



Bi- f -Harmonic Curves and Hypersurfaces

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Abstract. In the present paper, we study bi- f -harmonic maps which generalize not only f -harmonic maps, but also biharmonic maps. We derive bi- f -harmonic equations for curves in the Euclidean space, unit sphere, hyperbolic space, and for hypersurfaces of Riemannian manifolds.

1. Introduction

Harmonic maps between Riemannian manifolds, which can be viewed as a generalization of geodesics when the domain is 1-dimensional, or of harmonic functions when the ranges are Euclidean spaces, have an extensive study area and there exist many applications of such mappings in mathematics and physics. Dealing with the non-linear partial differential equations makes challenge to prove the existence of harmonic maps. A harmonic map may not always exist in a homotopy class, and if it exists, then it might not be unique.

Generalizing harmonic maps, J. Eells and J. H. Sampson introduced in [6] biharmonic maps between Riemannian manifolds. In [3], B. Y. Chen defined biharmonic submanifolds of the Euclidean space and stated a well-known conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, thus minimal. If one use the definition of biharmonic maps to Riemannian immersions into Euclidean space, it is easy to see that Chen's definition of biharmonic submanifold coincides with the definition given by using bienergy functional. In recent years, there has been an important literature on biharmonic submanifold theory including many results on the non-existence of biharmonic submanifolds in manifolds with non-positive sectional curvature. These non-existence consequences (see [8], [11]) as well as *Generalized Chen's conjecture*: Any biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal, which was proposed by R. Caddeo, S. Montaldo and C. Oniciuc [2], led the studies to spheres and other non-negatively curved spaces. But in recent years, the authors of [15] proved that the *Generalized Chen's conjecture* is not true by constructing examples of proper biharmonic hypersurfaces in a 5-dimensional space of non-constant negative sectional curvature. For some recent geometric studies of general biharmonic maps and biharmonic submanifolds see ([14], [2], [12], [15], [16], [9], [18], [19]) and the references therein.

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f -harmonic maps between Riemannian manifolds were introduced and studied by A. Lichnerowicz in 1970 (see also [5]). They have also some physical meanings by considering them as solutions of continuous spin systems and inhomogenous Heisenberg spin systems [1]. Moreover, there is a strong relationship between f -harmonic maps and gradient Ricci solitons [20].

There are two ways to formalize such a link between biharmonic maps and f -harmonic maps. The first formalization is that by simulating the theory for biharmonic maps, the authors of [21] extended bienergy functional to bi- f -energy functional and obtained a new type of harmonic maps called bi- f -harmonic maps. This idea has already been considered by S. Ouakkas, R. Nasri and M. Djaa [17]. They used the term “ f -biharmonic maps” for the critical points of bi- f -energy functional. As parallel to “biharmonic maps”, in [21], they considered that it is more appropriate to call them “bi- f -harmonic maps”. The second formalization is that by following the definition of f -harmonic map, to extend the f -energy functional to the f -bienergy functional and obtain another type of harmonic maps called f -biharmonic maps as critical points of f -bienergy functional.

The notion of f -biharmonic maps has been introduced by W.-J. Lu [10] as a generalization of biharmonic maps. A differentiable map between Riemannian manifolds is called f -biharmonic if it is a critical point of the f -bienergy functional defined by integral of f times the square-norm of the tension field, where f is a smooth positive function on the domain. If $f = 1$, then f -biharmonic maps are biharmonic. To avoid the confusion with the types of maps called by the same name in [17] and defined as critical points of the square-norm of the f -tension field, some authors (see [10], [13]) called the map defined in [17] as *bi- f -harmonic map*, which we shall study in this paper.

The following inclusions illustrate the relations between these notions:

harmonic maps \subset biharmonic maps \subset f -biharmonic maps,

harmonic maps \subset f -harmonic maps \subset bi- f -harmonic maps.

2. Bi- f -harmonic curves

In this section we derive the bi- f -harmonic equation for curves in Riemannian manifolds and discuss the particular cases of the Euclidean space, unit sphere and hyperbolic space.

Recall that *bi- f -harmonic maps* $\psi : (N, g) \rightarrow (\bar{N}, \bar{g})$ between two Riemannian manifolds are critical points of the bi- f -energy functional:

$$E_{f,2}(\psi) = \frac{1}{2} \int_{\Omega} |\tau_f(\psi)|^2 \vartheta_g, \quad (1)$$

where $\Omega \subset N$ is a compact domain, $\tau_f(\psi) \equiv f \tau(\psi) + d\psi(\text{grad} f)$ is the f -tension field of ψ , $\tau(\psi) \equiv \text{trace} \nabla d\psi$ is the tension field of ψ and ∇ is the connection induced from the Levi-Civita connection $\nabla^{\bar{N}}$ of \bar{N} and the pull-back connection ∇^{ψ} .

S. Ouakkas, R. Nasri and M. Djaa [17] gave the Euler-Lagrange equation of bi- f -harmonic maps, precisely:

Proposition 2.1. [17] *Let $\psi : (N, g) \rightarrow (\bar{N}, \bar{g})$ be a smooth map between Riemannian manifolds. Then, in terms of Euler-Lagrange equation, ψ is a bi- f -harmonic map if and only if its bi- f -tension field $\tau_{f,2}(\psi)$ vanishes, i.e.*

$$\text{trace} \left(\nabla^{\psi} f \left(\nabla^{\psi} \tau_f(\psi) \right) - f \nabla_{\nabla^{\psi}}^{\psi} \tau_f(\psi) + f R^{\bar{N}} \left(\tau_f(\psi), d\psi \right) d\psi \right) = 0, \quad (2)$$

where $f : I \rightarrow (0, \infty)$ is a smooth map defined on a real interval I .

From (2), we can easily see that bi- f -harmonic map is a much wider generalization of harmonic map, because it is not only a generalization of f -harmonic map (as $f \neq 1$ and $\tau_f(\psi) = 0$), but also a generalization of biharmonic map (as $f = 1$). Therefore, it would be interesting to know whether there is any non-trivial or proper bi- f -harmonic map which is neither harmonic map nor f -harmonic map with $f \neq \text{constant}$.

Definition 2.2. A submanifold in a Riemannian manifold is called a bi- f -harmonic submanifold if the isometric immersion defining the submanifold is a bi- f -harmonic map.

Let $\alpha : I \rightarrow (\bar{N}, \bar{g})$ be a curve in a Riemannian manifold (\bar{N}, \bar{g}) , defined on an open real interval I and parametrized by its arclength, and $\alpha' =: T$. We have

$$\begin{aligned} \tau(\alpha) &= \nabla_T^{\bar{N}} T \\ \tau_f(\alpha) &= f \nabla_T^{\bar{N}} T + f' T \end{aligned}$$

and in order to obtain the bi- f -tension field of α , we compute:

$$\begin{aligned} \text{trace}(\nabla^\alpha f(\nabla^\alpha \tau_f(\alpha)) - f \nabla_{\nabla^\alpha}^\alpha \tau_f(\alpha)) &= \nabla_{\frac{d}{dt}}^\alpha f \left(\nabla_{\frac{d}{dt}}^\alpha \tau_f(\alpha) \right) - f \nabla_{\nabla_{\frac{d}{dt}}^\alpha}^\alpha \tau_f(\alpha) \\ &= \nabla_T^{\bar{N}} f (\nabla_T^{\bar{N}} (f \nabla_T^{\bar{N}} T + f' T)) \\ &= (ff''' + f'f'')T + (3ff'' + 2(f')^2) \nabla_T^{\bar{N}} T \\ &\quad + 4ff' \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} T + f^2 \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} T \end{aligned} \tag{3}$$

and

$$\begin{aligned} \text{trace}(R^{\bar{N}}(\tau_f(\alpha), d\alpha)d\alpha) &= R^{\bar{N}}\left(\tau_f(\alpha), d\alpha\left(\frac{d}{dt}\right)\right)d\alpha\left(\frac{d}{dt}\right) \\ &= fR^{\bar{N}}(\nabla_T^{\bar{N}} T, T)T. \end{aligned} \tag{4}$$

From (3) and (4) we obtain

Proposition 2.3. Let $\alpha : I \rightarrow (\bar{N}, \bar{g})$ be a curve in a Riemannian manifold (\bar{N}, \bar{g}) , parametrized by its arclength, and $\alpha' = T$. Then α is a bi- f -harmonic curve if and only if

$$\begin{aligned} 0 &= (ff''' + f'f'')T + (3ff'' + 2(f')^2) \nabla_T^{\bar{N}} T \\ &\quad + 4ff' \nabla_T^{\bar{N}} T + f^2 \nabla_T^{\bar{N}} T + f^2 R^{\bar{N}}(\nabla_T^{\bar{N}} T, T)T, \end{aligned} \tag{5}$$

where $f : I \rightarrow (0, \infty)$ is a smooth map, $\nabla_T^2 T =: \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} T$ and $\nabla_T^3 T =: \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} T$.

Let $\{\bar{E}_1, \bar{E}_2, \dots, \bar{E}_n\}$ be the Frenet frame on the n -dimensional manifold \bar{N} , defined along α , where $\bar{E}_1 = \alpha' = T$ is the unit tangent vector field of α , \bar{E}_2 is the unit normal vector field of α , with the same direction as $\nabla_T^{\bar{N}} \bar{E}_1$ and the vector fields $\bar{E}_3, \dots, \bar{E}_n$ are the unit vector fields obtained from the Frenet equations for α :

$$\left\{ \begin{aligned} \nabla_T^{\bar{N}} \bar{E}_1 &= k_1 \bar{E}_2, \\ \nabla_T^{\bar{N}} \bar{E}_2 &= -k_1 \bar{E}_1 + k_2 \bar{E}_3, \\ &\dots \\ \nabla_T^{\bar{N}} \bar{E}_r &= -k_{r-1} \bar{E}_{r-1} + k_r \bar{E}_{r+1}, \quad r = 3, \dots, n-1, \\ &\dots \\ \nabla_T^{\bar{N}} \bar{E}_n &= -k_{n-1} \bar{E}_{n-1}, \end{aligned} \right. \tag{6}$$

where $k_1 = \|\nabla_T^{\bar{N}} \bar{E}_1\|$ and k_2, \dots, k_{n-1} are real valued non-negative maps.

From (6) we have

$$\nabla_T^2 T = \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} T = -k_1^2 \bar{E}_1 + k_1' \bar{E}_2 + k_1 k_2 \bar{E}_3, \tag{7}$$

$$\begin{aligned} \nabla_T^3 T &= \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} \nabla_T^{\bar{N}} T \\ &= -3k_1 k_1' \bar{E}_1 + (k_1'' - k_1^3 - k_1 k_2^2) \bar{E}_2 + (2k_1' k_2 + k_1 k_2') \bar{E}_3 + k_1 k_2 k_3 \bar{E}_4, \end{aligned} \tag{8}$$

$$R^{\bar{N}}(\nabla_T^{\bar{N}} T, T)T = k_1 R^{\bar{N}}(E_2, \bar{E}_1)E_1. \tag{9}$$

Using (6), (7), (8) and (9) in (5), we have

Theorem 2.4. *Let $\alpha : I \rightarrow (\bar{N}, \bar{g})$ be a curve in a Riemannian manifold (\bar{N}, \bar{g}) , parametrized by its arclength. Then α is a bi- f -harmonic curve if and only if*

$$\begin{aligned} 0 &= (-3k_1 k_1' f^2 - 4k_1^2 f f' + f f''' + f' f'') E_1 \\ &+ (-k_1^3 f^2 - k_1 k_2^2 f^2 + k_1'' f^2 + 4k_1' f f' + 3k_1 f f'' + 2k_1 (f')^2) \bar{E}_2 \\ &+ ((2k_1' k_2 f + k_1 k_2' f + 4k_1 k_2 f') f) \bar{E}_3 \\ &+ (k_1 k_2 k_3 f^2) \bar{E}_4 + k_1 f^2 R^{\bar{N}}(E_2, \bar{E}_1)E_1. \end{aligned} \tag{10}$$

Remark 2.5. *The property of a curve of being bi- f -harmonic in an n -dimensional space (with $n > 3$) does not depend on all its curvatures, but only on k_1, k_2 and k_3 .*

It is well known that in a Riemannian manifold (\bar{N}, \bar{g}) of constant sectional curvature c , the curvature tensor field $R^{\bar{N}}$ is of the form

$$R^{\bar{N}}(X, Y)Z = c(\bar{g}(Y, Z)X - \bar{g}(X, Z)Y),$$

for any $X, Y, Z \in \Gamma(T\bar{N})$ and we can state

Theorem 2.6. *Let $\alpha : I \rightarrow (\bar{N}(c), h)$ be a curve in a Riemannian space form $(\bar{N}(c), h)$, parametrized by its arclength. Then α is a bi- f -harmonic curve if and only if*

$$\begin{cases} -3k_1 k_1' f^2 - 4k_1^2 f f' + f f''' + f' f'' = 0, \\ -k_1^3 f^2 - k_1 k_2^2 f^2 + k_1'' f^2 + 4k_1' f f' + 3k_1 f f'' + 2k_1 (f')^2 + c k_1 f^2 = 0, \\ 2k_1' k_2 f + k_1 k_2' f + 4k_1 k_2 f' = 0, \\ k_1 k_2 k_3 = 0. \end{cases} \tag{11}$$

Let $\alpha : I \rightarrow \mathbb{E}^n$ be a curve in the n -dimensional Euclidean space, defined on an open real interval I and parametrized by its arclength. Since \mathbb{E}^n is a Riemannian space form with $c = 0$, from the bi- f -harmonic curve equation given by (11) we have

Theorem 2.7. *Let $\alpha : I \rightarrow \mathbb{E}^n$ be a curve in the n -dimensional Euclidean space, parametrized by its arclength. Then α is a bi- f -harmonic curve if and only if*

$$\begin{cases} -3k_1 k_1' f^2 - 4k_1^2 f f' + f f''' + f' f'' = 0, \\ -k_1^3 f^2 - k_1 k_2^2 f^2 + k_1'' f^2 + 4k_1' f f' + 3k_1 f f'' + 2k_1 (f')^2 = 0, \\ 2k_1' k_2 f + k_1 k_2' f + 4k_1 k_2 f' = 0, \\ k_1 k_2 k_3 = 0. \end{cases} \tag{12}$$

CASE I: If $k_1 = 0$, namely α is a geodesic curve, then from (12) we obtain that it is bi- f -harmonic if and only if $f f'' = \text{constant}$.

It is well known that geodesics are f -harmonic with $f = \text{constant}$ and they are automatically bi- f -harmonic. Remark that also for $f(t) = at + b, a, b \in \mathbb{R}$, any geodesic curve is bi- f -harmonic.

Theorem 2.8. *A geodesic curve is bi- f -harmonic if and only if $f f'' = \text{constant}$.*

CASE II: If $k_1 = \text{constant} \neq 0$ and $k_2 = 0$, then (12) reduces to

$$\begin{cases} -4k_1^2 f f' + f f''' + f' f'' = 0, \\ -k_1^2 f^2 + 3f f'' + 2(f')^2 = 0. \end{cases} \quad (13)$$

From the second equation above we obtain

$$f f'' = \frac{k_1^2 f^2 - 2(f')^2}{3}, \quad (14)$$

which implies

$$f' (5k_1^2 f + 2f'') = 0, \quad (15)$$

via the first equation of (13) and we get

Theorem 2.9. *Let $\alpha : I \rightarrow \mathbb{E}^n$ be a curve in the n -dimensional Euclidean space, parametrized by its arclength, with $k_1 = \text{constant} \neq 0$ and $k_2 = 0$. Then α is a bi- f -harmonic curve if and only if either f is a constant function or f is given by*

$$f(s) = c_1 \cos\left(\sqrt{\frac{5}{2}} k_1 s\right) + c_2 \sin\left(\sqrt{\frac{5}{2}} k_1 s\right),$$

for $s \in I$ and $c_1, c_2 \in \mathbb{R}$.

CASE III: If $k_1 = \text{constant} \neq 0$ and $k_2 = \text{constant} \neq 0$, then (12) reduces to

$$\begin{cases} -4k_1^2 f f' + f f''' + f' f'' = 0, \\ -k_1^2 f^2 - k_2^2 f^2 + 3f f'' + 2(f')^2 = 0, \\ f' = 0, \\ k_3 = 0, \end{cases} \quad (16)$$

which implies

$$\begin{cases} k_1^2 + k_2^2 = 0, \\ f' = 0, \\ k_3 = 0, \end{cases} \quad (17)$$

and we deduce

Theorem 2.10. *There is no bi- f -harmonic curve in the n -dimensional Euclidean space with $k_1 = \text{constant} \neq 0$ and $k_2 = \text{constant} \neq 0$.*

CASE IV: If $k_1 = \text{constant} \neq 0$ and $k_2 \neq \text{constant}$, then (12) reduces to

$$\begin{cases} -4k_1^2 f f' + f f''' + f' f'' = 0, \\ -k_1^2 f^2 - k_2^2 f^2 + 3f f'' + 2(f')^2 = 0, \\ k_2^2 f + 4k_2 f' = 0, \\ k_2 k_3 = 0, \end{cases} \quad (18)$$

and we have

Theorem 2.11. Let $\alpha : I \rightarrow \mathbb{E}^n$ be a curve in the n -dimensional Euclidean space, parametrized by its arclength, with $k_1 = \text{constant} \neq 0$ and $k_2 \neq \text{constant}$ and nowhere zero. Then α is a bi- f -harmonic curve if and only if $f = ck_2^{-\frac{1}{4}}$ (with c a positive constant), $k_3 = 0$ and the curvatures k_1 and k_2 satisfy:

$$\begin{cases} 32k_1^2k_2^2k_2' - 25(k_2')^3 + 32k_2k_2'k_2'' - 8k_2^2k_2''' = 0, \\ 16k_1^2k_2^2 + 16k_2^4 - 17(k_2')^2 + 12k_2k_2'' = 0. \end{cases} \quad (19)$$

CASE V: Concerning the case $k_1 \neq \text{constant}$ and $k_2 = 0$, we can state

Theorem 2.12. Let $\alpha : I \rightarrow \mathbb{E}^n$ be a curve in the n -dimensional Euclidean space, parametrized by its arclength, with $k_1 \neq \text{constant}$ and $k_2 = 0$. Then α is a bi- f -harmonic curve if and only if the curvatures k_1 and k_2 satisfy:

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \\ -k_1^3f^2 + k_1'f^2 + 4k_1'ff' + 3k_1ff'' + 2k_1(f')^2 = 0. \end{cases} \quad (20)$$

CASE VI: If $k_1 \neq \text{constant}$ and $k_2 = \text{constant} \neq 0$, then (12) reduces to

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \\ -k_1^3f^2 - k_1k_2^2f^2 + k_1'f^2 + 4k_1'ff' + 3k_1ff'' + 2k_1(f')^2 = 0, \\ k_1'f + 2k_1f' = 0, \\ k_1k_3 = 0, \end{cases} \quad (21)$$

and we have

Theorem 2.13. Let $\alpha : I \rightarrow \mathbb{E}^n$ be a curve in the n -dimensional Euclidean space, parametrized by its arclength, with $k_1 \neq \text{constant}$ and nowhere zero and $k_2 = \text{constant} \neq 0$. Then α is a bi- f -harmonic curve if and only if $f = ck_1^{-\frac{1}{2}}$ (with c a positive constant), $k_3 = 0$ and the curvatures k_1 and k_2 satisfy:

$$\begin{cases} 9(k_1')^3 + 4k_1^4k_1' - 10k_1k_1'k_1'' + 2k_1^2k_1''' = 0, \\ 3(k_1')^2 - 4k_1^4 - 4k_1^2k_2^2 - 2k_1k_1'' = 0. \end{cases} \quad (22)$$

CASE VII: Concerning the case $k_1 \neq \text{constant}$ and $k_2 \neq \text{constant}$, we can state

Theorem 2.14. Let $\alpha : I \rightarrow \mathbb{E}^n$ be a curve in the n -dimensional Euclidean space, parametrized by its arclength, with $k_1 \neq \text{constant}$ and $k_2 \neq \text{constant}$ and k_1, k_2 are nowhere zero. Then α is a bi- f -harmonic curve if and only if $f = ck_1^{-\frac{1}{2}}k_2^{-\frac{1}{4}}$ (with c a positive constant), $k_3 = 0$ and the curvatures k_1 and k_2 satisfy:

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \\ -k_1^3f^2 - k_1k_2^2f^2 + k_1'f^2 + k_1f^2 + 4k_1'ff' + 3k_1ff'' + 2k_1(f')^2 = 0. \end{cases} \quad (23)$$

Similar results hold for bi- f -harmonic curves in the n -dimensional sphere $S^n(1)$ and in the n -dimensional hyperbolic space $H^n(-1)$.

Theorem 2.15. Let $\alpha : I \rightarrow S^n(1)$ be a curve parametrized by its arclength. Then α is a bi- f -harmonic curve if and only if

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \\ -k_1^3f^2 - k_1k_2^2f^2 + k_1'f^2 + k_1f^2 + 4k_1'ff' + 3k_1ff'' + 2k_1(f')^2 = 0, \\ 2k_1'k_2f + k_1k_2'f + 4k_1k_2f' = 0, \\ k_1k_2k_3 = 0. \end{cases} \quad (24)$$

Theorem 2.16. Let $\alpha : I \rightarrow H^n(-1)$ be a curve parametrized by its arclength. Then α is a bi- f -harmonic curve if and only if

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \\ -k_1^3f^2 - k_1k_2^2f^2 + k_1'f^2 - k_1f^2 + 4k_1'ff' + 3k_1ff'' + 2k_1(f')^2 = 0, \\ 2k_1'k_2f + k_1k_2'f + 4k_1k_2f' = 0, \\ k_1k_2k_3 = 0. \end{cases} \quad (25)$$

Concerning the CASES IV–VII, we obtain similar conditions like in the Euclidean space and in the CASES I–III, we get the following characterizations of bi- f -harmonic curves in $S^n(1)$ and $H^n(-1)$, respectively.

Theorem 2.17. Let $\alpha : I \rightarrow \bar{N}$ be a curve in \bar{N} , parametrized by its arclength.

1. For $\bar{N} := S^n(1)$:

- (a) if $k_1 = 0$, then α is a bi- f -harmonic curve if and only if $ff'' = \text{constant}$;
- (b) if $k_1 = \text{constant} \neq 0$ and $k_2 = 0$, then α is a bi- f -harmonic curve if and only if either f is a constant function or f is given by

$$f(s) = c_1 \cos \left(\sqrt{\frac{5k_1^2 + 1}{2}} s \right) + c_2 \sin \left(\sqrt{\frac{5k_1^2 + 1}{2}} s \right),$$

for $s \in I$ and $c_1, c_2 \in \mathbb{R}$;

- (c) if $k_1 = \text{constant} \neq 0$ and $k_2 = \text{constant} \neq 0$, then α is a bi- f -harmonic curve if and only if f is a constant function, $k_1^2 + k_2^2 = 1$ and $k_3 = 0$.

2. For $\bar{N} := H^n(-1)$:

- (a) if $k_1 = 0$, then α is a bi- f -harmonic curve if and only if $ff'' = \text{constant}$;
- (b) if $k_1 = \text{constant} \neq 0$ and $k_2 = 0$, then α is a bi- f -harmonic curve if and only if either f is a constant function or f is given by one of the following expressions

$$f(s) = c_1s + c_2, \text{ for } k_1 = \pm \frac{\sqrt{5}}{5},$$

or

$$f(s) = c_1 \cos \left(\sqrt{\frac{5k_1^2 - 1}{2}} s \right) + c_2 \sin \left(\sqrt{\frac{5k_1^2 - 1}{2}} s \right),$$

$$\text{for } k_1 \in \left(-\infty, -\frac{\sqrt{5}}{5} \right) \cup \left(\frac{\sqrt{5}}{5}, \infty \right),$$

or

$$f(s) = c_1 e^{\sqrt{\frac{1-5k_1^2}{2}} s} + c_2 e^{-\sqrt{\frac{1-5k_1^2}{2}} s}, \text{ for } k_1 \in \left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right),$$

for $s \in I$ and $c_1, c_2 \in \mathbb{R}$;

- (c) if $k_1 = \text{constant} \neq 0$ and $k_2 = \text{constant} \neq 0$, then there is no bi- f -harmonic curve.

3. Bi-*f*-harmonic hypersurfaces

In this section we derive the bi-*f*-harmonic equation for hypersurfaces in Riemannian manifolds. Let N be an m -dimensional hypersurface of (\bar{N}, \bar{g}) with mean curvature vector $\eta = HV$, where V is the unit normal vector field of N . Denoting by g the Riemannian metric induced on N , by ∇^N and $\nabla^{\bar{N}}$ the Levi-Civita connections on (N, g) and (\bar{N}, \bar{g}) respectively, the Gauss and Weingarten formulas corresponding to N are given by:

$$\nabla_X^{\bar{N}} Y = \nabla_X^N Y + B(X, Y), \tag{26}$$

$$\nabla_X^{\bar{N}} V = -AX, \tag{27}$$

for any $X, Y \in \Gamma(TN)$, where B is the (symmetric) second fundamental tensor corresponding to V , A is the shape operator with respect to the unit normal vector field V , and let $b(X, Y) = \langle B(X, Y), V \rangle$, for any $X, Y \in \Gamma(TN)$.

The bi-*f*-tension field of the immersion $\psi : N \rightarrow \bar{N}$ is given by [17]:

$$\tau_{f,2}(\psi) = -f \operatorname{trace}(\nabla^\psi)^2 \tau_f(\psi) - f \operatorname{trace} R^{\bar{N}}(\tau_f(\psi), d\psi) d\psi - \nabla_{\operatorname{grad} f}^\psi \tau_f(\psi). \tag{28}$$

For an orthonormal frame field $\{e_1, e_2, \dots, e_m\} \subset \Gamma(TN)$, we have

$$\begin{aligned} \operatorname{trace}(\nabla^\psi f (\nabla^\psi \tau_f(\psi)) - f \nabla_{\nabla^N}^\psi \tau_f(\psi)) &= f \sum_{i=1}^m \left\{ \nabla_{e_i}^\psi \nabla_{e_i}^\psi \tau_f(\psi) - \nabla_{\nabla_{e_i}^N}^\psi \tau_f(\psi) \right\} \\ &\quad + \nabla_{\operatorname{grad} f}^\psi \tau_f(\psi). \end{aligned} \tag{29}$$

As a first step, we compute

$$\begin{aligned} \operatorname{trace}(\nabla^\psi)^2 \tau_f(\psi) &= \sum_{i=1}^m \left\{ \nabla_{e_i}^\psi \nabla_{e_i}^\psi \tau_f(\psi) - \nabla_{\nabla_{e_i}^N}^\psi \tau_f(\psi) \right\} \\ &= \sum_{i=1}^m \left\{ \nabla_{e_i}^\psi \nabla_{e_i}^\psi (f \tau(\psi) + d\psi(\operatorname{grad} f)) - \nabla_{\nabla_{e_i}^N}^\psi (f \tau(\psi) + d\psi(\operatorname{grad} f)) \right\} \\ &= \sum_{i=1}^m \left\{ \begin{aligned} &\nabla_{e_i}^\psi \nabla_{e_i}^\psi (f \tau(\psi)) + \nabla_{e_i}^\psi \nabla_{e_i}^\psi d\psi(\operatorname{grad} f) \\ &- \nabla_{\nabla_{e_i}^N}^\psi (f \tau(\psi)) - \nabla_{\nabla_{e_i}^N}^\psi d\psi(\operatorname{grad} f) \end{aligned} \right\} \\ &= \sum_{i=1}^m \left\{ \begin{aligned} &\nabla_{e_i}^\psi (e_i(f) \tau(\psi) + f \nabla_{e_i}^\psi \tau(\psi)) + \nabla_{e_i}^\psi \nabla_{e_i}^\psi d\psi(\operatorname{grad} f) \\ &- (\nabla_{e_i}^N e_i)(f) \tau(\psi) - f \nabla_{\nabla_{e_i}^N}^\psi \tau(\psi) - \nabla_{\nabla_{e_i}^N}^\psi d\psi(\operatorname{grad} f) \end{aligned} \right\} \\ &= \sum_{i=1}^m \left\{ \begin{aligned} &e_i(e_i(f)) \tau(\psi) + 2e_i(f) \nabla_{e_i}^\psi \tau(\psi) + f \nabla_{e_i}^\psi \nabla_{e_i}^\psi \tau(\psi) + \nabla_{e_i}^\psi \nabla_{e_i}^\psi d\psi(\operatorname{grad} f) \\ &- (\nabla_{e_i}^N e_i)(f) \tau(\psi) - f \nabla_{\nabla_{e_i}^N}^\psi \tau(\psi) - \nabla_{\nabla_{e_i}^N}^\psi d\psi(\operatorname{grad} f) \end{aligned} \right\} \\ &= (\Delta f) \tau(\psi) + 2 \nabla_{\operatorname{grad} f}^{\bar{N}} \tau(\psi) - f \Delta^\psi (\tau(\psi)) - \Delta^\psi (\operatorname{grad} f). \end{aligned} \tag{30}$$

Since the tension field of ψ is $\tau(\psi) = mHV$, we have

$$(\Delta f) \tau(\psi) = mH(\Delta f) V, \tag{31}$$

$$\begin{aligned} \nabla_{\operatorname{grad} f}^{\bar{N}} \tau(\psi) &= \nabla_{\operatorname{grad} f}^{\bar{N}} (mHV) = m \operatorname{grad} f(H)V - mHA(\operatorname{grad} f) \\ &= m \langle \operatorname{grad} f, \operatorname{grad} H \rangle V - mHA(\operatorname{grad} f), \end{aligned} \tag{32}$$

$$\begin{aligned}
 \Delta^\psi(\tau(\psi)) &= -\sum_{i=1}^m \left\{ \nabla_{e_i}^\psi \nabla_{e_i}^\psi (mHV) - \nabla_{\nabla_{e_i}^N e_i}^\psi (mHV) \right\} \\
 &= -m \sum_{i=1}^m \left\{ \nabla_{e_i}^\psi (e_i(H)V + H\nabla_{e_i}^N V) - (\nabla_{e_i}^N e_i)(H)V - H\nabla_{\nabla_{e_i}^N e_i}^N V \right\} \\
 &= -m \sum_{i=1}^m \left\{ e_i(e_i(H))V + 2e_i(H)\nabla_{e_i}^N V + H\nabla_{e_i}^N \nabla_{e_i}^N V - (\nabla_{e_i}^N e_i)(H)V - H\nabla_{\nabla_{e_i}^N e_i}^N V \right\} \\
 &= -m(\Delta H)V + 2mA(\text{grad}H) + mH\Delta^\psi(V).
 \end{aligned}
 \tag{33}$$

By using (31), (32) and (33) in (30) we obtain

$$\begin{aligned}
 \text{trace}(\nabla^\psi)^2 \tau_f(\psi) &= mH(\Delta f)V + 2m \langle \text{grad}f, \text{grad}H \rangle V - 2mHA(\text{grad}f) \\
 &\quad + mf(\Delta H)V - 2mfA(\text{grad}H) - mfH\Delta^\psi(V) - \Delta^\psi(\text{grad}f).
 \end{aligned}
 \tag{34}$$

As a second step, we compute:

$$\begin{aligned}
 \text{trace}R^{\bar{N}}(\tau_f(\psi), d\psi)d\psi &= \sum_{i=1}^m R^{\bar{N}}(f\tau(\psi) + d\psi(\text{grad}f), d\psi(e_i))d\psi(e_i) \\
 &= f \sum_{i=1}^m R^{\bar{N}}(\tau(\psi), e_i)e_i + \sum_{i=1}^m R^{\bar{N}}(\text{grad}f, e_i)e_i,
 \end{aligned}
 \tag{35}$$

which implies

$$\text{trace}R^{\bar{N}}(\tau_f(\psi), d\psi)d\psi = mfH \sum_{i=1}^m R^{\bar{N}}(V, e_i)e_i + \sum_{i=1}^m R^{\bar{N}}(\text{grad}f, e_i)e_i.
 \tag{36}$$

Also

$$\begin{aligned}
 \nabla_{\text{grad}f}^\psi \tau_f(\psi) &= \nabla_{\text{grad}f}^\psi (f\tau(\psi) + d\psi(\text{grad}f)) \\
 &= \nabla_{\text{grad}f}^{\bar{N}} (f\tau(\psi)) + \nabla_{\text{grad}f}^{\bar{N}} \text{grad}f \\
 &= \text{grad}f(f)\tau(\psi) + f\nabla_{\text{grad}f}^{\bar{N}} \tau(\psi) + \nabla_{\text{grad}f}^{\bar{N}} \text{grad}f,
 \end{aligned}
 \tag{37}$$

which gives

$$\begin{aligned}
 \nabla_{\text{grad}f}^\psi \tau_f(\psi) &= mH \langle \text{grad}f, \text{grad}f \rangle V + mf \langle \text{grad}f, \text{grad}H \rangle V \\
 &\quad - mfHA(\text{grad}f) + \frac{1}{2} \text{grad}(|\text{grad}f|^2) + B(\text{grad}f, \text{grad}f).
 \end{aligned}
 \tag{38}$$

By using (34), (36) and (38) in (28) we obtain the bi-*f*-tension field of ψ :

$$\begin{aligned}
\tau_{f,2}(\psi) &= -mfH(\Delta f)V - 3mf\langle \text{grad}f, \text{grad}H \rangle V + 3mfHA(\text{grad}f) \\
&\quad -mf^2(\Delta H)V + 2mf^2A(\text{grad}H) + mf^2H\Delta^\psi(V) \\
&\quad +f\Delta^\psi(\text{grad}f) - mH\langle \text{grad}f, \text{grad}f \rangle V - \frac{1}{2}\text{grad}(|\text{grad}f|^2) - B(\text{grad}f, \text{grad}f) \\
&\quad -mf^2H\sum_{i=1}^m R^{\bar{N}}(V, e_i)e_i - f\sum_{i=1}^m R^{\bar{N}}(\text{grad}f, e_i)e_i.
\end{aligned} \tag{39}$$

The tangential component of $\Delta^\psi(\text{grad}f)$ can be computed by

$$\begin{aligned}
(\Delta^\psi(\text{grad}f))^\top &= -\sum_{i,k=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad}f - \nabla_{\nabla_{e_i}^{\bar{N}} e_i}^{\bar{N}} \text{grad}f, e_k \right\rangle e_k \\
&= -\sum_{i,k=1}^m \left\langle \nabla_{e_i}^{\bar{N}} (\nabla_{e_i}^{\bar{N}} \text{grad}f + B(\text{grad}f, e_i)), e_k \right\rangle e_k \\
&\quad - \sum_{i,k=1}^m \left\langle -\nabla_{\nabla_{e_i}^{\bar{N}} e_i}^{\bar{N}} \text{grad}f - B(\nabla_{e_i}^{\bar{N}} e_i, \text{grad}f), e_k \right\rangle e_k \\
&= -\sum_{i,k=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad}f + B(\nabla_{e_i}^{\bar{N}} \text{grad}f, e_i) + \nabla_{e_i}^{\bar{N}} B(\text{grad}f, e_i), e_k \right\rangle e_k \\
&\quad - \sum_{i,k=1}^m \left\langle -\nabla_{\nabla_{e_i}^{\bar{N}} e_i}^{\bar{N}} \text{grad}f - B(\nabla_{e_i}^{\bar{N}} e_i, \text{grad}f), e_k \right\rangle e_k \\
&= -\sum_{i,k=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad}f + b(\nabla_{e_i}^{\bar{N}} \text{grad}f, e_i)V + \nabla_{e_i}^{\bar{N}} (b(\text{grad}f, e_i)V), e_k \right\rangle e_k \\
&\quad - \sum_{i,k=1}^m \left\langle -\nabla_{\nabla_{e_i}^{\bar{N}} e_i}^{\bar{N}} \text{grad}f - b(\nabla_{e_i}^{\bar{N}} e_i, \text{grad}f)V, e_k \right\rangle e_k \\
&= -\sum_{i,k=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad}f - \nabla_{\nabla_{e_i}^{\bar{N}} e_i}^{\bar{N}} \text{grad}f, e_k \right\rangle e_k \\
&\quad - \sum_{i,k=1}^m \left\langle b(\nabla_{e_i}^{\bar{N}} \text{grad}f, e_i)V + \nabla_{e_i}^{\bar{N}} (b(\text{grad}f, e_i)V) - b(\nabla_{e_i}^{\bar{N}} e_i, \text{grad}f)V, e_k \right\rangle e_k \\
&= -\sum_{i,k=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad}f - \nabla_{\nabla_{e_i}^{\bar{N}} e_i}^{\bar{N}} \text{grad}f + b(\nabla_{e_i}^{\bar{N}} \text{grad}f, e_i)V, e_k \right\rangle e_k \\
&\quad - \sum_{i,k=1}^m \left\langle e_i (b(\text{grad}f, e_i))V - b(\text{grad}f, e_i)Ae_i - b(\nabla_{e_i}^{\bar{N}} e_i, \text{grad}f)V, e_k \right\rangle e_k \\
&= \Delta(\text{grad}f) + \sum_{i,k=1}^m b(\text{grad}f, e_i) \langle Ae_i, e_k \rangle e_k,
\end{aligned} \tag{40}$$

which implies

$$(\Delta^\psi(\text{grad}f))^\top = \Delta(\text{grad}f) + A^2(\text{grad}f). \tag{41}$$

The normal component of $\Delta^\psi(\text{grad} f)$ can be computed by

$$\begin{aligned}
 (\Delta^\psi(\text{grad} f))^\perp &= -\sum_{i=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad} f - \nabla_{\nabla_{e_i}^{\bar{N}}}^{\bar{N}} \text{grad} f, V \right\rangle V \\
 &= -\sum_{i=1}^m \left\langle \begin{matrix} \nabla_{e_i}^{\bar{N}} (\nabla_{e_i}^{\bar{N}} \text{grad} f + B(\text{grad} f, e_i)) \\ -\nabla_{\nabla_{e_i}^{\bar{N}}}^{\bar{N}} \text{grad} f - B(\nabla_{e_i}^{\bar{N}} e_i, \text{grad} f) \end{matrix}, V \right\rangle V \\
 &= -\sum_{i=1}^m \left\langle \begin{matrix} \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad} f + B(\nabla_{e_i}^{\bar{N}} \text{grad} f, e_i) \\ +\nabla_{e_i}^{\bar{N}} B(\text{grad} f, e_i) - \nabla_{\nabla_{e_i}^{\bar{N}}}^{\bar{N}} \text{grad} f - B(\nabla_{e_i}^{\bar{N}} e_i, \text{grad} f) \end{matrix}, V \right\rangle V \\
 &= -\sum_{i=1}^m \left\langle \begin{matrix} \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad} f + b(\nabla_{e_i}^{\bar{N}} \text{grad} f, e_i)V \\ +\nabla_{e_i}^{\bar{N}} (b(\text{grad} f, e_i)V) - \nabla_{\nabla_{e_i}^{\bar{N}}}^{\bar{N}} \text{grad} f - b(\nabla_{e_i}^{\bar{N}} e_i, \text{grad} f)V \end{matrix}, V \right\rangle V \\
 &= -\sum_{i=1}^m \left\langle \begin{matrix} \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad} f - \nabla_{\nabla_{e_i}^{\bar{N}}}^{\bar{N}} \text{grad} f + b(\nabla_{e_i}^{\bar{N}} \text{grad} f, e_i)V \\ +\nabla_{e_i}^{\bar{N}} (b(\text{grad} f, e_i)V) - b(\nabla_{e_i}^{\bar{N}} e_i, \text{grad} f)V \end{matrix}, V \right\rangle V \\
 &= -\sum_{i=1}^m \left\langle \begin{matrix} \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} \text{grad} f - \nabla_{\nabla_{e_i}^{\bar{N}}}^{\bar{N}} \text{grad} f + b(\nabla_{e_i}^{\bar{N}} \text{grad} f, e_i)V \\ +e_i (b(\text{grad} f, e_i)) V - b(\text{grad} f, e_i)Ae_i - b(\nabla_{e_i}^{\bar{N}} e_i, \text{grad} f)V \end{matrix}, V \right\rangle V \\
 &= -\sum_{i=1}^m \left\langle b(\nabla_{e_i}^{\bar{N}} \text{grad} f, e_i)V + e_i (b(\text{grad} f, e_i)) V - b(\nabla_{e_i}^{\bar{N}} e_i, \text{grad} f)V, V \right\rangle V \\
 &= -\sum_{i=1}^m \{b(\nabla_{e_i}^{\bar{N}} \text{grad} f, e_i) + e_i (b(\text{grad} f, e_i)) - b(\nabla_{e_i}^{\bar{N}} e_i, \text{grad} f)\}V. \tag{42}
 \end{aligned}$$

The tangential component of $\Delta^\psi(V)$ can be computed by

$$\begin{aligned}
 (\Delta^\psi(V))^\top &= -\sum_{i,k=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} V - \nabla_{\nabla_{e_i}^{\bar{N}}}^{\bar{N}} V, e_k \right\rangle e_k \\
 &= \sum_{i,k=1}^m \left\langle \nabla_{e_i}^{\bar{N}} Ae_i - A(\nabla_{e_i}^{\bar{N}} e_i), e_k \right\rangle e_k \\
 &= \sum_{i,k=1}^m \{e_i \langle Ae_i, e_k \rangle - \langle Ae_i, \nabla_{e_i}^{\bar{N}} e_k \rangle - \langle A(\nabla_{e_i}^{\bar{N}} e_i), e_k \rangle\} e_k \\
 &= \sum_{i,k=1}^m \{e_i b(e_i, e_k) - b(e_i, \nabla_{e_i}^{\bar{N}} e_k) - b(\nabla_{e_i}^{\bar{N}} e_i, e_k)\} e_k \\
 &= \sum_{i,k=1}^m \{(\nabla_{e_i}^{\bar{N}} b)(e_k, e_i)\} e_k. \tag{43}
 \end{aligned}$$

By Codazzi-Mainardi equation, we have

$$\sum_{i=1}^m ((\nabla_{e_i}^{\bar{N}} b)(e_k, e_i) - (\nabla_{e_k}^{\bar{N}} b)(e_i, e_i)) = -\sum_{i=1}^m \langle R^{\bar{N}}(e_i, e_k)e_i, V \rangle = Ric^{\bar{N}}(V, e_k). \tag{44}$$

Putting the last equation into (43) we get

$$\begin{aligned}
 (\Delta^\psi(V))^\top &= \sum_{i,k=1}^m \{(\nabla_{e_i}^{\bar{N}}b)(e_k, e_i)\}e_k \\
 &= \sum_{k=1}^m \left\{ \sum_{i=1}^m (\nabla_{e_k}^{\bar{N}}b)(e_i, e_i) + Ric^{\bar{N}}(V, e_k) \right\} e_k \\
 &= m \text{grad}H + \sum_{k=1}^m Ric^{\bar{N}}(V, e_k)e_k.
 \end{aligned}
 \tag{45}$$

The normal component of $\Delta^\psi(V)$ can be computed by

$$\begin{aligned}
 (\Delta^\psi(V))^\perp &= -\sum_{i=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} V - \nabla_{\nabla_{e_i}^{\bar{N}} V}^{\bar{N}} V, V \right\rangle V \\
 &= -\sum_{i=1}^m \left\langle \nabla_{e_i}^{\bar{N}} \nabla_{e_i}^{\bar{N}} V, V \right\rangle V \\
 &= \sum_{i=1}^m \left\langle \nabla_{e_i}^{\bar{N}} V, \nabla_{e_i}^{\bar{N}} V \right\rangle V.
 \end{aligned}
 \tag{46}$$

On the other hand

$$\begin{aligned}
 \sum_{i=1}^m \left\langle \nabla_{e_i}^{\bar{N}} V, \nabla_{e_i}^{\bar{N}} V \right\rangle &= \sum_{i,j=1}^m \left\langle \nabla_{e_i}^{\bar{N}} V, \left\langle \nabla_{e_i}^{\bar{N}} V, e_j \right\rangle e_j \right\rangle \\
 &= \sum_{i,j=1}^m \left\langle \nabla_{e_i}^{\bar{N}} V, e_j \right\rangle^2 \\
 &= \sum_{i,j=1}^m \left\langle Ae_i, e_j \right\rangle^2 \\
 &= |A|^2,
 \end{aligned}
 \tag{47}$$

which implies together with (46)

$$(\Delta^\psi(V))^\perp = |A|^2 V.
 \tag{48}$$

The tangential and the normal components of the curvature terms are

$$\sum_{i,k=1}^m \left\langle R^{\bar{N}}(V, e_i)e_i, e_k \right\rangle e_k = \sum_{k=1}^m Ric^{\bar{N}}(V, e_k)e_k = (Ric^{\bar{N}}(V))^\top,
 \tag{49}$$

$$\sum_{i=1}^m \left\langle R^{\bar{N}}(V, e_i)e_i, V \right\rangle V = Ric^{\bar{N}}(V, V)V,
 \tag{50}$$

$$\sum_{i,k=1}^m \left\langle R^{\bar{N}}(\text{grad}f, e_i)e_i, e_k \right\rangle e_k = \sum_{k=1}^m Ric^{\bar{N}}(\text{grad}f, e_k)e_k = (Ric^{\bar{N}}(\text{grad}f))^\top,
 \tag{51}$$

$$\sum_{i=1}^m \left\langle R^{\bar{N}}(\text{grad}f, e_i)e_i, V \right\rangle V = Ric^{\bar{N}}(\text{grad}f, V)V.
 \tag{52}$$

By collecting all the tangential and normal components of the bi- f -tension field separately, we have

$$\begin{aligned} [\tau_{f,2}(\psi)]^T &= 3mfHA(\text{grad}f) + 2mf^2A(\text{grad}H) \\ &\quad + m^2f^2H\text{grad}H + f\Delta(\text{grad}f) \\ &\quad + fA^2(\text{grad}f) - \frac{1}{2}\text{grad}(|\text{grad}f|^2) \\ &\quad - mf^2H(\text{Ric}^{\bar{N}}(V))^\top - f(\text{Ric}^{\bar{N}}(\text{grad}f))^\top \end{aligned} \quad (53)$$

and

$$\begin{aligned} [\tau_{f,2}(\psi)]^\perp &= \{-mfH(\Delta f) - 3mf\langle \text{grad}f, \text{grad}H \rangle \\ &\quad - mf^2(\Delta H) + mf^2H|A|^2 + f(\Delta^\psi(\text{grad}f))^\perp \\ &\quad - mH|\text{grad}f|^2 - b(\text{grad}f, \text{grad}f) \\ &\quad - mf^2H\text{Ric}^{\bar{N}}(V, V) - f\text{Ric}^{\bar{N}}(\text{grad}f, V)\}V. \end{aligned} \quad (54)$$

Then we have

Theorem 3.1. Let (\bar{N}, \bar{g}) be an $(m+1)$ -dimensional Riemannian manifold and $\psi : N \rightarrow \bar{N}$ be an isometric immersion of codimension-one with mean curvature vector $\eta = HV$. Then ψ is a bi- f -harmonic map if and only if

$$\begin{aligned} 0 &= 3mfHA(\text{grad}f) + 2mf^2A(\text{grad}H) + m^2f^2H\text{grad}H \\ &\quad + f\Delta(\text{grad}f) + fA^2(\text{grad}f) - \frac{1}{2}\text{grad}(|\text{grad}f|^2) \\ &\quad - mf^2H(\text{Ric}^{\bar{N}}(V))^\top - f(\text{Ric}^{\bar{N}}(\text{grad}f))^\top \end{aligned} \quad (55)$$

and

$$\begin{aligned} 0 &= -mfH(\Delta f) - 3mf\langle \text{grad}f, \text{grad}H \rangle - mf^2(\Delta H) + mf^2H|A|^2 + f(\Delta^\psi(\text{grad}f))^\perp \\ &\quad - mH|\text{grad}f|^2 - b(\text{grad}f, \text{grad}f) - mf^2H\text{Ric}^{\bar{N}}(V, V) - f\text{Ric}^{\bar{N}}(\text{grad}f, V), \end{aligned} \quad (56)$$

where $\text{Ric}^{\bar{N}}$ denotes also the Ricci operator of the ambient space, A is the shape operator of the hypersurface with respect to the unit normal vector field V , Δ and grad are the Laplace and the gradient operator of the hypersurface, respectively, and Δ^ψ is the rough Laplace operator on sections of $\psi^{-1}TN$.

Theorem 3.2. Let N be a constant mean curvature hypersurface in an $(m+1)$ -dimensional Riemannian manifold \bar{N} . Then N is a bi- f -harmonic hypersurface if and only if

$$\begin{aligned} mf^2H(\text{Ric}^{\bar{N}}(V))^\top + f(\text{Ric}^{\bar{N}}(\text{grad}f))^\top &= 3mfHA(\text{grad}f) + f\Delta(\text{grad}f) \\ &\quad + fA^2(\text{grad}f) - \frac{1}{2}\text{grad}(|\text{grad}f|^2) \end{aligned} \quad (57)$$

and

$$\begin{aligned} mf^2H\text{Ric}^{\bar{N}}(V, V) + f\text{Ric}^{\bar{N}}(\text{grad}f, V) &= -mfH(\Delta f) + mf^2H|A|^2 + f(\Delta^\psi(\text{grad}f))^\perp \\ &\quad - mH|\text{grad}f|^2 - b(\text{grad}f, \text{grad}f). \end{aligned} \quad (58)$$

Then we have

Corollary 3.3. Let N be a constant mean curvature hypersurface in an $(m+1)$ -dimensional Ricci flat Riemannian manifold \bar{N} . Then N is a bi- f -harmonic hypersurface if and only if

$$fA^2(\text{grad}f) + 3mfHA(\text{grad}f) + f\Delta(\text{grad}f) - \frac{1}{2}\text{grad}(|\text{grad}f|^2) = 0 \quad (59)$$

and

$$mfH(\Delta f) + mH|\operatorname{grad}f|^2 - mf^2H|A|^2 + b(\operatorname{grad}f, \operatorname{grad}f) - f(\Delta^\psi(\operatorname{grad}f))^\perp = 0. \quad (60)$$

Corollary 3.4. *Let N be a hypersurface in an $(m + 1)$ -dimensional Einstein space \bar{N} . Then N is a bi- f -harmonic hypersurface if and only if*

$$\begin{aligned} f \frac{r}{m+1} \operatorname{grad}f &= 3mfHA(\operatorname{grad}f) + f\Delta(\operatorname{grad}f) \\ &+ fA^2(\operatorname{grad}f) - \frac{1}{2} \operatorname{grad}(|\operatorname{grad}f|^2) \end{aligned} \quad (61)$$

and

$$\begin{aligned} mf^2H \frac{r}{m+1} &= -mfH(\Delta f) + mf^2H|A|^2 + f(\Delta^\psi(\operatorname{grad}f))^\perp \\ &- mH|\operatorname{grad}f|^2 - b(\operatorname{grad}f, \operatorname{grad}f), \end{aligned} \quad (62)$$

where r is the scalar curvature of the ambient space.

Since an $(m + 1)$ -dimensional space of constant sectional curvature c is an Einstein space with scalar curvature $r = m(m + 1)c$, by using (61) and (62) we have

Corollary 3.5. *Let N be a hypersurface in an $(m + 1)$ -dimensional space \bar{N} of constant sectional curvature c . Then N is a bi- f -harmonic hypersurface if and only if*

$$mcf \operatorname{grad}f = 3mfHA(\operatorname{grad}f) + f\Delta(\operatorname{grad}f) + fA^2(\operatorname{grad}f) - \frac{1}{2} \operatorname{grad}(|\operatorname{grad}f|^2) \quad (63)$$

and

$$m^2c f^2H = -mfH(\Delta f) + mf^2H|A|^2 + f(\Delta^\psi(\operatorname{grad}f))^\perp - mH|\operatorname{grad}f|^2 - b(\operatorname{grad}f, \operatorname{grad}f). \quad (64)$$

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