\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_{2k-1}\right|)^2\right)^{\frac{1}{2}}+T(\|P_1(t)\|_{C[0,T]}+\|P_2(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\left|f_{2k-1}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+T\left\|q(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$\left.\left.+\left(\sum_{k=1}^{\infty}(\lambda_k\left\|f_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left\|q(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_{2k-1}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{42}$$

Assume that the data for the problem (1)-(3), (7) satisfy the following conditions:

(A_1) $\varphi(x)\in C^2[0,1]$, $\varphi'''(x)\in L_2(0,1)$, $\varphi(0)=\beta\varphi(1)$, $\varphi'(0)=\varphi'(1)$, $\varphi''(0)=\beta\varphi''(1)$;

(A_2) $\psi(x)\in C^2[0,1]$, $\psi'''(x)\in L_2(0,1)$, $\psi(0)=\beta\psi(1)$, $\psi'(0)=\psi'(1)$, $\psi''(0)=\beta\psi''(1)$;

(A_3) $f(x,t)\in C(D_T)$, $f_x(x,t)\in L_2(D_T)$, $f(0,t)=\beta f(1,t)$, $0\leq t\leq T$;

(A_4) $\beta\neq\pm1$, $P_i(t)\in C[0,T]$ $(i=1,2)$, $h(t)\in C^2[0,T]$, $h'(t)\neq 0$, $0\leq t\leq T$.

Then from (39)-(41), we find that

$$\|\tilde{u}_0(t)\|_{C[0,T]}\leq A_1(T)+B_1(T)\left\|q(t)\right\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}+C_1(T)\|u(x,t)\|_{B^3_{2,T}}, \tag{43}$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\|\tilde{u}_{2k-1}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}\leq A_2(T)+B_2(T)\left\|q(t)\right\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}+C_2(T)\|u(x,t)\|_{B^3_{2,T}}, \tag{44}$$

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\|\tilde{u}_{2k}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}\leq A_3(T)+B_3(T)\left\|q(t)\right\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}+C_3(T)\|u(x,t)\|_{B^3_{2,T}}, \tag{45}$$

$$\left\|\tilde{q}(t)\right\|_{C[0,T]}\leq A_1(T)+B_1(T)\left\|q(t)\right\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}+C_1(T)\|u(x,t)\|_{B^3_{2,T}}, \tag{46}$$

where

$$A_1(T)=2\left\|\varphi(x)\right\|_{L_2(0,1)}+2T\left\|\psi(x)\right\|_{L_2(0,1)}+2T\sqrt{T}\left\|f(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=T^2,$$

$$C_1(T)=T(\|P_1(t)\|_{C[0,T]}+T\|P_2(t)\|_{C[0,T]}),$$

$$A_2(T)=4\sqrt{3}\left\|\varphi'''(x)\right\|_{L_2(0,1)}+4\sqrt{6}\left\|\psi'''(x)\right\|_{L_2(0,1)}+4\sqrt{6T}\left\|f_x(x,t)\right\|_{L_2(D_T)},$$

$$B_2(T)=2\sqrt{3}T,$$

$$C_2(T)=T\sqrt{6}(\|P_1(t)\|_{C[0,T]}+\sqrt{2}\|P_2(t)\|_{C[0,T]}),$$

$$A_3(T) = 4\sqrt{7}\left\|\varphi'''(x)(1-b-ax) - 3a\varphi''(x)\right\|_{L_2(0,1)}$$
$$+4\sqrt{14}\left\|\psi'''(x)(1-b-ax) - 3a\psi''(x)\right\|_{L_2(0,1)}$$
$$+4\sqrt{14T}\left\|f_x(x,t)(1-b-sx) - af(x,t)\right\|_{L_2(D_T)}$$
$$+4\sqrt{14}aT\left\|\varphi'''(x)\right\|_{L_2(0,1)} + 4\sqrt{7}a(\sqrt{2}+T)\left\|\psi'''(x)\right\|_{L_2(0,1)}$$
$$+8\sqrt{7T}a(T+\sqrt{2})\left\|f_x(x,t)\right\|_{L_2(D_T)},$$
$$B_3(T) = 2\sqrt{7}T(1+\sqrt{2}a(T+\sqrt{2})T),$$
$$C_3(T) = T\sqrt{14}(1+\sqrt{2}aT)\,\|P_1(t)\|_{C[0,T]} + T\sqrt{14}(\sqrt{2}+a(\sqrt{2}+T))\,\|P_2(t)\|_{C[0,T]},$$
$$A_4(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) - f\left(\frac{1}{2},t\right)\right\|_{C[0,T]} + \left(\frac{1}{2}a+b\right)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\right.$$
$$\left.\times[2\sqrt{2}\left\|\varphi'''(x)\right\|_{L_2(0,1)} + 2\sqrt{2}\left\|\psi''(x)\right\|_{L_2(0,1)} + 2\sqrt{2T}\left\|f_x(x,t)\right\|_{L_2(D_T)}]\right\},$$
$$B_4(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\frac{1}{2}a+b\right)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(T+1),$$
$$C_4(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\frac{1}{2}a+b\right)\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}T(\|P_1(t)\|_{C[0,T]} + T\,\|P_2(t)\|_{C[0,T]}).$$

From (43)-(45), we conclude that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \le A_5(T) + B_5(T)\left\|q(t)\right\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_5(T)\,\|u(x,t)\|_{B^3_{2,T}}, \tag{47}$$

where

$$A_5(T) = A_1(T) + A_2(T) + A_3(T),\ B_5(T) = B_1(T) + B_2(T) + B_3(T),\ C_5(T) = C_1(T) + C_2(T) + C_3(T).$$

Finally, from inequalities (46) and (47), we deduce

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \left\|\tilde{q}(t)\right\|_{C[0,T]} \le A(T) + B(T)\left\|q(t)\right\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C(T)\,\|u(x,t)\|_{B^3_{2,T}}, \tag{48}$$

where

$$A(T) = A_4(T) + A_5(T),\ B(T) = B_4(T) + B_5(T),\ C(T) = C_4(T) + C_5(T).$$

Theorem 3.3. *Let the conditions* $(A_1) - (A_4)$*, and the condition*

$$(A(T)+2)(B(T)(A(T)+2) + C(T)) < 1 \tag{49}$$

be fulfilled. Then the problem (1)-(3), (7) has a unique solution in the ball $K = K_R(\|z\|_{E^3_T} \le A(T)+2)$ *of the space* E^3_T.

Proof. We represent the system of equations (35), (38) in the operator form:

$$z = \Phi z, \tag{50}$$

where $z = \{u, q\}$, the components $\Phi_i(u,q)$ $(i = 1,2)$ of operator $\Phi = \{\Phi_1(u,q), \Phi_2(u,q)\}$, defined by the right-hand side of equations (35) and (38).

Consider the operator $\Phi(u,q)$, in the ball $K = K_R$ of the space E_T^3. Let's show that the operator Φ mappings the elements of the ball $K = K_R$ into itself. Similarly to (48) we obtain that for any $z \in K_R$ the following inequalities hold

$$\|\Phi z\|_{E_T^3} \leq A(T) + B(T)\,\|u(x,t)\|_{B_{2,T}^3}\,\left\|q(t)\right\|_{C[0,T]} + C(T)\,\|u(x,t)\|_{B_{2,T}^3}$$

$$\leq A(T) + (A(T)+2)(B(T)(A(T)+2) + C(T)).$$

Hence, by (49), we get that the operator $\Phi(u,q)$ acts in the ball $K = K_R$. It is easy to show that the operator $\Phi(u,q)$ is contractive. Indeed, for any $z_1, z_2 \in K_R$ the following estimation is true

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq (B(T)(A(T)+2) + C(T))(\|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^3} + \left\|q_1(t) - q_2(t)\right\|_{C[0,T]}).$$

Since $B(T)(A(T)+2) + C(T) < (A(T)+2)[B(T)(A(T)+2) + C(T)]$, by virtue of (49), from the latter relation we obtain the operator $\Phi(u,q)$ is contractive. For that reason, the operator $\Phi(u,q)$ has a unique fixed point $\{z\} = \{u,q\}$ in the ball $K = K_R$, which is the solution of Equation (50) in the ball $K = K_R$, i.e. in the ball $K = K_R$ is a unique solution of the system (35), (38). Then the function $u(x,t)$ as an element of the space $B_{2,T}^3$, is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

It is easy to see that the function $u_t(x,t)$ is continuous in D_T.

Further, it is possible to verify that Equation (1) and conditions (2), (3), (7) are satisfied in the usual sense. Consequently, $\{u(x,t), q(t)\}$ is a solution of (1) - (3), (7), and by Corollary 3.2, it is unique in the ball $K = K_R$. The proof is complete. □

From Theorem 1.3 and Theorem 3.3, it follows directly the following assertion.

Theorem 3.4. *Suppose that all conditions of Theorem 3.3, and the compatibility conditions*

$$h(0) = \int_0^T P_1(t)h(t)dt + \varphi\left(\frac{1}{2}\right), \quad h'(0) = \int_0^T P_2(t)h(t)dt + \psi\left(\frac{1}{2}\right),$$

and

$$\left(T\,\|P_2(t)\|_{C[0,T]} + \|P_1(t)\|_{C[0,T]} + \frac{T}{2}(A(T)+2)\right)T < 1$$

hold. Then the problem (1)-(4) has a unique classical solution in the ball $K = K_R$ of the space E_T^3.

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