Filomat 33:16 (2019), 5237–5248
https://doi.org/10.2298/FIL1916237T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Geometries of Manifolds Equipped with a Ricci (Projection-Ricci) Quarter-Symmetric Connection

Wanxiao Tang[a], Jon Chol Yong[b], Ho Tal Yun[b], Guoqing He[c], Peibiao Zhao[d]

[a]*Dept. of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, P. R. China*
[b]*Faculty of Mathematics, Kim Il Sung University, Pyongyang, DPRK*
[c]*Department of Applied Mathematics, Anhui Normal University, Wuhu 241000, P. R. China*
[d]*Dept. of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, P. R. China*

Abstract. We first introduce a Ricci quarter-symmetric connection and a projective Ricci quarter-symmetric connection, and then we investigate a Riemannian manifold admitting a Ricci (projective Ricci) quarter-symmetric connection $(\mathcal{M}, g)$, and prove that a Riamannian manifold with a Ricci(projection-Ricci) quarter-symmetric connection is of a constant curvature manifold. Furthermore, we derive that an Einstein manifold $(\mathcal{M}, g)$ is conformally flat under certain condition.

1. Introduction

The study of geometric analysis of manifolds associated with certain semi symmetric connection or with Ricci operator following some semi symmetric properties has been an active field over the past five decades. The ideas and methods of these two themes overlap and promote each other, which makes their respective research achieve amazing results.

The geometric analysis of manifolds endowed with semi symmetric connections or endowed with Ricci operator following some semi symmetric properties, almost from the inception, was regarded as an extra-ordinary difficult interesting theory.

It is well known that A. Friedman and J. A. Schouten [6], in 1924, introduced the idea of a semi-symmetric linear connection in a differentiable manifold. H. A. Hayden [14], in 1930, introduced a metric connection with a non-zero torsion on a Riemannian manifold. In 1970, K. Yano [26] considered a semi-symmetric metric connection and studied some of its properties, he proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric metric connection for which the manifold is a group manifold, where a group manifold is a differentiable manifold admitting a linear connection ∇ such that it satisfies the relation $R = 0$ and $\nabla T = 0$. The semi-symmetric metric connection ∇ satisfies the relation

$$\nabla_Z g(X, Y) = 0, T(X, Y) = \pi(Y)x - \pi(X)Y \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 58E20; Secondary 53C21
Keywords. Ricci quarter-symmetric connection; Ricci quarter-symmetric projective connection; Schur's theorem.
Received: 07 September 2018; Accepted: 16 December 2018
Communicated by Mića Stanković
Research supported by the NNSF of China(No.11871275; No.11371194), by Jiangsu excellent graduate course"Portfolio management+micro course", by NUST Research Funding No.CXZZ11-0258, and by Research found for the doctoral program of Anhui Normal University(No.751841).
Corresponding author: Peibiao Zhao
Email addresses: wanxiaotang92@163.com (Wanxiao Tang), hgq1001@ahnu.edu.cn (Guoqing He), pbzhao@njust.edu.cn (Peibiao Zhao)

where π is a 1-form. In [1, 2, 9, 11], the semi-symmetric non-metric connection was studied. A semi-symmetric non-metric connection is exactly a geometrical model for scalar-tensor theories of gravitation was studied in [4]. Also other types of a semi-symmetric non-metric connection were studied in [2, 5, 18].

On the one hand, the Schur's theorem of a semi-symmetric non-metric connection is well-known in [15–17] based only on the second Bianchi identity. In 1975, S. Golab[8] defined and studied quarter-symmetric linear connections in differentiable manifolds. A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \pi(Y)\psi X - \pi(X)\psi Y \tag{2}$$

where ψ is a $(1,1)$-tensor. In [10, 17], a projective(projective-like) invariant of a quarter-symmetric metric connections was obtained. Afterwards, some properties of several types of a quarter-symmetric metric connection were studied ([7, 11, 20, 23–25, 27–31]). Recently, Han, Fu and Zhao [12, 13] studied similar problems in Sub-Riemannian manifolds.

On the other hand, the geometries of manifolds associated with some types of a quarter-symmetric non-metric connection ([3, 5, 19]), or with Ricci tensor satisfying certain condition were studied recently. As we know, for instance, that Y. J. Suh, C. A. Mantica and U. C. De [22] derived that a pesudo-Riemannian manifold is B-symmetric if and only if it is Ricci semi-symmetric; S. Mallick, A. Yildiz and U. C. De [21] investigated the geometric properties of mixed quasi-Einstein manifolds.

The so-called Ricci quarter-symmetric metric connection ∇ is defined as

$$\nabla_x Y = \overset{o}{\nabla}_X Y + \pi(Y)UX - U(X,Y)P \tag{3}$$

where $\overset{o}{\nabla}$ is the Levi-Civita connection, U is the Ricci operator, $U(X,Y)$ is the Ricci tensor and $\pi(X) = g(X,P)$. This connection satisfies the relation

$$\nabla_Z g(X,Y) = 0, T(X,Y) = \pi(Y)UX - \pi(X)UY \tag{4}$$

The local expressions of (3) and (4) are respectively

$$\Gamma^k_{ij} = \{^k_{ij}\} + \pi_j U^k_i - U_{ij}\pi^k \tag{5}$$

and

$$\nabla_k g_{ij} = 0, T^k_{ij} = \pi_j U^k_i - \pi_i U^k_j \tag{6}$$

The present paper is organized as follows. Section 2 investigates a Ricci quarter-symmetric connection and studies the geometrical characteristics of a manifolds associated with this connection; Section 3 studies projective Ricci quarter-symmetric connection, and we arrive at the physical properties for a Riemannian manifold with this connection.

2. Geometries Of Manifolds Associated With A Ricci Quarter-symmetric Connection

Let $(\mathcal{M}, g)$ be a Riemannian manifold ($dim\mathcal{M} \geq 2$), g be the Rieman-nian metric on $\mathcal{M}$, and $\overset{o}{\nabla}$ be the Levi-Civita connection with respect to g. Let $\mathcal{X}(\mathcal{M})$ denote the collection of all vector fields on $\mathcal{M}$.

Definition 2.1. *A connection ∇ is called a Ricci quarter-symmetric connection, if it satisfies*

$$\begin{aligned} \nabla_Z g(X,Y) &= (1-\alpha)\pi(Z)U(X,Y) + (\beta - \frac{1+\alpha}{2})\pi(X)U(Y,Z) + (\beta - \frac{1+\alpha}{2})\pi(Y)U(Z,X) \\ T(X,Y) &= \pi(Y)UX - \pi(X)UY \end{aligned} \tag{7}$$

where U is the Ricci operator and $U(X,Y)$ is the Ricci tensor, π is a 1-form and α, β are parameters for this connection.

Then there exists a unique connection ∇ in $\mathcal{M}$ given by

$$\nabla_X Y = \overset{o}{\nabla}_X Y + \frac{1+\alpha}{2}\pi(Y)UX - \frac{1-\alpha}{2}\pi(X)UY - \beta U(X,Y)P \tag{8}$$

where $\pi(X) = g(X, P)$.

Let (x^i) be the local coordinate, then $g, \overset{o}{\nabla}, \nabla, U, \pi,$ and T have local expressions g_{ij}, $\{^k_{ij}\}$, Γ^k_{ij}, U^k_j, π_i and T^k_{ij} respectively. At the same time, the expressions (7) and (8) can be rewritten as

$$\begin{aligned} \nabla_k g_{ij} &= (1-\alpha)\pi_k U_{ij} + (\beta - \frac{1+\alpha}{2})\pi_i U_{jk} + (\beta - \frac{1+\alpha}{2})\pi_j U_{ik} \\ T^k_{ij} &= \pi_j U^k_i - \pi_i U^k_j \end{aligned} \tag{9}$$

and

$$\Gamma^k_{ij} = \{^k_{ij}\} + \frac{1+\alpha}{2}\pi_j U^k_i - \frac{1-\alpha}{2}\pi_i U^k_j + \beta U_{ij}\pi^k \tag{10}$$

Remark 2.2. *By (8), it is obvious that there holds the following.*
When $\alpha = 1$ and $\beta = 1$, then ∇ is a Ricci quarter-symmetric metric connection([20])
When $\alpha = 1$ and $\beta = 0$, then ∇ satisfies

$$\nabla_k g_{ij} = -\pi_i U_{jk} - \pi_j U_{ik}, T^k_{ij} = \pi_j U^k_i - \pi_i U^k_j.$$

This connection is a semi-symmetric non-metric connection in an Einstain manifold ([1]).
When $\alpha = 1$ and $\beta = -1$, then ∇ satisfies

$$\nabla_k g_{ij} = -2\pi_i U_{jk} - 2\pi_j U_{ik}, T^k_{ij} = \pi_j U^k_i - \pi_i U^k_j.$$

This connection is a semi-symmetric non-metric connection satisfying the Schur's theorem in an Einstain manifold ([15]).

From (10), the curvature tensor of ∇, by a direct computation, is

$$\begin{aligned} R_{ijk}{}^l &= K_{ijk}{}^l - \frac{1+\alpha}{2}U^l_i a_{jk} - \beta b^l_i U_{jk} + \frac{1+\alpha}{2}c^l_{ij}\pi_k - \beta c_{ijk}\pi^l + \frac{1-\alpha}{2}d^l_{jk}\pi_i - \pi_{ij}U^l_k \\ &+ \frac{1+\alpha}{2}U^l_j a_{ik} + \beta b^l_j U_{ik} - \frac{1+\alpha}{2}c^l_{ji}\pi_k + \beta c_{jik}\pi^l - \frac{1-\alpha}{2}\pi_j d^l_{ik} + \pi_{ji}U^l_k \end{aligned} \tag{11}$$

where $K_{ijk}{}^l$ is a curvature tensor of Levi-Civita connection $\overset{o}{\nabla}$ and the other notations are given as follows

$$\begin{cases} a_{ik} = \overset{o}{\nabla}_i \pi_k - \frac{1+\alpha}{2}U^p_i \pi_p \pi_k + \frac{1-\alpha}{2}U^p_k \pi_p \pi_i + \beta U_{ik}\pi_p \pi^p \\ b_{ik} = \overset{o}{\nabla}_i \pi_k - \frac{1+\alpha}{2}U^p_k \pi_p \pi_i - \beta U_{ip}\pi^p \pi_k \\ c_{ijk} = \overset{o}{\nabla}_i U_{jk} \\ d^l_{ik} = \overset{o}{\nabla}_i U^l_k - \frac{1+\alpha}{2}U^p_i U^l_p \pi_k - \beta U_{ip}U^p_k \pi^l \\ \pi_{ij} = \frac{1-\alpha}{2}\overset{o}{\nabla}_i \pi_j \end{cases} \tag{12}$$

Let

$$A_{ijk}{}^l = \frac{1+\alpha}{2}U^l_i a_{jk} + \beta b^l_i U_{jk} - \frac{1+\alpha}{2}c^l_{ij}\pi_k + \beta c_{ijk}\pi^l - \frac{1-\alpha}{2}\pi_i d^l_{jk} + \pi_{ij}U^l_k \tag{13}$$

Then we from the expression (11) get

$$R_{ijk}{}^l = K_{ijk}{}^l - A_{ijk}{}^l + A_{jik}{}^l \tag{14}$$

So there exists the following

Theorem 2.3. *When $A_{ijk}{}^{l} = A_{jik}{}^{l}$, then the curvature tensor will keep unchanged under the connection transformation $\overset{o}{\nabla} \to \nabla$.*

From the expression (10), the coefficient of dual connection $\overset{*}{\nabla}$ of the Ricci quarter-symmetric connection ∇ is

$$\overset{*}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \frac{1-\alpha}{2}\pi_i U^k_j + \beta\pi_j U^k_i - \frac{1+\alpha}{2}U_{ij}\pi^k \tag{15}$$

By using the expression (15), the curvature tensor of dual connection $\overset{*}{\nabla}$ is

$$\begin{aligned}\overset{*}{R}_{ijk}{}^{l} \;=\; & K_{ijk}{}^{l} - \frac{1+\alpha}{2}a^l_i U_{jk} - \beta U^l_i b_{jk} + \beta C^l_{ij}\pi_k - \frac{1+\alpha}{2}C_{ijk}\pi^l - \frac{1-\alpha}{2}\pi_i d^l_{jk} + \pi_{ij}U^l_k \\ + \; & \frac{1+\alpha}{2}a^l_j U_{ik} + \beta U^l_j b_{ik} - \beta C^l_{ji}\pi_k + \frac{1+\alpha}{2}C_{jik}\pi^l - \frac{1-\alpha}{2}\pi_j d^l_{jk} - \pi_{ji}U^l_k\end{aligned} \tag{16}$$

It is well known that if a sectional curvature at a point P in a Riemannian manifold is independent of E (a 2-dimensional subspace of $T_p(\mathcal{M})$), the curvature tensor is

$$R_{ijk}{}^{l} = k(p)\big(\delta^l_i g_{jk} - \delta^l_j g_{ik}\big) \tag{17}$$

In this case, if $k(p) = const$, then the Riemannian manifold is a constant curvature manifold.

Theorem 2.4. *Suppose that $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ is a connected Riemannian manifold associated with an isotropic Ricci quarter-symmetric connection ∇ with $T^h_{hi} \neq 0$. If there holds*

$$\frac{1+\alpha}{2} + \beta = 0, \tag{18}$$

then $(\mathcal{M}, g, \nabla)$ is of a constant curvature manifold.

Proof. Substituting the expression (17) into the second Bianchi identity of the curvature tensor of the Ricci quarter-symmetric connection, that is, there holds the following

$$\nabla_h R_{ijk}{}^{l} + \nabla_i R_{jhk}{}^{l} + \nabla_j R_{hik}{}^{l} = T^m_{hi}R_{jmk}{}^{l} + T^m_{ij}R_{hmk}{}^{l} + T^m_{jh}R_{imk}{}^{l}$$

then we have

$$\begin{aligned}&\nabla_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \nabla_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \nabla_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk}) \\ &\quad + k(p)\big(\delta^l_i \nabla_h g_{jk} - \delta^l_j \nabla_h g_{ik} + \delta^l_j \nabla_i g_{hk} - \delta^l_h \nabla_i g_{jk} + \delta^l_h \nabla_j g_{ik} - \delta^l_i \nabla_j g_{hk}\big) \\ &= k(p)\Big[(\pi_i U^m_h - \pi_h U^m_i)(\delta^l_j g_{mk} - \delta^l_m g_{jk}) + (\pi_j U^m_i - \pi_i U^m_j)(\delta^l_h g_{mk} - \delta^l_m g_{hk}) \\ &\quad + (\pi_h U^m_j - \pi_j U^m_h)(\delta^l_i g_{mk} - \delta^l_m g_{ik})\Big]\end{aligned}$$

Using the expression (9), then we obtain

$$\begin{aligned}&\nabla_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \nabla_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \nabla_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk}) \\ &\quad + (\frac{3-\alpha}{2} - \beta)k(p)\Big[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})\Big] \\ &= k(p)\big(\delta^l_j \pi_i U_{hk} - \delta^l_i \pi_h U_{jk} - \pi_i U^l_h g_{jk} + \pi_h U^l_j g_{ik} + \delta^l_h \pi_i U_{jk} - \delta^l_i \pi_j U_{hk} - \pi_j U^l_i g_{hk} + \pi_i U^l_h g_{jk} \\ &\quad + \delta^l_i \pi_h U_{jk} - \delta^l_h \pi_j U_{ik} - \pi_h U^l_j g_{ik} + \pi_j U^l_i g_{hk}\big)\end{aligned}$$

Contracting the indices i and l, then we have

$$\begin{aligned}&(n-2)\Big[(g_{jk}\nabla_h k(p) - g_{hk}\nabla_j k(p)) + (\frac{3-\alpha}{2} - \beta)k(p)(\pi_h U_{jk} - \pi_j U_{hk})\Big] \\ &= (n-3)k(p)\Big[\pi_h U_{jk} - \pi_j U_{hk} + (U^i_i \pi_h - U^i_h \pi_i)g_{jk} - (U^i_i \pi_j - U^i_j \pi_i)g_{hk}\Big]\end{aligned}$$

Multiplying both sides of this expression by g^{jk} and contracting the indices j and k, then we arrive at

$$(n-2)\Big[(n-1)\nabla_h k(p) - \beta + \frac{1+\alpha}{2}(U^i_h\pi_i - U^i_i\pi_h)k(p)\Big] = 0.$$

From this equation above we obtain

$$\nabla_h k(p) = (\beta + \frac{1+\alpha}{2})\frac{k(p)}{n-1}T^i_{hi}.$$

Consequently, from $T^i_{hi} \neq 0$, we know that $k(p) = const$ if and only if $\beta + \frac{1+\alpha}{2} = 0$. This ends the proof of Theorem 2.4. □

The Ricci quarter-symmetric connection ∇ satisfying the Schur's theorem is denoted by $\overset{\alpha}{\nabla}$. The connection $\overset{\alpha}{\nabla}$ can be regarded as a special type of the Ricci quarter-symmetric connection ∇.

By Theorem 2.4 and using (18), the expression (9) shows

$$\begin{cases} \overset{\alpha}{\nabla}_k g_{ij} &= (1-\alpha)\pi_k U_{ij} - (1+\alpha)\pi_i U_{jk} - (1+\alpha)\pi_j U_{ik} \\ T^k_{ij} &= \pi_j U^k_i - \pi_i U^k_j \end{cases} \tag{19}$$

Similarly, the formula (10) for $\overset{\alpha}{\nabla}$ shows

$$\overset{\alpha}{\Gamma}{}^k{}_{ij} = \left\{{k \atop ij}\right\} + \frac{1+\alpha}{2}\pi_j U^k_i - \frac{1-\alpha}{2}\pi_i U^k_j + \frac{1+\alpha}{2}U_{ij}\pi^k \tag{20}$$

and the formula (15) for $\overset{\alpha^*}{\nabla}$ shows

$$\overset{\alpha^*}{\Gamma}{}^k{}_{ij} = \left\{{k \atop ij}\right\} - \frac{1+\alpha}{2}\pi_j U^k_i + \frac{1-\alpha}{2}\pi_i U^k_j - \frac{1+\alpha}{2}U_{ij}\pi^k \tag{21}$$

By the expressions (20) and (21), it is obvious that there holds the following Theorem.

Theorem 2.5. *The Levi-Civita connection $\overset{o}{\nabla}$ on a connected Riemannian manifold $(\mathcal{M}, g)$ (dim $\mathcal{M} > 2$) is the mean connection of the Ricci quarter-symmetric connection $\overset{\alpha}{\nabla}$ and its dual connection $\overset{\alpha^*}{\nabla}$.*

If $\alpha = -1$, then the Ricci quarter-symmetric connection $\overset{\alpha}{\nabla}$ satisfies the relation

$$\overset{\alpha}{\nabla}_k g_{ij} = 2\pi_k U_{ij},\ \overset{\alpha}{T}{}^k{}_{ij} = \pi_j U^k_i - \pi_i U^k_j,$$

and the coefficient of $\overset{\alpha}{\nabla}$

$$\overset{\alpha}{\Gamma}{}^k{}_{ij} = \left\{{k \atop ij}\right\} - \pi_i U^k_j$$

This connection is considered as a geometrical model for scalar-tensor theories of gravitation in an Einstain manifold ([4]).

Theorem 2.6. *Suppose that $\alpha = -1$, and the Riemannian manifold $(\mathcal{M}, g, \overset{\alpha}{\nabla})$(dim $\mathcal{M} > 2$) is of a constant curvature manifold associated with a Ricci quarter-symmetric connection $\overset{\alpha}{\nabla}$, then the Riemannian manifold $(\mathcal{M}, g, \overset{o}{\nabla})$ is of a constant curvature manifold.*

Proof. From the expression (18), if $\alpha = -1$, then $\beta = 0$. In this case from the expressions (11) and (16), the curvature tensors of ∇ and $\overset{*}{\nabla}$ are below

$$\overset{\alpha}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} - U^{l}_{k}(\pi_{ij} - \pi_{ji}) + \pi_i d^{l}_{jk} - \pi_j d^{l}_{ik}$$

$$\overset{\alpha^*}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + U^{l}_{k}(\pi_{ij} - \pi_{ji}) - \pi_i d^{l}_{jk} + \pi_j d^{l}_{ik}$$

respectively. From the two expressions above, we have

$$K_{ijk}{}^{l} = \frac{1}{2}(\overset{\alpha}{R}_{ijk}{}^{l} + \overset{\alpha^*}{R}_{ijk}{}^{l}) \tag{22}$$

By using the constant curvature assumption in Theorem 2.6, we have

$$\overset{\alpha}{R}_{ijk}{}^{l} = \overset{\alpha^*}{R}_{ijk}{}^{l} = k(\delta^{l}_{i} g_{jk} - \delta^{l}_{j} g_{ik}),\ (k = const).$$

Hence from the expression (22), we have

$$K_{ijk}{}^{l} = k(\delta^{l}_{i} g_{jk} - \delta^{l}_{j} g_{ik})$$

This means consequently that the Riemannian manifold $(\mathcal{M}, g, \overset{o}{\nabla})$ is of a constant curvature manifold. This completes the proof of Theorem 2.6. □

Theorem 2.7. *If an Einstain manifold $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ associated with a Ricci quarter-symmetric connection $\overset{\alpha}{\nabla}$ is a constant curvature manifold, then $(\mathcal{M}, g, \overset{o}{\nabla})$ is of a conformal flat manifold.*

Proof. From the expression (11), the curvature tensor of $\overset{\alpha}{\nabla}$ is

$$\begin{aligned} \overset{\alpha}{R}_{ijk}{}^{l} &= K_{ijk}{}^{l} + \frac{1+\alpha}{2} U^{l}_{j} a_{ik} - \frac{1+\alpha}{2} U^{l}_{i} a_{jk} + \frac{1+\alpha}{2} b^{l}_{i} U_{jk} + \frac{1-\alpha}{2} d^{l}_{jk} \pi_i - \pi_{ij} U^{l}_{k} \\ &+ \frac{1+\alpha}{2} c^{l}_{ij} \pi_k - \frac{1+\alpha}{2} b^{l}_{j} U_{ik} + \frac{1+\alpha}{2} c_{jik} \pi^{l} - \frac{1-\alpha}{2} \pi_j d^{l}_{ik} + \pi_{ji} U^{l}_{k} \end{aligned} \tag{23}$$

And from the expression (16), the curvature tensor of $\overset{\alpha^*}{\nabla}$ is

$$\begin{aligned} \overset{\alpha^*}{R}_{ijk}{}^{l} &= K_{ijk}{}^{l} - \frac{1+\alpha}{2} b^{l}_{i} U_{jk} + \frac{1+\alpha}{2} b^{l}_{j} U_{ik} + \frac{1+\alpha}{2} b_{jk} U^{l}_{i} - \frac{1+\alpha}{2} b_{ik} U^{l}_{j} + U^{l}_{k} \pi_{ij} \\ &\quad - \frac{1+\alpha}{2} C^{l}_{ij} \pi_k - \frac{1+\alpha}{2} C_{ijk} \pi^{l} - \frac{1-\alpha}{2} d^{l}_{jk} \pi_i + \frac{1-\alpha}{2} d^{l}_{ik} \pi_j - U^{l}_{k} \pi_{ji} \end{aligned} \tag{24}$$

Adding the expressions (23) and (24), we obtain

$$\overset{\alpha^*}{R}_{ijk}{}^{l} + \overset{\alpha}{R}_{ijk}{}^{l} = 2K_{ijk}{}^{l} + U^{l}_{j} \gamma_{ik} - U^{l}_{i} \gamma_{jk} + U_{ik} \gamma^{l}_{j} - U_{jk} \gamma^{l}_{i} \tag{25}$$

where $\gamma_{ik} = \frac{1+\alpha}{2}(a_{ik} - b_{ik})$. Using the assumption in the Theorem 2.5, that is, there holds

$$U_{ij} = \frac{k}{n} g_{ij} \tag{26}$$

where k is a scalar curvature of the the Levi-Civita connection. Using this expression, from the expressions (23) and (24), we have

$$\overset{\alpha^*}{R}_{ijk}{}^{l} + \overset{\alpha}{R}_{ijk}{}^{l} = 2K_{ijk}{}^{l} + \frac{k}{n}\left(\delta^{l}_{j} \gamma_{ik} - \delta^{l}_{i} \gamma_{jk} + g_{ik} \gamma^{l}_{j} - g_{jk} \gamma^{l}_{i}\right) \tag{27}$$

Contracting the indices i and l *w.r.t.* (27), we get

$$\overset{\alpha^*}{R}_{jk} + \overset{\alpha}{R}_{jk} = 2K_{jk} - \frac{k}{n}(n-2)\gamma_{jk} - \frac{k}{n} g_{jk}\gamma^l_l \tag{28}$$

Multiplying both sides of (28) by g_{jk} and contracting the indices j and k, then we arrive at

$$\overset{\alpha^*}{R} + \overset{\alpha}{R} = 2k - 2(n-2)\gamma^l_l.$$

From this expression above we have

$$\gamma^l_l = \frac{1}{2(n-1)k}\Big[2k - (\overset{\alpha^*}{R} + \overset{\alpha}{R})\Big]$$

Using the expression from (28) we obtain

$$\gamma_{jk} = \frac{n}{(n-2)k}\Big\{2K_{jk} - (\overset{\alpha^*}{R}_{jk} + \overset{\alpha}{R}_{jk}) - \frac{g_{jk}}{2(n-1)}[2K - (\overset{\alpha^*}{R} + \overset{\alpha}{R})]\Big\}$$

Substituting this expression into (25) and putting

$$\begin{aligned}
\overset{\alpha}{C}{}_{ijk}{}^{l} &= \overset{\alpha}{R}{}_{ijk}{}^{l} - \frac{1}{n-2}\Big(\delta^l_i \overset{\alpha}{R}_{jk} - \delta^l_j \overset{\alpha}{R}_{ik} + g_{jk}\overset{\alpha}{R}{}_i{}^l - g_{ik}\overset{\alpha}{R}{}_j{}^l\Big) - \frac{\overset{\alpha}{R}}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik}) \\
\overset{\alpha^*}{C}{}_{ijk}{}^{l} &= \overset{\alpha^*}{R}{}_{ijk}{}^{l} - \frac{1}{n-2}\Big(\delta^l_i \overset{\alpha^*}{R}_{jk} - \delta^l_j \overset{\alpha^*}{R}_{ik} + g_{jk}\overset{\alpha^*}{R}{}_i{}^l - g_{ik}\overset{\alpha^*}{R}{}_j{}^l\Big) - \frac{\overset{\alpha^*}{R}}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik}) \\
\overset{o}{C}{}_{ijk}{}^{l} &= K_{ijk}{}^{l} - \frac{1}{n-2}\Big(\delta^l_i K_{jk} - \delta^l_j K_{ik} + g_{jk}K_i{}^l - g_{ik}K_j{}^l\Big) - \frac{k}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik})
\end{aligned}$$

then by a direct computation, we obtain

$$\overset{\alpha}{C}{}_{ijk}{}^{l} + \overset{\alpha^*}{C}{}_{ijk}{}^{l} = 2\overset{o}{C}{}_{ijk}{}^{l}. \tag{29}$$

By using the constant curvature assumption of the connection $\overset{\alpha}{\nabla}$, we have $\overset{\alpha}{C}{}_{ijk}{}^{l} = \overset{\alpha^*}{C}{}_{ijk}{}^{l} = 0$. Hence we arrive at

$$\overset{o}{C}{}_{ijk}{}^{l} = 0.$$

This means consequently that the Einstein manifold is conformally flat. This confirms that Theorem 2.7 is tenable. □

3. Geometries Of Manifolds Associated With A Projective Ricci Quarter-symmetric Connection

Definition 3.1. *A connection $\overset{p}{\nabla}$ is called a projective Ricci quarter-symmetric connection family, if $\overset{p}{\nabla}$ is projective equivalent to a Ricci quarter-symmetric connection ∇.*

In a Riemannian manifold, a projective Ricci quarter-symmetric connection $\overset{p}{\nabla}$ satisfies the relation

$$\begin{cases}
\overset{p}{\nabla}_Z g(X,Y) &= -2\psi(Z)g(X,Y) - \psi(X)g(Y,Z) - \psi(Y)g(X,Z) + (1-\alpha)\pi(Z)U(X,Y) \\
&+ \ (\beta - \frac{1+\alpha}{2})\pi_(X)U(Y,Z) + (\beta - \frac{1+\alpha}{2})\pi(Y)U(X,Z) \\
\overset{p}{T}(X,Y) &= \pi(Y)UX - \pi(X)UY
\end{cases} \tag{30}$$

And the coefficient of $\overset{p}{\nabla}$ is

$$\overset{p}{\Gamma}{}^k_{ij} = \left\{ {k \atop ij} \right\} + \psi_i \delta^k_j + \psi_j \delta^k_i + \frac{1+\alpha}{2} \pi_j U^k_i - \frac{1-\alpha}{2} \pi_i U^k_j - \beta U_{ij} \pi^k \tag{31}$$

The local expression of the relation (30) is

$$\begin{cases} \overset{p}{\nabla}_k g_{ij} &= -2\psi_k g_{ij} - \psi_i g_{jk} - \psi_j g_{ik} + (1-\alpha)\pi_k U_{ij} \\ &+ (\beta - \frac{1+\alpha}{2})\pi_i U_{jk} + (\beta - \frac{1+\alpha}{2})\pi_j U_{ik} \\ \overset{p}{T}{}^k_{ij} &= \pi_j U^k_i - \pi_i U^k_j \end{cases} \tag{32}$$

From expression (31), the curvature tensor of $\overset{p}{\nabla}$, by a direct computation, is

$$\begin{aligned} \overset{p}{R}_{ijk}{}^l &= K_{ijk}{}^l - \frac{1+\alpha}{2} U^l_i \overset{p}{b}_{jk} - \beta \overset{p}{c}{}^l_i U_{jk} + \frac{1+\alpha}{2} \pi_i \overset{p}{d}{}^l_{jk} + \frac{1+\alpha}{2} e^l_{ij} \pi_k - \beta e_{ijk} \pi^l + \delta^l_k \psi_{ij} - U^l_k \pi_{ij} \\ &+ \delta^l_j \overset{p}{a}_{jk} + \frac{1+\alpha}{2} U^l_j \overset{p}{b}_{ik} + \beta \overset{p}{c}{}^l_j U_{ik} - \frac{1+\alpha}{2} \pi_j \overset{p}{d}{}^l_{ik} - \frac{1+\alpha}{2} e^l_{ji} \pi_k \\ &+ \beta e_{jik} \pi^l - \delta^l_k \psi_{ji} + U^l_k \pi_{ji} \end{aligned} \tag{33}$$

where $K_{ijk}{}^l$ is the curvature tensor of the Levi-Civita connection $\overset{o}{\nabla}$, and the other notations are given as follows

$$\begin{cases} \overset{p}{a}_{ik} &= \overset{o}{\nabla}_i \psi_k - \psi_i \psi_k - \frac{1+\alpha}{2} U^p_i \pi_p \pi_k + \frac{1-\alpha}{2} U^p_k \pi_p \pi_i + \beta U_{ik} \psi_p \pi^p \\ \overset{p}{b}_{ik} &= \overset{o}{\nabla}_i \pi_k - \frac{1+\alpha}{2} U^p_i \pi_p \pi_k + \frac{1+\alpha}{2} U^p_k \pi_p \pi_i - \pi_i \psi_k - \frac{1-\alpha}{1+\alpha} \pi_i \pi_k \\ \overset{p}{c}_{ik} &= \overset{o}{\nabla}_i \pi_k + \frac{1+\alpha}{2} U_{ik} \pi_p \pi^p - \frac{1-\alpha}{2} U_{kp} \pi^p \pi_i - \beta U_{ip} \pi^p \pi_k \\ \overset{p}{d}{}^l_{ik} &= \overset{o}{\nabla}_i U^l_k - \frac{1+\alpha}{2} U^l_p U^p_i \pi_k - \beta U_{ip} U^p_k \pi^l \\ \overset{p}{e}_{ijk} &= \overset{o}{\nabla}_i U_{jk} \\ \psi_{ik} &= \overset{o}{\nabla}_i \psi_k \end{cases}$$

For convenience, we set

$$B_{ijk}{}^l = \delta^l_i \overset{p}{a}_{jk} + \frac{1+\alpha}{2} U^l_j \overset{p}{b}_{jk} + \beta \overset{p}{c}{}^l_i U_{jk} - \frac{1-\alpha}{2} \pi_i \overset{p}{d}{}^l_{jk} - \frac{1+\alpha}{2} \pi_k \overset{p}{e}{}^l_{ij} + \beta \overset{p}{e}_{ijk} \pi^l - \delta^l_k \psi_{ij} + U^l_k \pi_{ij}.$$

Then we get

$$R_{ijk}{}^l = K_{ijk}{}^l - B_{ijk}{}^l + B_{jik}{}^l,$$

So there exists the following

Theorem 3.2. *When $B_{ijk}{}^l = B_{jik}{}^l$, then the curvature tensor will keep unchanged under the connection transformation $\overset{o}{\nabla} \to \overset{p}{\nabla}$.*

From the expression (31), the coefficient of dual connection $\overset{p*}{\nabla}$ of the projective Ricci quarter-symmetric connection $\overset{p}{\nabla}$ is

$$\overset{p*}{\Gamma}{}^k_{ij} = \left\{ {k \atop ij} \right\} - \psi_i \delta^k_j - g_{ij} \psi^k + \frac{1-\alpha}{2} \pi_i U^k_j - \frac{1+\alpha}{2} \pi^k U_{ij} + \beta U^k_i \pi_j \tag{34}$$

By using the expression (34), the curvature tensor of dual connection $\overset{p*}{\nabla}$ is

$$\begin{aligned}\overset{p*}{R}_{ijk}{}^{l} &= K_{ijk}{}^{l} - \overset{p}{a}{}^{l}_{i}g_{jk} - \frac{1+\alpha}{2}U_{jk}\overset{p}{b}{}^{l}_{i} - \beta\overset{p}{c}_{jk}U^{l}_{i} - \frac{1-\alpha}{2}\pi_i\overset{p}{d}{}^{l}_{jk} - \frac{1+\alpha}{2}\overset{p}{e}_{ijk}\pi^{l} + \beta\overset{p}{e}{}^{l}_{ij}\pi_k \\ &- \delta^{l}_{k}\psi_{ij} + U^{l}_{k}\pi_{ij} + g_{ik}\overset{p}{a}{}^{l}_{j} + \frac{1-\alpha}{2}U_{ik}\overset{p}{b}{}^{l}_{j} + \beta\overset{p}{c}_{ik}U^{l}_{j} + \frac{1-\alpha}{2}\pi_j\overset{p}{d}{}^{l}_{ik} + \frac{1+\alpha}{2}\overset{p}{e}_{jik}\pi^{l} \\ &- \beta\overset{p}{e}{}^{l}_{ji}\pi_k + \delta^{l}_{k}\psi_{ji} - U^{l}_{k}\pi_{ji}\end{aligned} \tag{35}$$

Theorem 3.3. *Suppose that $(\mathcal{M}, g)$ ($dim\mathcal{M} \geq 3$) is a connected Riemannian manifold associated with an isotropic projective Ricci quarter-symmetric connection $\overset{p}{\nabla}$. If there holds*

$$\psi_h = \frac{1}{n-1}\Big(\frac{1+\alpha}{2} + \beta\Big)T^{l}_{lh}, \tag{36}$$

then $(\mathcal{M}, g, \overset{p}{\nabla})$ is a constant curvature manifold(It is the so-called Schur's theorem of the projective Ricci quarter-symmetric connection).

Proof. Substituting the expression (15) into the second Bianchi identity of the curvature tensor of the projective Ricci quarter-symmetric connection, that is, there holds

$$\overset{p}{\nabla}_h R_{ijk}{}^{l} + \overset{p}{\nabla}_i R_{jhk}{}^{l} + \overset{p}{\nabla}_j R_{hik}{}^{l} = \overset{p}{T}{}^{m}_{hi}\overset{p}{R}_{jmk}{}^{l} + \overset{p}{T}{}^{m}_{ij}\overset{p}{R}_{hmk}{}^{l} + \overset{p}{T}{}^{m}_{jh}\overset{p}{R}_{imk}{}^{l}$$

then we get

$$\begin{aligned}&\overset{p}{\nabla}_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \overset{p}{\nabla}_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \overset{p}{\nabla}_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk}) \\ &+ k(p)\Big(\delta^l_i\overset{p}{\nabla}_h g_{jk} - \delta^l_j\overset{p}{\nabla}_h g_{ik} + \delta^l_j\overset{p}{\nabla}_i g_{hk} - \delta^l_h\overset{p}{\nabla}_i g_{jk} + \delta^l_h\overset{p}{\nabla}_j g_{ik} - \delta^l_i\overset{p}{\nabla}_j g_{hk}\Big) \\ &= k(p)\Big[(\pi_i U^m_h - \pi_h U^m_i)(\delta^l_j g_{mk} - \delta^l_m g_{jk}) + (\pi_j U^m_i - \pi_i U^m_j)(\delta^l_h g_{mk} - \delta^l_m g_{hk}) \\ &\quad + (\pi_h U^m_j - \pi_j U^m_h)(\delta^l_i g_{mk} - \delta^l_m g_{ik})\Big]\end{aligned}$$

Using the expression (32), then we obtain

$$\begin{aligned}&\overset{p}{\nabla}_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \overset{p}{\nabla}_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \overset{p}{\nabla}_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk}) \\ &- k(p)\Big\{\psi_h(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \psi_i(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \psi_j(\delta^l_h g_{ik} - \delta^l_i g_{hk}) \\ &+ (\frac{3-\alpha}{2} - \beta)k(p)\Big[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})\Big]\Big\} \\ &= k(p)\Big(\delta^l_j\pi_i U_{hk} - \delta^l_i\pi_h U_{jk} - \pi_i U^l_h g_{jk} + \pi_h U^l_i g_{jk} + \delta^l_h\pi_j U_{ik} - \delta^l_h\pi_i U_{jk} - \pi_j U^l_i g_{hk} + \pi_i U^l_j g_{hk} \\ &+ \delta^l_i\pi_h U_{jk} - \delta^l_i\pi_j U_{hk} - \pi_h U^l_j g_{ik} + \pi_j U^l_h g_{ik}\Big)\end{aligned}$$

Contracting the indices i and l, then we have

$$\begin{aligned}&(n-2)\Big[(g_{jk}\overset{p}{\nabla}_h k(p) - g_{hk}\overset{p}{\nabla}_j k(p)) - k(p)\Big(\psi_h g_{jk} - \psi_j g_{hk} - (\frac{3-\alpha}{2} - \beta)(\pi_h U_{jk} - \pi_j U_{hk})\Big)\Big] \\ &= k(p)\Big[(n-3)(\pi_h U_{jk} - \pi_j U_{hk}) + (U^i_i\pi_h - U^i_h\pi_i)g_{jk} - (U^i_i\pi_j - U^i_j\pi_i)g_{hk}\Big]\end{aligned}$$

Multiplying both sides of this expression by g^{jk}, then we arrive at

$$(n-1)\overset{p}{\nabla}_h k(p) - k(p)\Big[(n-1)\psi_h - (\frac{3-\alpha}{2} - \beta)(\pi_h U^i_i - U^i_h\pi_i)\Big] = 2k(p)(\pi_h U^i_i - U^i_h\pi_i)$$

From this expression, we have

$$\overset{p}{\nabla}_h k(p) - k(p)\Big[\psi_h - \frac{1}{n-1}(\frac{1+\alpha}{2}+\beta)T^i_{hi}\Big] = 0.$$

Consequently, we know that $k(p) = const(p \in \mathcal{M})$ if and only if

$$\psi_h = \frac{1}{n-1}(\frac{1+\alpha}{2}+\beta)T^i_{hi}.$$

This ends the proof of Theorem 3.3. □

Theorem 3.4. *An Einstein manifold* $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ *associated with a projective Ricci quarter-symmetric connection* $\overset{p}{\nabla}$ *possessing a constant curvature is of a conformal flat manifold.*

Proof. By using Einstein manifold assumption in Theorem 32, we have

$$U_{ik} = \frac{k}{n}g_{ik}$$

From this fact, we obtain $e_{ijk} = 0$. Adding the expressions (33) and (35), we obtain

$$\overset{p}{R}_{ijk}{}^l + \overset{p*}{R}_{ijk}{}^l = 2K_{ijk}{}^l + \delta^l_j\rho_{ik} - \delta^l_i\rho_{jk} + g_{ik}\rho^l_j - g_{jk}\rho^l_i \tag{37}$$

where $\rho_{ik} = a_{ik} + \frac{k}{n}(\frac{1+\alpha}{2}b_{ik} + \beta c_{ik})$. Contracting the indices i and l of (37), we get

$$\overset{p}{R}_{jk} + \overset{p*}{R}_{jk} = 2K_{jk} - (n-2)\rho_{jk} - g_{jk}\rho^i_i. \tag{38}$$

Multiplying both sides of (38) by g^{jk}, then we arrive at

$$\rho^i_i = \frac{1}{2(n-1)}[2k - (\overset{p}{R} + \overset{p*}{R})].$$

and

$$\rho_{jk} = \frac{1}{n-2}\Big\{2K_{jk} - (R_{jk} + \overset{p*}{R}_{jk}) - \frac{g_{jk}}{2(n-1)}[2k - (\overset{p}{R} + \overset{p*}{R})]\Big\}$$

Substituting ρ^i_i and ρ_{jk} into (37) and putting

$$\begin{aligned}
\overset{p}{C}_{ijk}{}^l &= \overset{p}{R}_{ijk}{}^l - \frac{1}{n-2}\Big(\delta^l_i\overset{p}{R}_{jk} - \delta^l_j\overset{p}{R}_{ik} + g_{jk}\overset{p}{R}_i{}^l - g_{ik}\overset{p}{R}_j{}^l\Big) - \frac{\overset{p}{R}}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik})\\
\overset{p*}{C}_{ijk}{}^l &= \overset{p*}{R}_{ijk}{}^l - \frac{1}{n-2}\Big(\delta^l_i\overset{p*}{R}_{jk} - \delta^l_j\overset{p*}{R}_{ik} + g_{jk}\overset{p*}{R}_i{}^l - g_{ik}\overset{p*}{R}_j{}^l\Big) - \frac{\overset{p*}{R}}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik})\\
\overset{o}{C}_{ijk}{}^l &= K_{ijk}{}^l - \frac{1}{n-2}\Big(\delta^l_i K_{jk} - \delta^l_j K_{ik} + g_{jk}K_i{}^l - g_{ik}K_j{}^l\Big) - \frac{k}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik})
\end{aligned}$$

then by a direct computation, we obtain

$$\overset{p}{C}_{ijk}{}^l + \overset{p*}{C}_{ijk}{}^l = 2\overset{o}{C}_{ijk}{}^l. \tag{39}$$

By using the constant curvature assumption in Theorem 32, we have

$$\overset{p}{C}_{ijk}{}^l = \overset{p*}{C}_{ijk}{}^l = 0.$$

This implies that

$$\overset{o}{C}_{ijk}{}^l = 0. \tag{40}$$

This means consequently that the Einstein manifold is of conformal flat. □

4. Examples

Example 4.1. *(ACMM: almost contact metric manifold)*

Assume that $(\mathcal{M}, \varphi, \xi, \eta)$ is a $(2n+1)$-dimensional almost contact manifold endowed with an almost contact strucuture (φ, ξ, η), where φ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1.$$

If an almost contact manifold equipped with a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

for all $X, Y \in \mathcal{X}(\mathcal{M})$, then (φ, ξ, η, g) is called an almost contact metric structure and $(\mathcal{M}, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold. We now define a linear connection on such manifold

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X,$$

where ∇ is the levi-Civita connection with respect to the Riemannian metric g, then we obtain

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \eta(X) = g(X, \xi).$$

which shows that $\tilde{\nabla}$ is a semi-symmetric non-metric connection. This connection is exactly a special case from Remark 21

5. Acknowledgements

This research was supported by NNSF of China(No.11871275; No.11371194) and by a Grant-in-Aid for Science Research from Nanjing University of Science and Technology(No.AE91574) and by the Research Found for the Doctoral Program of Anhui Normal University(No.751841).

References

[1] N. S. Agache and M. R. Chafle, A semi-symmetric non-metric connection on a Riemannian manifold, *Indian J. Pure Appl. Math.,* 1992, **23**, 399-409
[2] U. C. De, Y. L. Han and P. B. Zhao, A special type of semi- symmetric non-metric connection on a Riemannian manifold, *Facta Universitatis, Series, Math and Informatics,* 2016, **31**(2): 529-541
[3] De. U. C and Kamilya. D, Hypersurfaces of Kenmotsu manifolds Endowed with a quariter-symmetric non-metric connection, *Kuwait Journal of Sci. Eng,* 2012, **39**(1), A, 43-56
[4] K. A. Dunn, A Geometrical model for scalar-tensor of gravitation, *Tensor. N. S.,* 1975, **29**, 214-216
[5] Dwivedi. K. A, A Quarter symmetric non-metric connection in a Kenmotsu manifold. *International Electronic Journal of Geometry,* 2011, **4**(1): 115-124
[6] Fridman A and Schouten J. A., Über die Geometric der halb, symmerischen Übertragungen, *Math. Zeitschrit,* 1924, **21**, 211-233
[7] Fu F. Y., X. P. Yang and P. B. Zhao, Geometrical and physical characteristics of a class of conformal mappings, *Journal of Geometry and Physics,* 2012, **62**(6): 1467-1479
[8] S. Golab, On semi-symmetric and quarter-symmetric connections, *Tensor. N. S.* 1975, **29**, 249-254
[9] Y. L. Han, Ho T. Y. and F. Y. Fu, A classification of conformal-Weyl manifolds in view of non-metric connection, *Facta Universtitatis(Nis) Ser. Math. Inform,* 2016, **31**(2): 513-527
[10] Y. L. Han, Ho T. Y. and P. B. Zhao, Some invariant of quarter-symmetric metric connections under the projective transformation, *Filomat*, 2013, **27**(4): 679-691
[11] Y. L. Han, Ho T. Y. and P. B. Zhao, A Schur's lemmama based on a semi-symmetric non-metric connection, *International Journal of Geometry,* 2016, **5**(1):47-53
[12] Y. L. Han, F. Y. Fu and P. B. Zhao, A class of non-holonomic projective connections on sub-Riemannian manifolds, *Filomat,* 2017, **31**(7): 1295-1303
[13] Y. L. Han, F. Y. Fu and P. B. Zhao, On semi-symmetric metric connection in sub-Riemannian manifolds, *Tamkang Journal of Mathematics,* 2016, **47**(4): 373-384
[14] Hayden. H. A, Subspaces of a space with torsion, *Proc. London Math. Soc,* 1932, **34**, 27-50
[15] Ho Tal Yun. On a semi-symmetric non-metric connection satisfying the Schur's theorem on a Riemannian manifold, arXiv 1212.4748 v1

[16] Ho Tal Yun and Jen Cholyong and Jun Gungzhi, A projective Ricci quarter-symmetric conformal connection satisfying the Schur's theorem on a Riemannian manifold. *J. of Yanbin University (Natural Science)*, 2013, **40**(4): 290-294

[17] Ho Tal Yun, On the Projective semi-symmetric connection and conformal Ricci quarter-symmetric connection on the Riemannian manifold. *J. of Kim Il Sung University (Natural Science)*, 2013, **2**, 3 - 10

[18] Ho Tal Yun, An Jae Hyon, An Chang Gil, Some properties of mutual connection of Ricci quarter-symmetric metric connection and its Dual connection in a Riemannian manifold, *Acta Sceintiarum Naturalium Universitatis Nankaiensis*, 2013, **46**(4): 1-8

[19] J. Li, G. Q. He, P. B. Zhao, On submanifolds in a Riemannian manifold with a semi-symmetric non-metric connection, *Symmetry*, 2017, **9**, 112

[20] R. S. Mishra and S. N. Pandey, On Quarter-metric F-connections, *Tensor.*, 1980, **34**, 1-7

[21] S. Mallick, A. Yildiz and U. C. De, *Characterizations of mixed Einstein manifolds*, *International Journal of Geometric Methods in Modern Physics*, 2017, **14**(6), 1750096(14 pages)

[22] Y. J. Suh, C. A. Mantica, U. C. De and P. Pal, Pseudo B-symmetric manifolds, *International Journal of Geometric Methods in Modern Physics*, 2017, **14**(9), 1750119(14 pages)

[23] W. X. Tang, T. Y. Ho, F. Y. Fu, P. B. Zhao, On a quarter-symmetric projective conformal connection, *International Electronic Journal of Geometry*, 2017, **10**(2): 37-45

[24] W. X. Tang, T. Y. Ho, K. I. Ri, F. Y. Fu, P. B. Zhao, On a generalized quarter symmetric metric recurrent connection, *Filomat*, 2018, **32**(1): 207-215

[25] M. M. Tripathi, A New connection in a Riemannian manifold, *International Electronic Journal of Geometry*, 2008, **1**(1): 15-24

[26] K. Yano, On semi- symmetric metric connection, *Rev. Roum. Math. Pure Apple.*, 1970, **15**, 1579-1586

[27] K. Yano, Imai. J, Quarter-symmetric connection and their curvature tensors, *Tensor. N. S.* 1982, **38**, 13-18

[28] P. B. Zhao, Some Properties of projective semi-symmetric connection, *International Mathematical Forum*, 2008, **3**(7): 341-347

[29] P. B. Zhao, The invariant of projective transformation of semi-symmetric metric recurrent connections and curvature tensor expressions. *J. of Engineering Math.*, 2001, **17**(2): 105-108

[30] P. B. Zhao and H. Z. Song, An invariant of the projective semi-symmetric connection, *Chin. Quart. J. of Math.*, 2001, **16**(4): 49-54

[31] P. B. Zhao and H. Z. Song and X. P. Yang, Some invariant properties of Ricci semi-symmetric metric recurrent connections and curvature tensor expressions. *Chin. Quart. J. of Math.*, 2004, **19**(4): 355-361