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On the Riesz Summability of the Eigenfunction Expansions on a Closed Domain

Abdumalik Rakhimov[a], Erkin Saydamatov[b], Shuhrat Sheraliev[b]

[a]*Department of Science in Engineering, IIUM*
[b]*Lomonosov named MSU Tashkent branch*

Abstract. In the present paper we study the Riesz summability on a closed domain expansions in eigenfunctions of the first boundary value problem for the Schrodinger operator with a weak singular potential.

1. Introduction and preliminaries

Let Ω be a finite two dimensional domain with a smooth boundary $\partial\Omega$. By $\overline{\Omega}$ denote the closer of the domain Ω. Let $u_n(x)$ be an eigenfunction of the first boundary value problem corresponding to the eigenvalue λ_n :

$$\triangle u(x) + (\lambda + q(x))u(x) = 0,\ x \in \Omega \tag{1}$$

$$u\mid_{\partial\Omega} = 0, \quad u \in C^{(1)}(\overline{\Omega}) \cap W_2^2(\Omega), \tag{2}$$

where a potential function $q(x)$ is from the Sobolev space $W_2^1(\Omega)$ and may have singularities at one point of the domain Ω (or in finite numbers of the points) and enough smooth out of this point.

A system of eigenfunctions $\{u_n(x)\}$ is a complete orthogonal in $L_2(\Omega)$ corresponding to the sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < .... < \lambda_n \to \infty$. For the $\mu_o \to \infty$ the following estimation for the eigenfunctions $u_n(x)$ is valid [13]

$$\sum_{|\sqrt{\lambda_n}-\mu_0| \leq 1} u_n^2(x) = O(\mu_o \ln^2 \mu_0) \tag{3}$$

uniformly on the closed domain $\overline{\Omega}$.

Let $f \in L_2(\Omega)$. Recall the Riesz means of order s of the partial sums of the Fourier series

$$E_\lambda^s f(x) = \sum_{\lambda_n<\lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s f_n u_n(x), \tag{4}$$

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Email addresses: abdumalik@iium.edu.my (Abdumalik Rakhimov), saydamatov@mail.ru (Erkin Saydamatov)

here $\lambda > 0$, $f_n = (f, u_n)$ are the Fourier coefficients of the function f with respect to the system $\{u_n(x)\}$. Note that if $s = 0$, then (4) is just the partial sum of the Fourier series of the function f.

Further in this paper we consider convergence of $E^s_\lambda f(x)$ in the domains that are separated from the singular point(s) of $q(x)$. Also we will suppose that the support of the function f is not containing the singular point(s).

When $q(x) = 0$, then precise conditions of uniform convergence of the Riesz means (4) of the functions from the Sobolev spaces $W^\alpha_p(\Omega)$ on compact subsets of the domain Ω as follows [7]

$$\alpha + s \geq \frac{1}{2}, \ \ \alpha p > 2, \ \ s \geq 0, \ \ p \geq 1. \tag{5}$$

The conditions (5)are sufficient for the functions from the Nikol'skii spaces $H^\alpha_p(\Omega)$ [4]. If the second condition in (5) is replaced by $\alpha p = 2$, then it is necessary additional to assume that the function f is continuous (see [3]) and the inequalities (5) will be as follows:

$$\alpha + s > \frac{1}{2}, \ \ \alpha p = 2, \ \ s \geq 0, \ \ p \geq 1. \tag{6}$$

In this case the first condition $\alpha + s > \frac{1}{2}$ in (6) is also precise meaning that if $x_0 \in \Omega$ and

$$\alpha + s = \frac{1}{2}, \ \ \alpha p = 2, \ \ s \geq 0, \ \ p \geq 1, \tag{7}$$

then there exists a continuous in Ω function $f \in \mathring{W}^\alpha_p(\Omega)$ and such that [3]

$$\overline{\lim_{\lambda\to\infty}} E^s_\lambda f(x_0) = +\infty. \tag{8}$$

2. Main theorem

Let f is a function with compact support from the space $W^{\frac{1}{2}}_p(\Omega)$, $p > 4$, such that the series

$$\sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} (\ln \lambda_n)^{2+\varepsilon} f_n^2$$

converges. Then its expansion via eigenfunctions associated with the elliptic operators of second order for the first boundary value problem converges uniformly on the closed domain $\overline{\Omega}$ [8].

For the continuous functions $f(x)$ from the Nikolskii space $\mathring{H}^\alpha_p(\Omega)$ with compact support in the domain Ω (where $\mathring{H}^\alpha_p(\Omega)$ is the closure of the space $C^\infty_0(\Omega)$ with the respect to the norm of the space $H^s_p(\Omega)$) the conditions for the uniformly on the closed domain $\overline{\Omega}$ convergence of the Riesz means (4) are [5]

$$\alpha + s > \frac{1}{2}, \ \ \alpha p \geq 2, \ \ s \geq 0, \ \ p \geq 1. \tag{9}$$

Note that if $\alpha p = 2$ the condition $\alpha + s > (1)/2$ is precise. In the case $\alpha p > 2$ a problem of the sharpness of the inequality $\alpha + s > (1)/2$ is still open.

Recall $\|f\|_{L_{pq}(\Omega)}$ the norm in the Lebesgue spaces with mixed norm [20] $L_{pq}(\Omega)$ and define H^α_{pq} the Banach space of all measurable functions f with respect to the norm

$$\|f\|_{H^\alpha_{pq}(\Omega)} = \|f\|_{L_{pq}(\Omega)} + \sum_{|k|=\ell} \sup_z |z|^{-\kappa} \|\Delta^2_z \partial^k f(y)\|_{L_{pq}(\Omega_{|z|})},$$

where $h > 0$ such that the set $\Omega_h = \{x \in \Omega : dist(x, \partial\Omega) > h\}$ is proper subset of the domain Ω, $\alpha = \ell + \kappa$, ℓ is a non negative integer, $0 < \kappa \le 1$, $p, q \ge 1$, $k = (k_1, k_2)$ multi-index, $|k| = k_1 + k_2$, and $\partial^k f$ denotes the weak derivative

$$\partial^k f(y) = \frac{\partial^{|k|} f(y)}{\partial y_1^{k_1} \partial y_2^{k_2}},$$

and the symbol $\Delta_z^2 \partial^k f(y)$ denotes the second difference of the function $\partial^k f(y)$:

$$\Delta_z^2 \partial^k f(y) = \partial^k f(y+z) - 2\partial^k f(y) + \partial^k f(y).$$

By $\mathring{H}^\alpha_{pq}(\Omega)$ denote the closure of the space $C_0^\infty(\Omega)$ with respect to the norm of the space $H^\alpha_{pq}(\Omega)$.

In [13] it is established te following sufficient conditions for the uniform convergence on the closed domain of the eigenfunction expansions associated with the first boundary value problems of the functions from the space $H^\alpha_{pq}(\Omega)$

$$\alpha > \frac{1}{2} - s, \quad \alpha = \frac{1}{q} + \frac{1}{p}, \quad 2 \le p < q. \tag{10}$$

The main result of the present paper is to prove the following

Theorem 2.1. *Let the conditions (10) are valid and let f be a continuous function with compact support in the domain Ω belonging to the space $\mathring{H}^\alpha_{pq}(\Omega)$. Then uniformly on $\overline{\Omega}$*

$$\lim_{\lambda \longrightarrow \infty} E^s_\lambda f(x) = f(x).$$

As can be seen in this paper we study convergence in the spaces of smooth functions. One can study this problem in the spaces of singular functions. Some results for the spectral expansions of the singular distributions can be found in [9], [10], [11], [15], [16] and [12]. These questions for the expansions associated with the Laplace Beltrami operator on the sphere studied in [1], [6], [17].

3. Proof of the theorem

First we obtain estimation of the Riesz means of the spectral expansions $E^s_\lambda f(x)$. Let's choose a small positive number h such that a set Ω_h is a non-empty proper subset of Ω, and distance $r = |x - y|$ between $x \in \Omega_h$ and $y \in \partial\Omega$ less than h. Let $y \in \overline{\Omega}$ and consider the following function of the distance r :

$$V(r,\lambda) = \begin{cases} \frac{\Gamma(s+1)2^{s-1}\lambda^{\frac{1-s}{2}} J_{1+s}(r\sqrt{\lambda})}{\pi r^{1+s}}, & r \le R \\ 0, & r > R \end{cases}, \tag{11}$$

where $0 < R < h/4$ and $J_\nu(t)$ the Bessel function of the first kind of order ν.

Let

$$S_\tau(\phi) = \int_0^{2\pi} \phi(x_1 + \tau cos\theta, x_2 + \tau sin\theta) d\theta.$$

We use the following mean value formula for the eigenfunctions $u_n(x)$ [7]:

$$S_r(u_n) = 2\pi J_0(r\sqrt{\lambda_n}) u_n(x) + \frac{\pi}{2} \int_0^r \{J_0(t\sqrt{\lambda_n}) Y_0(r\sqrt{\lambda_n}) - Y_0(t\sqrt{\lambda_n}) J_0(r\sqrt{\lambda_n})\} S_t(qu_n) t dt$$

where $x = (x_1, x_2) \in \Omega_h$.

Using this mean value formula we get the following representation for the Fourier coefficients of the function $V(r,\lambda)$

$$v_n^\lambda(x) = 2^s \Gamma(s+1) \lambda^{\frac{1-s}{2}} u_n(x) \int_0^R J_{1+s}(\sqrt{\lambda} r) J_0(\sqrt{\lambda_n} r) r^{-s} dr +$$

$$+\, 2^{s-2}\Gamma(s+1)\lambda^{\frac{1-s}{2}} \int_0^R r^{-s} J_{1+s}(\sqrt{\lambda}r)\times$$

$$\times \int_0^r \{J_0(t\sqrt{\lambda_n})Y_0(r\sqrt{\lambda_n}) - Y_0(t\sqrt{\lambda_n})J_0(r\sqrt{\lambda_n})\}S_t(qu_n)tdtdr \tag{12}$$

First integral on the right-hand side of (12) we will split into two as

$$\int_0^R = \int_0^\infty - \int_R^\infty,$$

then using the following well-known formula for the Bessel functions [19] :

$$\int_0^\infty J_{a+s}(\sqrt{\lambda}r)J_{a-1}(\sqrt{\lambda_n}r)r^{-s}dr = \begin{cases} \frac{(1-\frac{\lambda_n}{\lambda})^s \lambda^s \lambda_n^{\frac{a-1}{2}}}{2^s\Gamma(s+1)\lambda^{\frac{a+s}{2}}}, & \lambda_n \le \lambda \\ 0, & \lambda_n > \lambda \end{cases}. \tag{13}$$

we will get

$$v_n^\lambda(x) = \delta_n^\lambda u_n(x)(1-\frac{\lambda_n}{\lambda})^s - 2^s\Gamma(s+1)\lambda_n^{\frac{-1}{4}}\lambda^{\frac{1}{4}-\frac{s}{2}}u_n(x)I_0(\lambda_n,\lambda)+$$

$$+\, 2^{s-2}\Gamma(s+1)\lambda^{\frac{1-s}{2}} \int_0^R r^{-s}J_{1+s}(\sqrt{\lambda}r)\int_0^r \{J_0(t\sqrt{\lambda_n})Y_0(r\sqrt{\lambda_n}) - Y_0(t\sqrt{\lambda_n})J_0(r\sqrt{\lambda_n})\}S_t(qu_n)tdtdr \tag{14}$$

where $I_0(\lambda_n, \lambda)$ is a function defined by the following integral

$$I_0(\lambda_n,\lambda) = (\lambda\lambda_n)^{\frac{1}{4}} \int_R^\infty J_{1+s}(\sqrt{\lambda}r)J_0(\sqrt{\lambda_n}r)r^{-s}dr, \tag{15}$$

and $\delta_n^\lambda = \begin{cases} 1, & \lambda_n < \lambda \\ 0, & \lambda_n \ge \lambda. \end{cases}$

Lemma 3.1. *For any function $f \in L_2(\Omega)$ the integral*

$$\int_\Omega f(x)V(x,y,\lambda)dx$$

is continuous in y on the closed domain $\overline{\Omega}$.

Proof. From estimate (3) it follows that for any positive number ε uniformly with respect to $y \in \overline{\Omega}$

$$\sum_{\lambda_n<\lambda} u_n^2(y)\lambda_n^{\varepsilon-1} = O(\lambda^\varepsilon \ln^2\lambda),$$

$$\sum_{\lambda_n>\lambda} u_n^2(y)\lambda_n^{-\varepsilon-1} = O(\lambda^{-\varepsilon}\ln^2\lambda). \tag{16}$$

This can be achieved if the sum in the left-hand side of the second formula in (16) represent as

$$\sum_{\lambda_n>\lambda} u_n^2(y)\lambda_n^{-\varepsilon-1} \le \sum_{k=0}^\infty \sum_{\sqrt{\lambda}+k<\sqrt{\lambda_n}<\sqrt{\lambda}+k+1} u_n^2(y)\lambda_n^{-\varepsilon-1} \le \sum_{k=0}^\infty (\sqrt{\lambda}+k)^{-2\varepsilon-2} \sum_{\sqrt{\lambda}+k<\sqrt{\lambda_n}<\sqrt{\lambda}+k+1} u_n^2(y).$$

Similarly from the estimate of integral (15) (see [19])

$$|I_0(\lambda,\lambda_n)| \le \frac{c}{1+|\sqrt{\lambda}-\sqrt{\lambda_n}|} \tag{17}$$

(where $c > 0$ is independent of λ and n) and estimate (16) with the application of Cauchy-Schwarz inequality it follows that series

$$\sum_{n=1}^{\infty} f_n u_n(y) \lambda_n^{\frac{-1}{4}} I_0(\lambda, \lambda_n)$$

converges uniformly on the closed domain $\overline{\Omega}$. □

We will multiply equality (14) by $u_n(y)$ then take summation with the respect n, get the following representation of the function (11) in the sense of L_2 space with the variable y

$$V(r,\lambda) = \Theta^s(x,y,\lambda) - 2^s\Gamma(s+1)\lambda^{\frac{1}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \lambda_n^{\frac{-1}{4}} I_0(\lambda,\lambda_n) u_n(x) u_n(y) + 2^{s-2}\Gamma(s+1)\lambda^{\frac{1-s}{2}} \int_0^R r^{-s} J_{1+s}(\sqrt{\lambda} r) \sum_{n=1}^{\infty} u_n(x) \times$$

$$\times \int_0^r \{J_0(t\sqrt{\lambda_n}) Y_0(r\sqrt{\lambda_n}) - Y_0(t\sqrt{\lambda_n}) J_0(r\sqrt{\lambda_n})\} S_t(qu_n) t dt dr \tag{18}$$

where

$$\Theta^s(x,y,\lambda) = \sum_{\lambda_n<\lambda} (1 - \frac{\lambda_n}{\lambda})^s u_n(x) u_n(y)$$

is known as the Riesz means of the spectral function [2].

Let $f \in L_2(\Omega)$ and $supp f \subset \Omega_h$. Then for $y \in \overline{\Omega}$ the Riesz means (4) of the partial sums of the Fourier series of the function f via eigenfunctions $u_n(x)$ can be written as follows by multiplication both side of (18) and then integration with the respect to x

$$E_\lambda^s f(y) = \int_{\Omega_h} f(x) V(r,\lambda) dx + +2^s\Gamma(s+1)\lambda^{\frac{1}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} f_n u_n(y) \lambda_n^{\frac{-1}{4}} I_0(\lambda,\lambda_n) +$$

$$+ 2^{s-2}\Gamma(s+1)\lambda^{\frac{1-s}{2}} \int_0^R r^{-s} J_{1+s}(\sqrt{\lambda} r) \sum_{n=1}^{\infty} f_n \times$$

$$\times \int_0^r \{J_0(t\sqrt{\lambda_n}) Y_0(r\sqrt{\lambda_n}) - Y_0(t\sqrt{\lambda_n}) J_0(r\sqrt{\lambda_n})\} S_t(qu_n) t dt dr \tag{19}$$

In order to estimate (19) we will study some properties of the eigenfunctions on the closed domain.

Note that from (3), (16) and (17) it follows that uniformly with the respect to $y \in \overline{\Omega}$ the following estimation is valid [14]

$$\sum_{n=1}^{\infty} u_n^2(y)\, \lambda_n^{\frac{-1}{2}}\, [I_0(\lambda,\lambda_n)]^2 \;\leq\; C\, ln^2\lambda, \tag{20}$$

where a constant $C > 0$ is independent of λ and y.

Further we always suppose that α satisfying (10).

Lemma 3.2. *Let $f \in \overset{\circ}{H}{}_2^{\alpha}(\Omega)$. Then uniformly with respect to $y \in \overline{\Omega}$*

$$\sum_{n=1}^{\infty} f_n u_n(y)\, \lambda_n^{-\frac{1}{4}}\, I_0(\lambda,\lambda_n) \;=\; O\Big(\frac{ln\lambda}{\lambda^{\frac{\alpha}{2}}}\Big) ||f||_{H_2^\alpha}. \tag{21}$$

Proof. Note, that for any $\varepsilon > 0$ [3]

$$\sum_{\lambda<\lambda_n<4\lambda} f_n^2\, \lambda_n^{\alpha} \;\leq\; c\, ||f||^2_{H_2^\alpha}, \tag{22}$$

where $c > 0$ is independent of f and λ.

Then using (17) and (20) obtain

$$\sum_{1<\lambda_n<\frac{\lambda}{4}} f_n u_n(y)\, \lambda_n^{-\frac{1}{4}}\, I_0(\lambda,\lambda_n) \le c\frac{ln\lambda}{\lambda^{\frac{\alpha}{2}}}\|f\|_{H_2^\alpha}, \tag{23}$$

where $c > 0$ is independent of f and λ.

If $\lambda_n > \frac{9\lambda}{4}$ we second estimate in (16)) and obtain the estimate for such n.

The proof for the numbers n for which $\frac{\lambda}{4} < \lambda_n < \frac{9\lambda}{4}$ is as follows. Let k be the least natural number satisfying $2^{k+1} \ge \sqrt{\lambda}$. Then using (3), (17) and (22) we get

$$\Big|\sum_{\frac{\lambda}{4}<\lambda_n<\frac{9\lambda}{4}} f_n u_n(y)\, \lambda_n^{-\frac{1}{4}}\, I_0(\lambda,\lambda_n)\Big| \le$$

$$\le \Big(\sum_{m=1}^{k}\ \sum_{2^{m-1}\le|\sqrt{\lambda_n}-\sqrt{\lambda}|\le 2^m} u_n^2(y)\, \lambda_n^{\frac{-1}{2}-\alpha}\, [I_0(\lambda,\lambda_n)]^2\Big)^{\frac{1}{2}}\Big(\sum_{\frac{\lambda}{4}<\lambda_n<\frac{9\lambda}{4}} f_n^2\, \lambda_n^{\alpha}\Big)^{\frac{1}{2}} \le$$

$$\le c\, \lambda^{\frac{-\alpha}{2}}\, ln\lambda\, \|f\|_{H_2^\alpha}.$$

where $c > 0$ is independent from f and λ. □

From this lemma we get the following estimate for the second term in the expression (19) of the Riesz means of the spectral expansions $E_\lambda^s f(y)$

$$2^s\Gamma(s+1)\lambda^{\frac{1}{4}-\frac{s}{2}}\sum_{n=1}^{\infty} f_n u_n(y)\lambda_n^{\frac{-1}{4}} I_0(\lambda,\lambda_n) \le c\, \lambda^{\frac{-\varepsilon}{2}}\, ln\lambda\, \|f\|_{H_2^\alpha} \tag{24}$$

where $\varepsilon = s+\alpha-\frac{1}{2}$ and from (10) it follows that this number is positive. Third term in (19) can be estimated similarly to (24) using appropriate estimates of the Bessel functions (see for example in [19]).

Thus, for the function $f \in \mathring{H}_2^\alpha(\Omega)$ with $supp f \subset \Omega_h$ uniformly with respect to $y \in \overline{\Omega}$ we get

$$E_\lambda^s f(y) = \int_{\Omega_h} f(x)V(r,\lambda)dx + O(\frac{ln\lambda}{\lambda^{\frac{-\varepsilon}{2}}})\,\|f\|_{H_2^\alpha} \tag{25}$$

Then choosing $R < \frac{h}{4}$ from (11) obtain

$$E_\lambda^s f(y) = c_s\, \lambda^{\frac{1-s}{2}} \int_0^R r^{-s}J_{1+s}(r\sqrt{\lambda})\, S_r(f)\, dr + O(\frac{ln\lambda}{\lambda^{\frac{-\varepsilon}{2}}})\,\|f\|_{H_2^\alpha}, \tag{26}$$

where $c_s = \frac{\Gamma(s+1)2^{s-1}}{\pi}$.

Let f be a continuous function with compact support in the domain Ω belonging to the space $\mathring{H}_{pq}^\alpha(\Omega)$ and (10) holds.

Note that because $2 \le p < q$ from the embedding $H_{p,q}^\alpha \to H_2^\alpha$ we write (26) as follows

$$E_\lambda^s f(y) = c_s\, \lambda^{\frac{1-s}{2}} \int_0^R r^{-s}J_{1+s}(r\sqrt{\lambda})\, S_r(f)\, dr + O(\frac{ln\lambda}{\lambda^{\frac{-\varepsilon}{2}}})\,\|f\|_{H_{pq}^\alpha}. \tag{27}$$

Let $\beta = \frac{1}{2} - s$. Then for the integral in (27) we have [18]

$$\int_0^R r^{-s}J_{1+s}(r\sqrt{\lambda})S_r(f)dr \le C\left(\|f\|_{H_{p,q}^\beta} + \|f\|_{L_\infty}\right)\lambda^{\frac{-\beta}{2}-\frac{1}{4}},$$

where $C > 0$ is independent of f and λ.

It is clear that $\beta < \alpha$ and that is why from $H^{\alpha}_{p,q} \to H^{\beta}_{p,q}$ we obtain the following estimate for the spectral expansions

$$E^s_\lambda f(y) \;\leq\; C \left(\|f\|_{H^{\alpha}_{p,q}} \;+\; \|f\|_{C(\overline{\Omega})} \right). \tag{28}$$

Note that the Fourier series via eigenfunction expansions of a function $f \in C_0^\infty(\Omega)$ converges uniformly and absolutely in the closed domain $\overline{\Omega}$. Due to the density of $C_0^\infty(\Omega)$ in the space $\mathring{H}^{\alpha}_{pq}(\Omega)$, the statement of Theorem 2.1 follows from the inequality (28).

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