Meromorphic Functions Partially Sharing 1CM+1IM Concerning Periodicities and Shifts

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Abstract. The purpose of this article is to deal with the uniqueness problems of meromorphic functions partially sharing values. It is showed that two entire functions f and g with $\rho_2(f) < 1$ and periodic restriction must be identically if $E(0, f(z)) = E(0, g(z))$ except for a possible set G_1 and $\overline{E}(1, f(z)) = \overline{E}(1, g(z))$ except for a possible set G_2 with $N(r, G_i) = O(r^\lambda)$, $(i = 1, 2)$, where $\lambda(< 1)$ is a fixed constant. This result is a generalization of some previous works of Chen in [5] and Cai and Chen in [7].

1. Introduction and main results

The paper mainly concerns the uniqueness problem of the periodic meromorphic function partially sharing "1CM + 1IM" with another meromorphic function. This problem was originated from the well-known Nevanlinna five theorem in 1920s (see, e.g., [17]-[21]), which states that two non-constant meromorphic functions must be identically if they share five values IM. Here, we say two meromorphic functions f and g share a IM if $f - a$ and $g - a$ have the same zeros or $\overline{E}(a, f) = \overline{E}(a, g)$. And the notation $\overline{E}(a, f)$ denotes the set of all the zeros of $f(z) - a$, where a zero is counted one time. In addition, we say f and g share a CM if $f - a$ and $g - a$ has the same zeros with multiplicities or $E(a, f) = E(a, g)$. And the notation $E(a, f)$ denotes the set of all the zeros of $f(z) - a$, where a zero with multiplicity m is counted m times. Later on, Nevanlinna gave the famous four values theorem. Since then, the uniqueness theory of meromorphic functions has become an extensive subfield of the value distribution theory. Many researchers focused their efforts on the aspect (see, e.g., [9, 10]). In 1989, in his Ph.D thesis, Brosch considered the uniqueness problem in another direction. He firstly studied the periodicity relationship of two meromorphic functions if they shared three values CM. In fact, Brosch derived that if a periodic meromorphic function f shares three values CM with another meromorphic function g, then g is also periodic. In Brosch's work, the forms of f and g were not given. So, it becomes an interesting work to seek the forms of f and g. The work has been done by Zheng in [23]. More precisely, Zheng derived the following result.

Theorem A. (see [23]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with the same period $c(\neq 0)$. Suppose that $\mu_2(f) < 1$. If $f(z)$ and $g(z)$ share $0, 1, \infty$ CM, then $f(z) \equiv g(z)$ or $f(z)$ and $g(z)$ assume

the following form $f(z) = \frac{e^{a_1 z + b_1} - 1}{e^{a_2 z + b_2} - 1}$, $g(z) = \frac{e^{-a_1 z - b_1} - 1}{e^{-a_2 z - b_2} - 1}$, where $a_1 = \frac{2m\pi i}{c}$, $a_2 = \frac{2k\pi i}{c}$, b_1, b_2 are constants, and m, k are integers.

Recently, the difference analogues to Nevanlinna's theory was established by Halburd and Korhonen [14, 15], Chiang and Feng [8], independently, and improved by Halburd, Korhonen, and Tohge [16] from finite order of meromorphic functions to infinite order (hyper-order strictly less than 1). And it becomes a powerful theoretical tool to study the uniqueness problems of meromorphic functions taking into account their shifts (see, e.g., [3, 12, 13]) or difference operators (see, e.g., [18, 22]), and so on. Due to the difference analogues to Nevanlinna's theory, Chen and his co-worker further discussed the above uniqueness problem in [4]. And the follow-up work was also due to Chen. In 2017, he derived the following theorem.

Theorem B. (see [5, Theorem 1.3]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with the same period $c(\neq 0)$. Suppose that $\rho_2(f) < 1$. If $f(z)$ and $g(z)$ share $0, \infty$ CM and 1 IM, then either (i) $f(z) \equiv g(z)$; or (ii) $f(z) = e^{az+b}g(z)$ and $\mu(f) = \mu(g) = 1$, where $a = \frac{2k\pi i}{c}$, b are constants, and k is some integer.

It is pointed out that for entire functions, Chen in [6] weakened the shared values condition "2CM + 1IM" to that of "1CM + 2IM" in his PhD thesis. Later, Cai and Chen further improved Chen's result by weakening the condition "1CM + 2IM" to that of "1CM + 1IM" as follows.

Theorem C. (see[7, Theorem 1.6.]) Let $f(z)$ and $g(z)$ be two non-constant entire functions. Let $c \in \mathbb{C}\backslash\{0\}$, and let a_1, a_2 be two distinct finite complex numbers. Suppose that $\mu(f) \neq 1$, $\rho_2(f) < 1$, and $f(z) = f(z + c)$ for all $z \in \mathbb{C}$. If $f(z)$ and $g(z)$ share a_1 CM, a_2 IM, then $f(z) \equiv g(z)$.

By studying the above theorem, one may ask whether the conclusion of Theorem C still holds or not if the condition $E(a_1, f) = E(a_1, g)$ is weakened to $E(a_1, f)\backslash G = E(a_1, g)\backslash G$, where $G \subseteq E(a_1, f) \cup E(a_1, g)$ is a set such that $div_{f-a_1}(z) \neq div_{g-a_1}(z)$ for $z \in G$. Here we will use $div_f(z)$ to denote the multiplicity of a zero z of f. (It is pointed out $div_f(z)$ may be zero, which implies that z is not a zero of f). We emphasize that the point z in G is counted $\max\{div_{f-a_1}(z), div_{g-a_1}(z)\}$ times. Below, we also need the notation $\overline{E}(a_1, f)\backslash G = \overline{E}(a_1, g)\backslash G_1$, where $G_1 \subseteq E(a_1, f) \cup E(a_1, g)$ is a set such that either $div_{f-a_1}(z) = 0$ or $div_{g-a_1}(z) = 0$ for $z \in G_1$. And the point z in G_1 is also counted $\max\{div_{f-a_1}(z), div_{g-a_1}(z)\}$ times.

We agree to say $E(a_1, f) = E(a_1, g)$ (resp $\overline{E}(a_1, f) = \overline{E}(a_1, g)$) allowed the exceptional set G (resp. G_1) if $E(a_1, f)\backslash G = E(a_1, g)\backslash G$ (resp. $\overline{E}(a_1, f)\backslash G = \overline{E}(a_1, g)\backslash G_1$) holds. The size of the set G is measured by the function in $n(r, G)$, the number of these points in $G \cap \{z : |z| < r\}$ counted with multiplicities. And denote by $N(r, G)$ (resp. $\overline{N}(r, G)$) the counting function (resp. the reduced counting function) of the set G. In the paper, we pay attention to the above problem and give an affirmative answer to it. More specifically, we prove the following.

Theorem 1. Let $f(z)$ and $g(z)$ be two non-constant entire functions, and let $c \in \mathbb{C}\backslash\{0\}$. Suppose that $\mu(f) \neq 1$, $\rho_2(f) < 1$, and $f(z) = f(z + c)$ for all $z \in \mathbb{C}$. If $f(z)$ and $g(z)$ satisfy $E(0, f(z)) = E(0, g(z))$ except for a possible set G_1 and $\overline{E}(1, f(z)) = \overline{E}(1, g(z))$ except for a possible set G_2, where $N(r, G_i) = O(r^\lambda)$, $(i = 1, 2)$ and $\lambda(< 1)$ is a constant, then $f(z) \equiv g(z)$.

As a matter of fact, we get a more general result, and Theorem 1 is a special case of it.

Theorem 2. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that $N(r, f) = O(r^\lambda)$, $N(r, g) = O(r^\lambda)$, where $\lambda(< 1)$ is a constant, and let $c \in \mathbb{C}\backslash\{0\}$. Suppose that $\mu(f) \neq 1$, $\rho_2(f) < 1$, and $f(z) = f(z + c)$ for all $z \in \mathbb{C}$. If $f(z)$ and $g(z)$ satisfy $E(0, f(z)) = E(0, g(z))$ except for a possible set G_1 and $\overline{E}(1, f(z)) = \overline{E}(1, g(z))$ except for a possible set G_2, where $N(r, G_i) = O(r^\lambda)$, $(i = 1, 2)$, then $f(z) \equiv g(z)$.

Clearly, Theorem 1 and 2 are generalizations of Theorem C. In order to prove the above theorems, we

need the following theorem, which is of its own interest.

Theorem 3. Let $f(z)$ and $f(z+c)$ be two non-constant meromorphic functions of $\rho_2(f) < 1$ and $N(r, f) = S(r, f)$. If $f(z)$ and $f(z + c)$ satisfy $E(0, f(z)) \subseteq E(0, f(z + c))$ except for a possible set G_1 and $\overline{E}(1, f(z)) \subseteq \overline{E}(1, f(z + c))$ except for a possible set G_2, where $N(r, G_i) = S(r, f)$, $(i = 1, 2)$, then $f(z) = f(z + c)$.

Remark 1. The condition $E(0, f(z)) \subseteq E(0, f(z + c))$ except for a possible set G_1 should be understood as above, where $G_1 \subseteq E(0, f(z))$ is a set such that $div_f(z) > div_g(z)$ for $z \in G_1$. And the point z in G_1 is counted $div_f(z)$ times. We emphasise that the conclusion of Theorem 3 is still valid if the values $0, 1$ are replaced by two distinct periodic functions a_1 and a_2 with a period of c and $T(r, a_j) = S(r, f)$. In fact, we only need to make the transformation $F(z) = \frac{f(z) - a_1}{a_2 - a_1}$.

Remark 2. There are many results which are related with Theorem 3. It was Heittokangas et. al. in [13] who firstly studied the uniqueness problem of $f(z)$ and $f(z + c)$ when they shared three distinct functions a_i $(i = 1, 2, 3)$ with period c, and $T(r, a_j) = S(r, f)$. Later on, Heittokangas et. al. [12] improved the uniqueness result by replacing the condition "3CM" with "2CM + 1IM". Recently, Chen in [6] further generalized the above theorems with the idea of partially sharing values. We say $f(z)$ and $f(z + c)$ partially share 1 CM if $E(1, f(z)) \subseteq E(1, f(z+c))$. Clearly, Theorem 3 is an improvement of Chen's result in some sense.

Before to proceed, we spare the reader for a moment and assume his/her familiarity with the basics of Nevanlinna's theory of meromorphic functions in $\mathbb{C}$ such as the *first* and *second* fundamental theorems, and the usual notations such as the *characteristic function* $T(r, f)$, the *proximity function* $m(r, f)$ and the *counting function* $N(r, f)$. $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, except possibly on a set of finite logarithmic measure - not necessarily the same at each occurrence (see e.g. [17, 19, 21]). We also need the following definition.

Definition 1.1 The order $\rho(f)$, hyper-order $\rho_2(f)$, lower order $\mu(f)$ and low hyper-order $\mu_2(f)$ of the meromorphic function $f(z)$ are defined in turn as follows:

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \quad \rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r},$$

$$\mu(f) = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}, \quad \mu_2(f) = \liminf_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

Proof of Theorem 3. The proof is based on the idea of Chen in [6]. For the convenience of the reader, we present our proof in all detail. On the contrary, suppose that $f(z+c) \not\equiv f(z)$. We below derive a contradiction. Set

$$\frac{f(z + c)}{f(z)} = H. \tag{1}$$

If $H \equiv 1$, then $f(z + c) = f(z)$, a contradiction. So, $H \not\equiv 1$. By the logarithmic derivative lemma, one has

$$m(r, H) = m\left(r, \frac{f(z + c)}{f(z)}\right) = S(r, f). \tag{2}$$

Moreover, by the condition that $E(0, f(z)) \subset E(0, f(z + c))$ except for a possible set G_1, one has

$$N(r, H) \le N(r, G_1) + N(r, f(z + c)) \le N(r, G_1) + N(r, f(z)) + S(r, f) = S(r, f), \tag{3}$$

since $N(r, f(z+c)) \leq N(r, f(z)) + S(r, f)$ if $\rho_2(f) < 1$, which can be seen in [16, Lemma 8.3]. Therefore, combining (2) and (3) yields that $T(r, H) = m(r, H) + N(r, H) = S(r, f)$. Further, the assumption that $\overline{E}(1, f(z)) \subset \overline{E}(1, f(z+c))$ except for a possible set G_2 implies that

$$\overline{N}(r, \frac{1}{f(z)-1}) \leq \overline{N}(r, G_2) + \overline{N}(r, \frac{1}{H(z)-1}) \leq \overline{N}(r, G_2) + T(r, H) \quad \leq T(r, H(z)) + S(r, f) = S(r, f). \tag{4}$$

By using the fact $\overline{N}(r, \frac{1}{f(z+c)}) \leq \overline{N}(r, \frac{1}{f(z)}) + S(r, f)$ in [16, Lemma 8.3], we have

$$\overline{N}(r, \frac{1}{f(z+c)-1}) \leq \overline{N}(r, \frac{1}{f(z)-1}) + S(r, f) = S(r, f). \tag{5}$$

Rewrite (1) to

$$f(z+c) - 1 = H(f(z) - \frac{1}{H}).$$

This means

$$\overline{N}(r, \frac{1}{f(z+c)-1}) = \overline{N}(r, \frac{1}{f(z)-\frac{1}{H}}) + S(r, f). \tag{6}$$

Formulas (5) and (6) illustrate that

$$\overline{N}(r, \frac{1}{f(z)-\frac{1}{H}}) = S(r, f).$$

According to the second fundamental theorem, one has that

$$T(r, f(z)) \leq \overline{N}(r, f(z)) + \overline{N}(r, \frac{1}{f(z)-1}) + \overline{N}(r, \frac{1}{f(z)-\frac{1}{H}}) + S(r, f) \leq S(r, f),$$

which is impossible. So $f(z+c) \equiv f(z)$.

Thus, the proof is finished.

To end this section, we give another proof of Theorem 3, which may be a little complicated. But, it has its own interest. We firstly introduce the following auxiliary function, which can be found in [7].

$$\phi = [f(z) - f(z+c)][\frac{f'(z)}{f(z)(f(z)-1)} - \frac{f'(z+c)}{f(z+c)(f(z+c)-1)}]. \tag{7}$$

Suppose that $\phi \not\equiv 0$. By the logarithmic derivative lemma again, we can obtain

$$\begin{aligned}
m(r, \phi) &= m(r, (f(z) - f(z+c))(\frac{f'(z)}{f(z)(f(z)-1)} - \frac{f'(z+c)}{f(z+c)(f(z+c)-1)})) \\
&\leq m(r, \frac{(f(z)-f(z+c))f'(z)}{f(z)(f(z)-1)}) + m(r, \frac{(f(z)-f(z+c))f'(z+c)}{f(z+c)(f(z+c)-1)}) + O(1) \\
&\leq m(r, \frac{f(z)-f(z+c)}{f(z)}) + m(r, \frac{f'(z)}{f(z)-1}) \\
&\quad + m(r, \frac{f(z)-f(z+c)}{f(z+c)}) + m(r, \frac{f'(z+c)}{f(z+c)-1}) + O(1) = S(r, f).
\end{aligned} \tag{8}$$

According to (7), we can get two properties of ϕ by simple calculation, which can be found in [7].

Property 1. For $\eta_0 \in \mathbb{C}$, if η_0 is a zero of both $f(z)$ and $f(z + c)$ with multiplicity p, then η_0 is a zero of ϕ with multiplicity at least p;

Property 2. For $\eta_1 \in \mathbb{C}$, if $f(\eta_1) = f(\eta_1 + c) = 1$, then $\phi(\eta_1) \neq \infty$.

Now, we define some sets as follows.

$$E_1 = \{z : f(z) = 0, z \notin G_1\}, \quad E_2 = \{z : f(z) = 0, z \in G_1\},$$
$$E_3 = E(0, f(z + c)) \backslash E_1, \text{counting multiplicities}.$$

By [Lemma 8.3] in [16] again and the condition that $E(0, f(z)) \subset E(0, f(z + c))$ except for a possible set G_1, we have

$$N(r, \frac{1}{f(z + c)}) = N(r, E_1) + N(r, E_3)$$

$$\leq N(r, \frac{1}{f(z)}) + S(r, f) = N(r, E_1) + N(r, E_2) + S(r, f)$$

$$\leq N(r, E_1) + N(r, G_1) + S(r, f) = N(r, E_1) + S(r, f),$$

which implies that $N(r, E_3) = S(r, f)$. Set

$$\alpha_1 = \{z : f(z) = 1, z \notin G_2\} \quad \alpha_2 = \{z : f(z) = 1, z \in G_2\},$$
$$\alpha_3 = E(1, f(z + c)) \backslash \alpha_1, \text{ignoring multiplicities}.$$

The similar proceed leads that $\overline{N}(r, \alpha_3) = S(r, f)$. Therefore, all the above discussions yields that

$$\begin{aligned} N(r, \phi) &\leq N(r, f) + N(r, f(z + c)) + N(r, G_1) + N(r, G_2) \\ &\quad + \overline{N}(r, \alpha_3) + \overline{N}(r, E_3) = S(r, f). \end{aligned} \tag{9}$$

From (8) and (9), we can get $T(r, \phi) = m(r, \phi) + N(r, \phi) = S(r, f)$.
Moreover, the property 1 shows that

$$N(r, \frac{1}{f(z)}) \leq N(r, G_1) + N(r, \frac{1}{\phi}) \leq T(r, \phi) + S(r, f) = S(r, f). \tag{10}$$

From (4), (10) and $N(r, f) = S(r, f)$, we can deduced by the second fundamental theorem that

$$T(r, f(z)) \leq \overline{N}(r, \frac{1}{f(z) - 1}) + \overline{N}(r, \frac{1}{f(z)}) + \overline{N}(r, f(z)) + S(r, f) = S(r, f),$$

which is a contradiction. Therefore, $\phi \equiv 0$ and

$$\frac{f'(z)}{f(z)(f(z) - 1)} = \frac{f'(z + c)}{f(z + c)(f(z + c) - 1)}. \tag{11}$$

We next prove that $f(z)$ and $f(z + c)$ share 0 CM. Suppose that η_0 is a zero of $f(z)$ of multiplicity p. The above equation yields that η_0 must be a zero of $f(z + c)$ or $f(z + c) - 1$. Suppose η_0 is a zero of $f(z + c) - 1$ of multiplicity q. Then, the Laurent expansions of $f(z)$ and $f(z + c) - 1$ at η_0 are as follows.

$$f(z) = s_p(z - \eta_0)^p + s_{p+1}(z - \eta_0)^{p+1} + s_{p+2}(z - \eta_0)^{p+2} + \cdots, \tag{12}$$

$$f(z + c) - 1 = t_q(z - \eta_0)^q + t_{q+1}(z - \eta_0)^{q+1} + t_{q+2}(z - \eta_0)^{q+2} + \cdots, \tag{13}$$

where s_i $(i = p, p + 1, \cdots)$ and t_i $(i = q, q + 1, \cdots)$ are finite complex numbers with $s_p \neq 0$ and $t_q \neq 0$. Further, we get

$$f'(z) = ps_p(z - \eta_0)^{p-1} + (p + 1)s_{p+1}(z - \eta_0)^p + (p + 2)s_{p+2}(z - \eta_0)^{p+1} + \cdots, \tag{14}$$

$$f'(z + c) = qt_q(z - \eta_0)^{q-1} + (q + 1)t_{q+1}(z - \eta_0)^q + (q + 2)t_{q+2}(z - \eta_0)^{q+1} + \cdots, \tag{15}$$

Combining (12) and (14) yields that

$$
\begin{aligned}
&\frac{f'(z)}{f(z)(f(z) - 1)} \\
&= \frac{1}{(0 - 1)} \cdot \frac{ps_p(z - \eta_0)^{p-1} + (p + 1)s_{p+1}(z - \eta_0)^p + (p + 2)s_{p+2}(z - \eta_0)^{p+1} + \cdots}{s_p(z - \eta_0)^p + s_{p+1}(z - \eta_0)^{p+1} + s_{p+2}(z - \eta_0)^{p+2} + \cdots} \\
&= \frac{p}{(0 - 1)(z - \eta_0)} + \frac{s_{p+1}}{s_p(0 - 1)} + O(z - \eta_0).
\end{aligned}
\tag{16}
$$

On the other hand, by (13) and (15) we get

$$
\begin{aligned}
&\frac{f'(z + c)}{f(z + c)(f(z + c) - 1)} \\
&= \frac{1}{(1 - 0)} \cdot \frac{qt_q(z - \eta_0)^{q-1} + (q + 1)t_{q+1}(z - \eta_0)^q + (q + 2)t_{q+2}(z - \eta_0)^{q+1} + \cdots}{t_q(z - \eta_0)^q + t_{q+1}(z - \eta_0)^{q+1} + t_{q+2}(z - \eta_0)^{q+2} + \cdots} \\
&= \frac{q}{(1 - 0)(z - \eta_0)} + \frac{t_{q+1}}{t_q(1 - 0)} + O(z - \eta_0).
\end{aligned}
\tag{17}
$$

Combining (16) and (17) yields $-p = q$, which is impossible. So, η_0 must be a zero of $f(z + c)$. Then, repeating the above argument can show that the multiplicity of $f(z + c)$ at the point η_0 is p, which implies that $E(0, f(z)) \subseteq E(0, f(z + c))$. Similarly, one can obtain $E(0, f(z + c)) \subseteq E(0, f(z))$. Therefore, $f(z)$ and $f(z + c)$ share 0 CM. Furthermore, we can derive that $f(z)$ and $f(z + c)$ share $1, \infty$ CM. We omit the proof here. Then, from [13, Theorem 2.1], we can show that $f(z) \equiv f(z + c)$, a contradiction.

Thus, we finish the proof of Theorem 3.

2. Proof of Theorem 2

In this section, we shall prove the theorem 2. Before to its proof, we first give the following results, where the first one is Theorem 5.1 of Halburd-Korhonen-Tohge in [16], the second one is Lemma 3.3 of Bergweiler and Langley in [2].

Lemma 1. Let $f(z)$ be a meromorphic function of of hyper order $\rho_2 < 1$, and $\eta \in \mathbb{C}$. Then for each $\varepsilon > 0$, we have

$$m(r, \frac{f(z + \eta)}{f(z)}) = o(\frac{T(r, f)}{r^{1 - \rho_2 - \varepsilon}}) = S(r, f),$$

for all r outside of a set of finite logarithmic measure.

Lemma 2. Let $g(z)$ be a non-constant meromorphic function in the plane of order less than 1, and let $h > 0$. Then there exists an ε-set E such that

$$\frac{g(z + \eta)}{g(z)} \to 1, \text{ as } z \to \infty \text{ in } \mathbb{C} \backslash E,$$

uniformly in η for $|\eta| < h$.

Proof of Theorem 2 The proof is based on the ideas in [5, 7]. On the contrary, suppose that $f(z) \not\equiv g(z)$. We below derive a contradiction. Note that $f(z)$ is a non-constant periodic meromorphic function. Then

$\mu(f) \geq 1$. The fact can be found in [21, Lemma 5.1]. Together with $N(r, f) = O(r^\lambda)$, one has $N(r, f) = S(r, f)$. By the second fundamental theorem and the assumptions of Theorem 2, we have

$$
\begin{aligned}
T(r, f) &\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-1}) + \overline{N}(r, f) + S(r, f) \\
&= \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g-1}) + \overline{N}(r, G_1) + \overline{N}(r, G_2) + S(r, f) \\
&\leq T(r, \frac{1}{g}) + T(r, \frac{1}{g-1}) + S(r, f) \\
&= 2T(r, g) + S(r, f) \quad (r \to \infty, r \notin E_0, \ mesE_0 < \infty).
\end{aligned}
\tag{18}
$$

Similarly, one has that

$$
T(r, g) \leq 2T(r, f) + S(r, g) \quad (r \to \infty, r \notin E_1, \ mesE_1 < \infty). \tag{19}
$$

Combining (18) and (19) yields

$$
1 < \mu(g) = \mu(f) \leq \rho(g) = \rho(f), \quad \rho_2(g) = \rho_2(f) < 1. \tag{20}
$$

For convenience, we set $S(r) := S(r, f) = S(r, g)$. By Hadamard factorization theorem, we can assume that

$$
\frac{f(z)}{g(z)} = z^k \frac{\pi_1(z)}{\pi_2(z)} e^{\alpha(z)} = P(z)e^{\alpha(z)}, \tag{21}
$$

where k is an integer, $\alpha(z)$ is an entire function, and $\pi_1(z), \pi_2(z)$ are the canonical products of $\frac{f(z)}{g(z)}$ formed with the non-null zeros and poles of $f(z)$ and $g(z)$, respectively. Obviously, by (21), one has

$$
\frac{f(z+c)}{g(z+c)} = (z+c)^k \frac{\pi_1(z+c)}{\pi_2(z+c)} e^{\alpha(z+c)} = P(z+c)e^{\alpha(z+c)}. \tag{22}
$$

We claim that $g(z) \equiv g(z+c)$, which implies that g is also a periodic function with period c. Observe that $E(0, f(z)) = E(0, g(z))$ except for a possible set G_1 with $N(r, G_1) = O(r^\lambda)$. So, there exists a set G_3 such that $N(r, G_3) = O(r^\lambda)$ and $E(0, f(z+c)) = E(0, g(z+c))$ except for a possible set G_3. In view of the periodic function $f(z)$, one has $E(0, f(z)) = E(0, f(z+c))$. All the above discussion shows that $E(0, g(z)) = E(0, g(z+c))$ except for a possible set $G_4 = G_1 \cup G_3$. Clearly, $N(r, G_4) = O(r^\lambda)$. Applying the same argument to the set of $\overline{E}(1, g(z))$ and $\overline{E}(1, g(z+c))$, one obtains that $\overline{E}(0, g(z)) = \overline{E}(0, g(z+c))$ except for a possible set G_5 with $N(r, G_5) = O(r^\lambda)$. Plus the fact $N(r, g) = O(r^\lambda)$, we see that the function $g(z)$ and $g(z+c)$ satisfy all the conditions of Theorem 3, which implies that $g(z) \equiv g(z+c)$ and the claim is right. Furthermore, combining (21) and (22) yields that

$$
e^{\alpha(z)-\alpha(z+c)} = \frac{P(z+c)}{P(z)}.
$$

By the well-know result due to Borel (see [21, Theorem 2.3]), we know that the order $\rho(\pi_j)$ of $\pi_j(z)$ $(j = 1, 2)$ is equal to the exponent of convergence $\lambda(\pi_j)$ of the zeros of $\pi_j(z)$ $(j = 1, 2)$. Again from (21), we see that all the zeros of π_1(resp. π_2) come from the points of G_1 and the poles of $g(z)$ (resp $f(z)$). Thus, we can deduce that

$$
\rho(\pi_1) = \lambda(\pi_1) = \limsup_{r\to\infty} \frac{\log N(r, \frac{1}{\pi_1})}{\log r} = \limsup_{r\to\infty} \frac{\log[N(r, g) + N(r, G_1)]}{\log r} \leq \lambda < 1,
$$

$$
\rho(\pi_2) = \lambda(\pi_2) = \limsup_{r\to\infty} \frac{\log N(r, \frac{1}{\pi_2})}{\log r} = \limsup_{r\to\infty} \frac{\log[N(r, f) + N(r, G_1)]}{\log r} \leq \lambda < 1.
$$

The form $P(z) = z^k \frac{\pi_1(z)}{\pi_2(z)}$ yields that $\rho(P(z)) < 1$. Together with Lemma 1, we get

$$T(r, e^{\alpha(z)-\alpha(z+c)}) = m(r, e^{\alpha(z)-\alpha(z+c)}) = m(r, \frac{P(z+c)}{P(z)}) = O(r^{\rho(P)-1+\varepsilon}) = O(r^\varepsilon),$$

which implies that $\alpha(z) - \alpha(z + c)$ must be a constant. Let $\alpha(z) - \alpha(z + c) \equiv h$ for some constant $h \in \mathbb{C}$. Then

$$e^h = \frac{P(z+c)}{P(z)},$$

and $\alpha'(z) - \alpha'(z + c) \equiv 0$. Consequently $\alpha'(z)$ is a periodic function with period c. Suppose that $\alpha'(z)$ is not constant. Then, $\rho(\alpha(z)) = \rho(\alpha'(z)) \geq 1$. On the other hand, by (21), we have $\rho(\alpha(z)) = \rho(\alpha'(z)) \leq \rho_2(f) < 1$, a contradiction. Therefore, $\alpha'(z)$ is constant. Then, $\alpha(z)$ is a linear function, say $\alpha(z) = az + b$ with constants $a, b \in \mathbb{C}$. In view of $\rho(P(z)) < 1$ and Lemma 2, there exists a set E of finite logarithmic measure such that for $|z| = r \notin E$,

$$e^h = \frac{P(z)}{P(z+c)} \to 1, \ as \ |z| \to \infty.$$

Thus $e^h = 1$ and $P(z) = P(z + c)$. The fact $\rho(P(z)) < 1$ forces that P is a constant, say A. We rewrite (21) as

$$\frac{f(z)}{g(z)} = Ae^{\alpha(z)} = Ae^{az+b}. \tag{23}$$

In the following, we will prove that

$$N(r, \frac{1}{f-1}) = N(r, \frac{1}{g-1}) = O(r). \tag{24}$$

We consider into two cases.

Case 1. $a = 0$.

Then, $\frac{f(z)}{g(z)} = Ae^b$ is a constant. If there exists a point z_0 such that $f(z_0) = g(z_0) = 1$, then substituting z_0 into the above equation leads to $f(z) \equiv g(z)$, a contradiction. Thus, $f - 1$ and $g - 1$ has no common zeros. Therefore, $E(1, f) \subseteq G_2$ and $E(1, g) \subseteq G_2$, which shows that

$$N(r, \frac{1}{f-1}) \leq N(r, G_2) = O(r^\lambda) = O(r), \ N(r, \frac{1}{g-1}) \leq N(r, G_2) = O(r^\lambda) = O(r).$$

Case 2. $a \neq 0$.

In this case, we will employ a result of Chen in [5], which is stated as follows.

Proposition. Suppose that $f(z) = f(z + c)$, $g(z) = g(z + c)$ and $\frac{f(z)}{g(z)} = Ae^{az+b}$, where $a \neq 0$ and b are constant. If the point a_0 is common zero of $f - 1$ and $g - 1$ with multiplicities p and q, respectively, then there exists a positive integer M (which is independent of a_0) such that $p, q \leq M$.

Suppose that b_0 is common zero of $f - 1$ and $g - 1$, then $Ae^{az+b}|_{b_0} = 1$. By the above proposition, one has

$$N(r, \frac{1}{f-1}) \leq MN(r, \overline{E}(1, f) \backslash G_2) + N(r, G_2) \leq MN(r, \frac{1}{Ae^{az+b}-1}) + O(r^\lambda) = O(r).$$

Similarly, we can deduce $N(r, \frac{1}{g-1}) = O(r)$. From the cases 1 and 2, we prove the formula (24) holds. In addition, we set

$$\frac{f(z)-1}{g(z)-1} = z^n \frac{\pi_3(z)}{\pi_4(z)} e^{\beta(z)} = Q(z)e^{\beta(z)}, \tag{25}$$

where n is an integer, $\beta(z)$ is an entire function, and $\pi_3(z)$, $\pi_4(z)$ are the canonical products of $\frac{f(z)-1}{g(z)-1}$ formed with the non-null zeros and poles of $f(z) - 1$ and $g(z) - 1$, respectively.

It follows from (25) that

$$\frac{f(z+c)-1}{g(z+c)-1} = (z+c)^n \frac{\pi_3(z+c)}{\pi_4(z+c)} e^{\beta(z+c)} = Q(z+c)e^{\beta(z+c)}. \tag{26}$$

Combining (25) and (26) yields

$$e^{\beta(z)-\beta(z+c)} = \frac{Q(z+c)}{Q(z)}. \tag{27}$$

Consequently, using the same argument as in the proof of $\rho(P(z)) < 1$, we have

$$\rho(\pi_3) = \lambda(\pi_3) = \limsup_{r\to\infty} \frac{\log N(r, \frac{1}{\pi_3})}{\log r} \leq \limsup_{r\to\infty} \frac{\log[N(r,g) + N(r,\frac{1}{f-1})]}{\log r} \leq 1,$$

$$\rho(\pi_4) = \lambda(\pi_4) = \limsup_{r\to\infty} \frac{\log N(r, \frac{1}{\pi_4})}{\log r} \leq \limsup_{r\to\infty} \frac{\log[N(r,f) + N(r,\frac{1}{g-1})]}{\log r} \leq 1.$$

The form $Q(z) = z^n \frac{\pi_3(z)}{\pi_4(z)}$ implies that $\rho(Q(z)) \leq 1$. The same proceed as above can show that $\beta(z) = c_1 z + d$ with two constants c_1, d. Further,

$$\frac{f(z)-1}{g(z)-1} = Q(z)e^{c_1 z + d}. \tag{28}$$

Combining (23) and (28) gives that

$$f(z) = \frac{-Q(z)e^{c_1 z + d} + 1}{1 - Q(z)A^{-1}e^{(c_1-a)z+d-b}}. \tag{29}$$

Then it follows from (29) that

$$\rho(f) \leq max\{\rho(Q), \rho(e^{c_1 z + d}), \rho(e^{(c_1-a)z+d-b})\} \leq 1,$$

which contradicts the fact $\rho(f) \geq \mu(f) > 1$.

Thus, the proof is finished.

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