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On Some Vector Valued Multiplier Spaces with Statistical Cesáro Summability

Ramazan Kama[a]

[a]*Department of Mathematics and Physical Sciences Education, Faculty of Education, Siirt University, The Kezer Campus, Kezer, 56100–Siirt, Turkey*

Abstract. In the present paper we define and study some vector valued spaces within the frame of the Statistical Cesàro convergence and the Statistically Cesàro summability in normed spaces. Also, we characterize the continuous, compact and sequential continuous summing operators on Statistical Cesàro multiplier sequence spaces.

1. Introduction

The idea of statistical convergence which is independently introduced by Fast [12] and Steinhaus [26] has been studied by various mathematicians. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Connor [9], Fridy [13], Salat [24] and many others [4, 7, 8, 10, 11, 15, 16, 18–22, 25, 29].

A subset A of $\mathbb{N}$ is said to have density $\delta(A)$ if

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k)$$

exists, where χ_A is the characteristic function of A. Then, any finite set has zero natural density and $\delta(A^c) = 1 - \delta(A)$, where A^c is the complement of the set A. If a property $P(k)$ satisfies for all $k \in A$ with $\delta(A) = 1$, we say that P satisfies for "almost all k", and we abbreviate this by "a.a.k".

Let X be a real normed space. A sequence (x_k) in X is statistical convergent to $x \in X$, and we write $St - \lim_k x_k = x$, if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : \|x_k - x\| < \varepsilon\}) = 1,$$

and also a sequence (x_k) in X is weak statistical convergent to $x \in X$, and we write $wSt - \lim_k x_k = x$, if for every $\varepsilon > 0$ and every $f \in X^*$(dual of X)

$$\delta(\{k \in \mathbb{N} : |f(x_k) - f(x)| < \varepsilon\}) = 1.$$

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Email address: ramazankama@siirt.edu.tr (Ramazan Kama)

Analogously, a sequence (x_k) in X is statistical Cauchy if for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists an integer $m \geq n$ such that

$$\delta(\{k \in \mathbb{N} : \|x_k - x_m\| < \varepsilon\}) = 1.$$

The Cesàro matrix C with *Cesàro mean of order one,* which is a well-known method of summability and is defined by the matrix $C = (c_{nk})$ as follows

$$c_{nk} = \begin{cases} \frac{1}{n}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

The C-transform of a sequence $x = (x_k)$ is defined by

$$Cx = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)$$

for all $n \in \mathbb{N}$ [6]. A series $\sum_k x_k$ in X is Cesàro summing [14] if there exists $x_0 \in X$ such that $\lim_n \frac{1}{n}(S_1 + S_2 + \ldots + S_n) = x_0$, where $S_n = \sum_{k=1}^{n} x_k$. We will denote it by $C - \sum_k x_k = x_0$ and this is equivalent to

$$\lim_n \left(\frac{1}{n} \sum_{k=1}^{n} (n-k+1) x_k \right) = x_0.$$

A series $\sum_k x_k$ is weakly Cesàro summing if there exists $x_0 \in X$ such that $\lim_n \frac{1}{n} f(S_1 + S_2 + \ldots + S_n) = f(x_0)$ for each $f \in X^*$. We will denote it by $wC - \sum_k x_k = x_0$.

The consept of statistically Cesàro summing also makes sense in the above. We will say that a series $\sum_k x_k$ is statistically Cesàro summing [1] if there exists $x_0 \in X$ such that $St - \lim_n \frac{1}{n} \sum_{k=1}^{n}(n-k+1)x_k = x_0$. We will denote it by $StC - \sum_k x_k = x_0$ and this is equivalent to saying that there exists $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that $\lim_{n \in A} \left(\frac{1}{n} \sum_{k=1}^{n}(n-k+1)x_k \right) = x_0$. In the similar way, we denote $StwC - \sum_k x_k$.

The space of bounded sequences in X will be denoted as usual by $\ell_\infty(X)$. It is well known that if X is a Banach space, then $\ell_\infty(X)$ becomes a Banach space with the sup norm. The usual subspaces $\ell_\infty(X)$ are the space of evetually null sequences, of null sequences and of convergent sequences in X denoted by $c_{00}(X)$, $c_0(X)$ and $c(X)$, respectively.

Let Y be a real normed space and the usual space of continuous linear operators from X into Y will be denoted by $L(X, Y)$. If λ be a vector space of X-valued sequences and $T_i \in L(X, Y)$ for $i \in \mathbb{N}$, the series $\sum_i T_i$ is λ-multiplier convergent (Cauchy) [28] if the series $\sum_i T_i x_i$ is convergent (Cauchy) in Y for every sequence $x = (x_i) \in \lambda$. If $\lambda = \ell_\infty(X)$, a series $\sum_i T_i$ is said to be $\ell_\infty(X)-$multiplier (Cauchy) convergent, and if $\lambda = c_0(X)$, a series $\sum_i T_i$ is said to be $c_0(X)-$multiplier (Cauchy) convergent.

In [2, 3, 5, 17, 23, 27, 28], there are important results that relate the case of a convergence method to classical properties of scalar and vector-valued multiplier spaces. However, in the case of the statistical Cesàro convergence, such spaces have not yet been introduced. Only in [1] the authors give the Orlicz-Pettis theorem associated to the statistical Cesàro convergence for scalar case.

In this manuscript we introduce and study spaces of vector valued sequences defined by the statistical Cesàro convergence and statistical Cesàro summability, and following the lines suggested in the above references we characterize classical properties such us completeness, compactness, continuity, sequentially continuity in terms of the statistical Cesàro convergence.

2. Spaces of sequences defined by the statistical Cesàro convergent

In this section, we define the sets $\Gamma_{stC}(X)$ and $\Gamma_{stwC}(X)$, which are called the sets of all bounded statistical Cesàro convergent sequences and of all bounded statistical weakly Cesàro convergent sequences in X, respectively.

For a sequence $x = (x_i)$ in a normed space X, the sets $\Gamma_{stC}(X)$ and $\Gamma_{stwC}(X)$ are defined by

$$\Gamma_{stC}(X) \quad := \quad \left\{ x = (x_i) \in l_\infty(X) : stC - \lim_i x_i \text{ exists} \right\}$$

and

$$\Gamma_{stwC}(X) \quad := \quad \left\{ x = (x_i) \in l_\infty(X) : stwC - \lim_i x_i \text{ exists} \right\},$$

respectively. Since the inclusion $\Gamma_{stC}(X) \subseteq \Gamma_{stwC}(X) \subset l_\infty(X)$ is hold, we show that these spaces are complete with the sup norm.

Theorem 2.1. *Let X be a real Banach space. Then $\Gamma_{stC}(X)$ and $\Gamma_{stwC}(X)$ are closed subspaces of $l_\infty(X)$ endowed with the sup norm.*

Proof. We will only show the closedness of $\Gamma_{stwC}(X)$. Let $(x^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $\Gamma_{stwC}(X)$ and then, there exists $x^{(0)} \in l_\infty(X)$ such that $x^{(n)} \to x^{(0)}$. We will show that $x^{(0)} \in \Gamma_{stwC}(X)$. Since $(x^{(n)})_{n\in\mathbb{N}} \subset \Gamma_{stwC}(X)$, there exists $x_n \in X$ for each $n \in \mathbb{N}$ such that $stw - \lim_k \frac{1}{k} \sum_{i=1}^{k} x_i^{(n)} = x_n$. On the other hand, there exists $n_0 \in \mathbb{N}$ such that $\|x^{(p)} - x^{(q)}\|_\infty \leq \frac{\varepsilon}{3}$ for $\varepsilon > 0$ and each $p, q > n_0$. We consider a functional $\phi \in S_{X^*}$ such that $\|x_p - x_q\| = |\phi(x_p) - \phi(x_q)|$. Fix $p, q \geq n_0$, we have that

$$\left|\phi(x_p) - \frac{1}{k}(\phi(x_1^{(p)}) + \phi(x_2^{(p)}) + \phi(x_3^{(p)}) + \ldots + \phi(x_k^{(p)}))\right| \leq \frac{\varepsilon}{3} \;\; a.a.k$$

and

$$\left|\phi(x_q) - \frac{1}{k}(\phi(x_1^{(q)}) + \phi(x_2^{(q)}) + \phi(x_3^{(q)}) + \ldots + \phi(x_k^{(q)}))\right| \leq \frac{\varepsilon}{3} \;\; a.a.k.$$

If choose $k \in \mathbb{N}$, we obtain that

$$\begin{aligned} \|x_p - x_q\| &\leq \left|\phi(x_p) - \frac{1}{k}(\phi(x_1^{(p)}) + \phi(x_2^{(p)}) + \ldots + \phi(x_k^{(p)}))\right| \\ &+ \left|\frac{1}{k}(\phi(x_1^{(p)} - x_1^{(q)}) + \phi(x_2^{(p)} - x_2^{(q)}) + \ldots + \phi(x_k^{(p)} - x_k^{(q)}))\right| \\ &+ \left|\phi(x_q) - \frac{1}{k}(\phi(x_1^{(q)}) + \phi(x_2^{(q)}) + \ldots + \phi(x_k^{(q)}))\right| \\ &\leq \varepsilon. \end{aligned}$$

Since X is a Banach space, there exists $x_0 \in X$ such that $x_n \to x_0$. Finally, we will show that $stwC{-}\lim_i x_i^{(0)} = x_0$. Now, we can fix $p \in \mathbb{N}$ satisfying $\|x_0 - x_p\| \leq \frac{\varepsilon}{3}$ and $\|x^{(0)} - x^{(p)}\| \leq \frac{\varepsilon}{3}$. Since $stw - \lim_k \frac{1}{k} \sum_{i=1}^{k} x_i^{(p)} = x_p$, there exists $K \subset \mathbb{N}$ with $\delta(K) = 1$ such that for $k \in K$

$$\left|\frac{1}{k}(\phi(x_1^{(p)}) + \phi(x_2^{(p)}) + \phi(x_3^{(p)}) + \ldots + \phi(x_k^{(p)})) - \phi(x_p)\right| < \frac{\varepsilon}{3}.$$

Therefore,

$$\begin{aligned} \left|\phi(x_0) - \frac{1}{k}(\phi(x_1^{(0)}) + \phi(x_2^{(0)}) + \ldots + \phi(x_k^{(0)}))\right| &\leq |\phi(x_0) - \phi(x_p)| \\ &+ \left|\phi(x_p) - \frac{1}{k}(\phi(x_1^{(p)}) + \phi(x_2^{(p)}) + \phi(x_3^{(p)}) + \ldots + \phi(x_k^{(p)}))\right| \\ &+ \left|\frac{1}{k}(\phi(x_1^{(p)} - x_1^{(0)}) + \phi(x_2^{(p)} - x_2^{(0)}) + \ldots + \phi(x_k^{(p)} - x_k^{(0)})\right| \\ &\leq \varepsilon \end{aligned}$$

for each $k \in K$, which completes the proof. □

We give the following corollary:

Corollary 2.2. *The normed space X is a Banach space if and only if $\Gamma_{stC}(X)$ $(\Gamma_{stwC}(X))$ is a Banach space.*

Proof. It is easy to check that if $\Gamma_{stC}(X)$ $(\Gamma_{stwC}(X))$ is complete, then X is closed subspaces of $l_\infty(X)$. □

In the following remark, we characterize the statistical weakly Cesàro convergence in $c_0(X)$ by means of the coordinate-wise statistical Cesàro convergence.

Remark 2.3. *Let X be a real normed space, M be a dense subspace in X^* and $(x_i)_{i\in\mathbb{N}} \subset c_0(X)$ with $\|x_i\| \leq L$ for each $i \in \mathbb{N}$. We take $\phi \in X^*$ and $\varepsilon > 0$. Since M is dense in X^*, there exists a $\psi \in M$ such that $\|\phi - \psi\| < \frac{\varepsilon}{2L}$. If $stC - \lim_i \psi(x_i) = 0$, then*

$$\begin{aligned} \frac{1}{k}\left|\phi\left(\sum_{i=1}^{k} x_i\right)\right| &\leq \frac{1}{k}\left(\left|(\phi-\psi)\left(\sum_{i=1}^{k} x_i\right)\right| + \left|\psi\left(\sum_{i=1}^{k} x_i\right)\right|\right) \\ &\leq \frac{1}{k}\frac{\varepsilon}{2L}kL + \frac{1}{k}\left|\psi\left(\sum_{i=1}^{k} x_i\right)\right| \\ &= \frac{\varepsilon}{2} + \frac{1}{k}\left|\psi\left(\sum_{i=1}^{k} x_i\right)\right|, \end{aligned}$$

and hence $stC - \lim_i \phi(x_i) = 0$. Since ϕ is arbitrary, we have that $stwC - \lim_i x_i = 0$.

Now, we give the consept of statistical weak* Cesàro convergence.

Definition 2.4. *We will say that a sequence $(\phi_i)_{i\in\mathbb{N}} \subset X^*$ is statistical weak* Cesàro convergent to ϕ_0 if $st - \lim_k \frac{1}{k}\sum_{i=1}^{k} \phi_i(x) = \phi_0(x)$ for each $x \in X$. We will denote by $stw^*C - \lim_i \phi_i = \phi_0$.*

If we use the previous definition, then we have

$$\Gamma_{stC}^{w^*}(X^*) := \left\{\phi = (\phi_i) \in l_\infty(X^*) : stw^*C - \lim_i \phi_i \text{ exists}\right\}.$$

Theorem 2.5. *If X is a real normed space, then $\Gamma_{stC}^{w^*}(X^*)$ is a closed subspace of $l_\infty(X^*)$.*

Proof. Let $(\phi^{(n)})_{n\in\mathbb{N}}$ be a sequence in $\Gamma_{stC}^{w^*}(X^*)$ with $\phi^{(n)} \to \phi^{(0)} \in l_\infty(X^*)$. Then, there exists $(\phi_n) \subset X^*$ such that $stw^* - \lim_k \frac{1}{k}\sum_{i=1}^{k} \phi_i^{(n)} = \phi_n$ for each $n \in \mathbb{N}$. We will check that (ϕ_n) is a Cauchy sequence. Let $\varepsilon > 0$, and since $(\phi^{(n)})$ be a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that for $p, q > n_0$ $\|\phi^{(p)} - \phi^{(q)}\| \leq \frac{\varepsilon}{6}$. If we fix $p, q \geq n_0$, then there exists a vector $x \in S_X$ such that

$$\|\phi_p - \phi_q\| - \frac{\varepsilon}{2} < |(\phi_p - \phi_q)(x)| \leq \|\phi_p - \phi_q\|.$$

Also, we have

$$\left|\phi_p(x) - \frac{1}{k}(\phi_1^{(p)}(x) + \phi_2^{(p)}(x) + \phi_3^{(p)}(x) + \ldots + \phi_k^{(p)}(x))\right| \leq \frac{\varepsilon}{6} \ a.a.k$$

and

$$\left|\phi_q(x) - \frac{1}{k}(\phi_1^{(q)}(x) + \phi_2^{(q)}(x) + \phi_3^{(q)}(x) + \ldots + \phi_k^{(q)}(x))\right| \leq \frac{\varepsilon}{6} \ a.a.k.$$

Therefore, we can choose $k \in \mathbb{N}$ such that

$$\begin{aligned} \|\phi_p - \phi_q\| - \frac{\varepsilon}{2} &\leq \left|\phi_p(x) - \frac{1}{k}(\phi_1^{(p)}(x) + \phi_2^{(p)}(x) + \phi_3^{(p)}(x) + \ldots + \phi_k^{(p)}(x))\right| \\ &+ \left|\frac{1}{k}((\phi_1^{(p)} - \phi_1^{(q)})(x) + (\phi_2^{(p)} - \phi_2^{(q)})(x) + \ldots + (\phi_k^{(p)} - \phi_k^{(q)})(x))\right| \\ &+ \left|\phi_q(x) - \frac{1}{k}(\phi_1^{(q)}(x) + \phi_2^{(q)}(x) + \phi_3^{(q)}(x) + \ldots + \phi_k^{(q)}(x))\right| \\ &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Thus, (ϕ_n) is a Cauchy sequence in X^* and hence there exists $\phi_0 \in X^*$ such that $\phi_n \to \phi_0$. Now, we will check that $stw^*C - \lim_{k\to\infty} \phi_k^{(0)} = \phi_0$. Then we can fix $p \in \mathbb{N}$ satisfying $\|\phi^{(0)} - \phi^{(p)}\| \leq \frac{\varepsilon}{3}$ and $\|\phi_p(x) - \phi_0(x)\| \leq \frac{\varepsilon}{3}$. Also, since $stw^*C - \lim_{k\to\infty} \phi_k^{(p)} = \phi_p$, there exists a $K \subset \mathbb{N}$ with $\delta(K) = 1$ such that

$$\left|\phi_p(x) - \frac{1}{k}(\phi_1^{(p)}(x) + \phi_2^{(p)}(x) + \ldots + \phi_k^{(p)}(x))\right| \leq \frac{\varepsilon}{3}.$$

Thus, for each $k \in K$, it follows that

$$\begin{aligned} \left|\phi_0(x) - \frac{1}{k}(\phi_1^{(0)}(x) + \phi_2^{(0)}(x) + \ldots + \phi_k^{(0)}(x))\right| &\leq |\phi_0(x) - \phi_p(x)| \\ &+ \left|\phi_p(x) - \frac{1}{k}(\phi_1^{(p)}(x) + \phi_2^{(p)}(x) + \ldots + \phi_k^{(p)}(x))\right| \\ &+ \left|\frac{1}{k}((\phi_1^{(p)} - \phi_1^{(0)})(x) + \ldots + (\phi_k^{(p)} - \phi_k^{(0)})(x))\right| \\ &\leq \varepsilon, \end{aligned}$$

and hence $\phi^{(0)} \in \Gamma_{stC}^{w^*}(X^*)$. □

The following remark shows that in $\ell_1(X^*)$ the statistical weakly Cesàro convergence can be characterized by means of the coordinate-wise statistical Cesàro convergence.

Remark 2.6. *As in Remark 2.3, X be a real normed space, M be a dense subspace of X and $(\phi_k)_{k\in\mathbb{N}}$ be a sequence in $\ell_1(X^*)$. If $stC - \lim_{k\to\infty} \phi_k(z) = 0$ for each $z \in M$, then we obtain that $stw^*C - \lim_{k\to\infty} \phi_k = 0$.*

3. Spaces of sequences defined by statistical Cesàro summability

Let $\sum_i T_i$ be a series in $L(X, Y)$. We will let $\Gamma_{StC}^{\infty}(\sum_i T_i)$ denote the vector valued multiplier space of statistical Cesàro convergent of the series $\sum_i T_i$, that is,

$$\Gamma_{StC}^{\infty}(\sum_i T_i) = \left\{x = (x_i) \in \ell_\infty(X) : StC - \sum_{i=1}^{k} T_i x_i \text{ exists}\right\}.$$

Likewise we can also consider the vector valued multiplier space of statistical weakly Cesàro convergent of the series $\sum_i T_i$, that is,

$$\Gamma_{StwC}^{\infty}(\sum_i T_i) = \left\{x = (x_i) \in \ell_\infty(X) : StwC - \sum_{i=1}^{k} T_i x_i \text{ exists}\right\}.$$

Notice that $\Gamma_{StC}^{\infty}(\sum_i T_i) \subset \Gamma_{StwC}^{\infty}(\sum_i T_i) \subset \ell_\infty(X)$. Also, we give a characterization of $c_0(X)$−multiplier Cauchy series obtained in [28].

Proposition 3.1. *The series $\sum_i T_i$ is $c_0(X)$−multiplier Cauchy if and only if the set $E = \left\{\sum_{i=1}^{n} T_i x_i : \|x_i\| \leq 1, n \in \mathbb{N}\right\}$ is bounded.*

A natural problem to wonder is on the completeness of the previous two spaces. We take care of this in the following results.

Theorem 3.2. *Let X and Y are normed spaces and $\sum_i T_i$ is a series in $L(X,Y)$. If*

(i) X and Y are Banach spaces,
(ii) The series $\sum_i T_i$ is $c_0(X)-$multiplier Cauchy,

then $\Gamma^{\infty}_{StC}(\sum_i T_i)$ is a Banach space.

Proof. We suppose that (*i*) is hold and let $(x^{(m)})$ be a Cauchy sequence in $\Gamma^{\infty}_{StC}(\sum_i T_i)$ with $\lim_m x^{(m)} = x^{(0)}$ in $\ell_\infty(X)$, where $x^{(m)} = (x_i^{(m)})$. Then, there exists sequence $y_m \in Y$ for each $m \in \mathbb{N}$ such that

$$St-\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i x_i^{(m)} = y_m. \tag{1}$$

First, we show that (y_m) is a Cauchy sequence. By (*ii*) and Proposition 3.1, there exists $M > 0$ such that $M = \sup\left\{\left\|\sum_{i=1}^{k} T_i x_i\right\| : \|x_i\| \le 1, k \in \mathbb{N}\right\}$. If $\varepsilon > 0$ is given, then there exists $m_0 \in \mathbb{N}$ such that $\left\|x^{(p)} - x^{(q)}\right\| < \frac{\varepsilon}{3M}$ for $p, q \ge m_0$. Therefore, $\frac{3M}{\varepsilon}\left\|x^{(p)} - x^{(q)}\right\| < 1$, and hence

$$\left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i(x_i^{(p)} - x_i^{(q)})\right\| < \frac{\varepsilon}{3} \tag{2}$$

for $p, q \ge m_0$ and $k \in \mathbb{N}$. From limit in (1), we obtain

$$\delta(A_p) = \delta\left(\left\{k \in \mathbb{N} : \left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i x_i^{(p)} - y_p\right\| < \frac{\varepsilon}{3}\right\}\right) = 1$$

and

$$\delta(A_q) = \delta\left(\left\{k \in \mathbb{N} : \left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i x_i^{(q)} - y_q\right\| < \frac{\varepsilon}{3}\right\}\right) = 1.$$

If we take $A = A_p \cap A_q$, then $\delta(A) = 1$. If pick $k \in A$, then we have

$$\begin{aligned}\|y_p - y_q\| &= \left\|y_p - \frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i x_i^{(p)}\right\| + \left\|y_q - \frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i x_i^{(q)}\right\| \\ &+ \left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i(x_i^{(p)} - x_i^{(q)})\right\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for any $p > q \ge m_0$. Hence, (y_m) is a Cauchy sequence, and thus there exists $y_0 \in Y$ such that $\lim_m y_m = y_0 \in Y$.

Next, we show that $St-\lim_k \frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i x_i^{(0)} = y_0$ and so $x^{(0)} \in \Gamma^{\infty}_{StC}(\sum_i T_i)$. From (2), there exists $m \in \mathbb{N}$ such that

$$\left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i(x_i^{(m)} - x_i^{(0)})\right\| < \frac{\varepsilon}{3}.$$

Also, we can choose $m \in \mathbb{N}$ such that

$$\|y_m - y_0\| < \frac{\varepsilon}{3}.$$

On the other hand, since $St-\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_ix_i^{(m)}=y_m$, we have

$$\delta(K)=\delta\left(\left\{k\in\mathbb{N}:\left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_ix_i^{(m)}-y_m\right\|<\frac{\varepsilon}{3}\right\}\right)=1.$$

Consequently,

$$\begin{aligned}\left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_ix_i^{(0)}-y_0\right\| &\leq \left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_i(x_i^{(m)}-x_i^{(0)})\right\| \\ &+ \left\|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)T_ix_i^{(m)}-y_m\right\|+\|y_m-y_0\| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\end{aligned}$$

for each $k\in K$. Hence, the proof is complete. □

Theorem 3.3. *Let X and Y be Banach spaces. If the series $\sum_i T_i$ is $c_0(X)-$multiplier Cauchy, then $\Gamma^{\infty}_{StwC}(\sum_i T_i)$ is a Banach space.*

Proof. Let us suppose that $(x^{(m)})$ be a Cauchy sequence in $\Gamma^{\infty}_{StwC}(\sum_i T_i)$ and $x^{(m)}\to x^{(0)}\in\ell_\infty(X)$. We show that $x^{(0)}\in\Gamma^{\infty}_{StwC}(\sum_i T_i)$. If we take $f\in S_{Y^*}$ (unit sphere in Y^*), then there exists sequence $y_m\in Y$ for each $m\in\mathbb{N}$ such that

$$St-\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_ix_i^{(m)})=f(y_m). \tag{3}$$

To show that

a. (y_m) is weak convergent to y_0 in Y.

b. $St-\lim_k\frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_ix_i^{(0)})=f(y_0)$.

a. Since the series $\sum_i T_i$ is $c_0(X)-$multiplier Cauchy, by Proposition 3.1, there exists $M>0$ such that $M=\sup\left\{\left\|\sum_{i=1}^{k}T_ix_i\right\|:\|x_i\|\leq 1,k\in\mathbb{N}\right\}$. If given $\varepsilon>0$, then there exists $m_0\in\mathbb{N}$ such that

$$\left|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i(x_i^{(p)}-x_i^{(q)}))\right|<\frac{\varepsilon}{3} \tag{4}$$

for $p,q\geq m_0$ and $k\in\mathbb{N}$. From (3), we have

$$\delta(A_p)=\delta\left(\left\{k\in\mathbb{N}:\left|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_ix_i^{(p)})-f(y_p)\right|<\frac{\varepsilon}{3}\right\}\right)=1$$

and

$$\delta(A_q)=\delta\left(\left\{k\in\mathbb{N}:\left|\frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_ix_i^{(q)})-f(y_q)\right|<\frac{\varepsilon}{3}\right\}\right)=1.$$

Let $A = A_p \cap A_q$, and so $\delta(A) = 1$. If we choose $k \in A$, then

$$\begin{aligned} \|y_p - y_q\| &= |f(y_p) - f(y_q)| \\ &= \left| f(y_p) - \frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i x_i^{(p)}) \right| \\ &+ \left| f(y_q) - \frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i x_i^{(q)}) \right| \\ &+ \left| \frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i(x_i^{(p)} - x_i^{(q)})) \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for any $p > q \geq m_0$. Therefore, (y_m) is a Cauchy sequence, and we have $\lim_m f(y_m) = f(y_0)$.
b. From (3), we obtain that

$$\delta(K) = \delta\left(\left\{ k \in \mathbb{N} : \left| \frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i x_i^{(m)}) - f(y_m) \right| < \frac{\varepsilon}{3} \right\}\right) = 1.$$

Since $x^{(m)} \to x^{(0)}$ and from (4), there exists $m \in \mathbb{N}$ such that

$$\left| \frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i(x_i^{(m)} - x_i^{(0)})) \right| < \frac{\varepsilon}{3},$$

and also we can choose $m \in \mathbb{N}$ such that

$$|f(y_m) - f(y_0)| < \frac{\varepsilon}{3}.$$

Then, for each $k \in K$,

$$\begin{aligned} \left| \frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i x_i^{(0)}) - f(y_0) \right| &\leq \left| \frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i(x_i^{(m)} - x_i^{(0)})) \right| \\ &+ \left| \frac{1}{k}\sum_{i=1}^{k}(k-i+1)f(T_i x_i^{(m)}) - f(y_m) \right| + |f(y_m) - f(y_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This completes the proof of the theorem. □

Remark 3.4. *If $\Gamma^{\infty}_{StC}(\sum_i T_i)$ is a Banach space, then $c_0(X) \subseteq \Gamma^{\infty}_{StC}(\sum_i T_i)$. We suppose that $c_0(X) \nsubseteq \Gamma^{\infty}_{StC}(\sum_i T_i)$. Then, there exists $x^{(0)} = (x_i^{(0)}) \in c_0(X)$ such that the series $\sum_i T_i x_i^{(0)}$ is not statistical Cesàro convergent. On the other hand, there exists a sequence $x^{(m)} = (x_i^{(m)})$ in $c_{00}(X)$ such that $\lim_m x^{(m)} = x^{(0)}$. Also, it is clear that $c_{00}(X) \subset \Gamma^{\infty}_{StC}(\sum_i T_i)$, and hence $x^{(m)} = (x_i^{(m)})$ in $\Gamma^{\infty}_{StC}(\sum_i T_i)$. Thus $\Gamma^{\infty}_{StC}(\sum_i T_i)$ is not complete.*

By the previous theorems and remark above, we can give the following corollary:

Corollary 3.5. *Let X be a Banach space and $\sum_i T_i$ is a series in $L(X, Y)$. If Y is a Banach space, then the following are equivalent:*

(i) The series $\sum_i T_i$ is $c_0(X)$−multiplier Cauchy.

(ii) $\Gamma^{\infty}_{StC}(\sum_i T_i)$ is a Banach space.

(iii) $c_0(X) \subseteq \Gamma^{\infty}_{StC}(\sum_i T_i)$.

(iv) $\Gamma^{\infty}_{StwC}(\sum_i T_i)$ *is a Banach space.*

(v) $c_0(X) \subseteq \Gamma^{\infty}_{StwC}(\sum_i T_i)$.

Remark 3.6. *In the above corollary, if Y is not complete, then we can show that there exists a $c_0(X)-$multiplier Cauchy series $\sum_i T_i$ such that $\Gamma^{\infty}_{StC}(\sum_i T_i)$ $\left(\Gamma^{\infty}_{StwC}(\sum_i T_i)\right)$ is not complete. Indeed, let $\sum_i y_i$ be a series in Y so that $\|y_i\| < \frac{1}{3^{2i}}$ and $\sum_i y_i = y^{**} \in Y^{**} \setminus Y$ for every $i \in \mathbb{N}$. Clearly, it will be StC-$\sum_i y_i = y^{**}$. We take $x_0 \in X$ with $\|x_0\| = 1$. By Hahn-Banach theorem, we choose $x_0^* \in X^*$ such that $x_0^*(x_0) = \|x_0\|$. Now, we consider the series $\sum_i T_i$ defined as $T_i x = x_0^*(x)3^i y_i$ for each $i \in \mathbb{N}$. Let $x = (x_i) \in c_0(X)$. Then,*

$$\left\|\sum_{i=p}^{q} T_i x_i\right\| \leq \sum_{i=p}^{q} \|T_i\|\|x_i\| \leq \sup_{p\leq i\leq q} \|x_i\| \sum_{i=p}^{q} \|T_i\| \to 0$$

for $p < q$, and hence $\sum_i T_i$ is $c_0(X)-$multiplier Cauchy series. On the other hand, if consider sequence $x = (x_0/3^i) \in c_0(X)$, then $x^{(m)} = \sum_{i=1}^{m} e^{(i)} \otimes x_0/3^i \in \Gamma^{\infty}_{StC}(\sum_i T_i)$ and $x^{(m)} \to x_0/3^i$ in $\|.\|_\infty$. But, since we have that

$$StC - \sum_i T_i x_i = StC - \sum_i \frac{1}{3^i} x_0^*(x_0) 3^i y_i = StC - \sum_i y_i = y^{**},$$

$(x_0/3^i) \notin \Gamma^{\infty}_{StC}(\sum_i T_i)$. Therefore, $\Gamma^{\infty}_{StC}(\sum_i T_i)$ is not complete.

For the case of $\Gamma^{\infty}_{StwC}(\sum_i T_i)$, it is also enough to observe $\ell_\infty(X)-$multiplier Cauchy of the series $\sum_i T_i$ and to use the following proposition.

Proposition 3.7. *Let X and Y be normed spaces. If $\sum_i T_i$ is $\ell_\infty(X)-$multiplier Cauchy series, then $\Gamma^{\infty}_{StC}(\sum_i T_i) = \Gamma^{\infty}_{StwC}(\sum_i T_i)$.*

Proof. Let us suppose that $x = (x_i) \in \Gamma^{\infty}_{StwC}(\sum_i T_i)$. Then there exists $y \in Y$ such that

$$St - \frac{1}{k}\sum_{i=1}^{k}(k - i + 1) f(T_i x_i) = f(y)$$

for every $f \in Y^*$. On the other hand, $\left\{\sum_{i=1}^{k} T_i x_i\right\}_{k\in\mathbb{N}}$ is a Cauchy sequence in Y, and hence there exists $F \in Y^{**}$ (second dual of Y) such that

$$St - \frac{1}{k}\sum_{i=1}^{k}(k - i + 1) T_i x_i = F.$$

Therefore, we have $y = F$ beacuse of the uniqueness of limit, that is $x = (x_i) \in \Gamma^{\infty}_{StC}(\sum_i T_i)$. □

Remark 3.8. *Let $\sum_i T_i$ be a series in $L(X, Y)$. The following spaces endowed with the sup norm have been studied in [5]:*

$$M^{\infty}_{C}(\sum_i T_i) = \left\{x = (x_i) \in \ell_\infty(X) : \sum_{i=1}^{k} T_i x_i \text{ is Cesàro convergent}\right\}$$

and

$$M^{\infty}_{wC}(\sum_i T_i) = \left\{x = (x_i) \in \ell_\infty(X) : \sum_{i=1}^{k} T_i x_i \text{ is weakly Cesàro convergent}\right\}.$$

It is well know that the inclusions $M^{\infty}_{C}(\sum_i T_i) \subset \Gamma^{\infty}_{StC}(\sum_i T_i)$ and $M^{\infty}_{wC}(\sum_i T_i) \subset \Gamma^{\infty}_{StwC}(\sum_i T_i)$ are provided, but we do not know what conditions allow us to obtain the equality of both spaces. On the other hand, in order to obtain *a relative between the spaces $\Gamma^{\infty}_{StC}(\sum_i T_i)$ and $M^{\infty}_{wC}(\sum_i T_i)$, we will need extra conditions. Let us suppose that X and Y are Banach spaces and $\sum_i T_i$ is $c_0(X)-$multiplier convergent series. From Corollary 3.5, $c_0(X) \subset \Gamma^{\infty}_{StC}(\sum_i T_i)$ and hence $x = (x_i) \in \Gamma^{\infty}_{StC}(\sum_i T_i)$. Then, $x = (x_i) \in \Gamma^{\infty}_{StwC}(\sum_i T_i)$, and we suppose that $StC - \sum_i f(T_i x_i) = f(y_0)$ for $y_0 \in Y$ and every $f \in Y^*$. Therefore, we obtain the following equalities*

$$C - \sum_i f(T_i x_i) = StC - \sum_i f(T_i x_i) = f(y_0).$$

This means that $x = (x_i) \in M^{\infty}_{wC}(\sum_i T_i)$, that is $\Gamma^{\infty}_{StC}(\sum_i T_i) \subset M^{\infty}_{wC}(\sum_i T_i)$.

4. The summing operators for operator valued series

In this section we introduce and study the summing operators related to the series $\sum_i T_i$ defined on $\Gamma^{\infty}_{StC}(\sum_i T_i)$ and $\Gamma^{\infty}_{StwC}(\sum_i T_i)$.

Let $\sum_i T_i$ be a series in $L(X, Y)$. We define the summing operator σ by

$$\sigma : \Gamma^{\infty}_{StC}(\sum_i T_i) \to Y, \quad \sigma(x) = StC - \sum_i T_i x_i.$$

Also, we define the summing operator μ by

$$\mu : \Gamma^{\infty}_{StwC}(\sum_i T_i) \to Y, \quad \mu(x) = StwC - \sum_i T_i x_i.$$

Now, we give a characterization of the continuity of summing operators σ and μ.

Theorem 4.1. *Let X and Y be normed spaces. Then, the following are equivalent:*

(i) $\sum_i T_i$ *is* $c_0(X)-$*multiplier Cauchy series.*

(ii) $\sigma : \Gamma^{\infty}_{StC}(\sum_i T_i) \to Y$ *is continuous.*

(iii) $\mu : \Gamma^{\infty}_{StwC}(\sum_i T_i) \to Y$ *is continuous.*

Proof. $(i) \Rightarrow (ii)$. If the series $\sum_i T_i$ is $c_0(X)-$multiplier Cauchy, then from Proposition 3.1, the set $\{\sum_{i=1}^n T_i x_i : \|x_i\| \leq 1, n \in \mathbb{N}\}$ is bounded. We suppose that

$$H \geq \sup_k \left\{ \left\| \sum_{i=1}^k T_i x_i \right\| : \|x_i\| \leq 1, k \in \mathbb{N} \right\}.$$

Also, by Theorem 3.2 and Remark 3.4, it is easily see that $x = (x_i) \in \Gamma^{\infty}_{StC}(\sum_i T_i)$ and hence $StC - \sum_{i=1}^k T_i x_i$ exists. Therefore,

$$\|\sigma_k(x)\| = \left\| StC - \sum_{i=1}^k T_i x_i \right\| \leq H$$

for $k \in \mathbb{N}$. This part is complete.

$(ii) \Rightarrow (iii)$. Since σ is continuous and $c_{00}(X) \subset \Gamma^{\infty}_{StC}(\sum_i T_i)$, there exists $H > 0$ such that

$$\left\| \sigma\left(\sum_{i=1}^k e^{(i)} \otimes x_i \right) \right\| = \left\| StC - \sum_i T_i x_i \right\| \leq H$$

for $\|x_i\| \leq 1$ and $k \in \mathbb{N}$, where $e^{(i)} \otimes x$ denote the series with x in the ith coordinate and zero in the other coordinates. If we take $x = (x_i) \in \Gamma^{\infty}_{StwC}(\sum_i T_i)$ and $y^* \in B_{Y^*}$, then

$$\begin{aligned} \|\mu(x)\| &= \left\| StwC - \sum_{i=1}^{\infty} T_i x_i \right\| = \left| StC - \sum_{i=1}^{\infty} y^*(T_i x_i) \right| \\ &= \left| StC - \lim_k \sum_{i=1}^{k} y^*(T_i x_i) \right| \leq H\|x\|. \end{aligned}$$

That is, μ is continuous.

(iii) ⇒ (i). We consider the sequence $x = (x_i) \in c_{00}(X)$. Since $c_{00}(X) \subset \Gamma^{\infty}_{StwC}(\sum_i T_i)$ and μ is continuous, we obtain the following inequality

$$\left\| \sum_{i=1}^{k} T_i x_i \right\| = \left\| StwC - \sum_{i=1}^{\infty} T_i x_i \right\| \leq \|\mu\|.$$

Then, $\sum_i T_i$ is $c_0(X)$−multiplier Cauchy series by Proposition 3.1. □

In the following theorem, we characterize the compactness of summing operators σ and μ.

Theorem 4.2. *If Y is a Banach space, then the following are equivalent:*

(i) $\sum_i T_i$ *is* $\ell_\infty(X)$*−multiplier convergent series.*

(ii) $\sigma : \Gamma^{\infty}_{StC}(\sum_i T_i) \to Y$ *is compact.*

(iii) $\mu : \Gamma^{\infty}_{StwC}(\sum_i T_i) \to Y$ *is compact.*

Proof. (i) ⇒ (ii). Let $\sum_i T_i$ is $\ell_\infty(X)$−multiplier convergent series. Then, by Corollary 11.11 in [27] the series $\sum_i T_i x_i$ is uniformly statistical Cesàro convergent for $\|x_i\| \leq 1$, that is for every $\varepsilon > 0$ there exists $A \subset \mathbb{N}$ with $\delta(A) = 1$ and there exists $N = N(\varepsilon) \in A$ such that

$$\left\| \frac{1}{k} \sum_{i=k+1}^{\infty} (k - i + 1) T_i x_i \right\| \leq \varepsilon \tag{5}$$

for all $k \geq N$ with $k \in A$ and $\|x_i\| \leq 1$. Now, we define the finite rank operators $\sigma_k : \Gamma^{\infty}_{StC}(\sum_i T_i) \to Y$ by

$$\sigma_k(x) = StC - \sum_{i=1}^{k} T_i x_i$$

for $k \in \mathbb{N}$, by Theorem 4.1, (σ_k) is bounded, and hence is compact. Then, by (5)

$$\begin{aligned} \|\sigma_k - \sigma\| &= \left\| StC - \sum_{i=1}^{k} T_i x_i - StC - \sum_{i=1}^{\infty} T_i x_i \right\| \\ &= \left\| St - \frac{1}{k} \sum_{i=k+1}^{\infty} (k - i + 1) T_i x_i \right\| \to 0 \end{aligned}$$

for $\|x_i\| \leq 1$. This means that σ is compact.

(ii) ⇒ (iii). Let $x = (x_i) \in \Gamma^{\infty}_{StC}(\sum_i T_i)$ and

$$M = \left\{ \sum_{i \in \pi} e^{(i)} \otimes x_i : \pi \text{ finite }, \|x_i\| \leq 1 \right\}.$$

Then, since the set $M \subset \Gamma^{\infty}_{StC}(\sum_i T_i)$ is bounded and σ is compact, the set

$$\sigma(M) = \left\{ StC - \sum_{i \in \pi} T_i x_i : \pi \text{ finite }, \|x_i\| \leq 1 \right\} \tag{6}$$

is relatively compact. On the other hand, we take $x = (x_i) \in \Gamma^{\infty}_{StwC}(\sum_i T_i)$ and also, since $M \subset \Gamma^{\infty}_{StwC}(\sum_i T_i)$, we obtain that $\mu(M) = \{StwC - \sum_{i\in\pi} T_i x_i : \pi \text{ finite }, \|x_i\| \leq 1\}$. Therefore, by (6), one can easily see that μ is a compact operator.

$(iii) \Rightarrow (i)$. We take a sequence $x = (x_i)$ in $\ell_\infty(X)$. Since $\mu(M)$, defined in (6), is weakly relatively compact set, the series $\sum_i T_i x_i$ is subseries statistical weakly Cesàro convergent ([28, Theorem 2.48]). From Orlicz-Pettis theorem for statistical Cesàro convergence, the series $\sum_i T_i$ is $\ell_\infty(X)-$multiplier convergent. □

Now, we denote the weak topology on X by $\tau(X, X^*)$. Since $\tau(\ell_\infty(X), \ell_1(X^*))$ is a dual pair and $\Gamma^{\infty}_{StC}(\sum_i T_i)$, $\Gamma^{\infty}_{StwC}(\sum_i T_i) \subset \ell_\infty(X)$, $\tau(\Gamma^{\infty}_{StC}(\sum_i T_i), \ell_1(X^*))$ and $\tau(\Gamma^{\infty}_{StwC}(\sum_i T_i), \ell_1(X^*))$ are also dual pairs. In the two next theorems we obtain some results for sequential continuity related to $\tau(\Gamma^{\infty}_{StC}(\sum_i T_i), \ell_1(X^*))$ and $\tau(\Gamma^{\infty}_{StwC}(\sum_i T_i), \ell_1(X^*))$ of the summing operators σ and μ, respectively.

Theorem 4.3. *If $\sigma : \Gamma^{\infty}_{StC}(\sum_i T_i) \to Y$ is sequentially $\tau(\Gamma^{\infty}_{StC}(\sum_i T_i), \ell_1(X^*))-$ continuous, then $\sum_i T_i$ is $\ell_\infty(X)-$multiplier Cauchy series.*

Proof. Let $x = (x_i) \in \ell_\infty(X)$ and we consider the set $x^{(m)} = \chi_{\{1,2,\ldots,m\}} x = (x_1, x_2, \ldots, x_m, 0, 0, \ldots)$. Then, it is clearly that $x^{(m)} \in \Gamma^{\infty}_{StC}(\sum_i T_i)$ for each $m \in \mathbb{N}$. On the other hand, let us take $f = (f_i) \in \ell_1(X^*)$, then we have

$$(f, x^{(m)}) = \sum_{i=1}^{m} (f_i, x_i) \to \sum_{i=1}^{\infty} (f_i, x_i).$$

That is, $x^{(m)}$ is $\tau(\Gamma^{\infty}_{StC}(\sum_i T_i), \ell_1(X^*))$-Cauchy. Since σ is sequentially $\tau(\Gamma^{\infty}_{StC}(\sum_i T_i), \ell_1(X^*))-$continuous, $\tau(x^{(m)}) = StC - \sum_{i=1}^{m} T_i x_i$ is norm Cauchy. By the monotonity of $\ell_\infty(X)$, the series $\sum_{i=1}^{m} T_i x_i$ is subseries statistical Cesàro Cauchy, and hence $\sum_i T_i$ is $\ell_\infty(X)-$multiplier Cauchy series from Orlicz-Pettis theorem. □

Corollary 4.4. *Under the same hypotheses of the last theorem, if Y is complete, then $\sum_i T_i$ is $\ell_\infty(X)-$multiplier convergent series.*

Next, we consider the converse to the previous theorem using completely continuous operators.

Theorem 4.5. *Let T_i be completely continuous operators for each $i \in \mathbb{N}$ and $\sum_i T_i$ is $\ell_\infty(X)-$multiplier convergent series. Then, $\sigma : \Gamma^{\infty}_{StC}(\sum_i T_i) \to Y$ is sequentially $\tau(\Gamma^{\infty}_{StC}(\sum_i T_i), \ell_1(X^*))-$continuous.*

Proof. Let us suppose that $\varepsilon > 0$ and $\tau(\Gamma^{\infty}_{StC}(\sum_i T_i), \ell_1(X^*)) - \lim_i x^{(i)} = 0$. By Lemma 1.16 in [28], we suppose that $\|x^{(i)}\| \leq 1$ for every $i \in \mathbb{N}$. Since $\sum_i T_i$ is $\ell_\infty(X)-$multiplier convergent series, the series $\sum_i T_i x_i$ is uniformly statistical Cesàro convergent for $\|x_i\| \leq 1$. Thus, there exists $m \in \mathbb{N}$ such that

$$\left\| StC - \sum_{i=m}^{\infty} T_i x_i \right\| < \frac{\varepsilon}{2}$$

for $\|x_i\| \leq 1$. By Lemma 1.17 in [28], $\tau(X, X^*) - \lim_i x_j^{(i)} = 0$ for each $j \in \mathbb{N}$, and hence, by hypothesis, $\lim_j \|T_j x_j^{(i)}\| = 0$. Then, there exists $m_0 \in \mathbb{N}$ such that

$$\left\| StC - \sum_{j=1}^{m-1} T_j x_j^{(i)} \right\| < \frac{\varepsilon}{2}$$

for $i \geq m_0$. Consequently, we obtain that

$$\|\sigma x^{(i)}\| = \left\| StC - \sum_{j=1}^{\infty} T_j x_j^{(i)} \right\| \leq \left\| StC - \sum_{j=1}^{m-1} T_j x_j^{(i)} \right\| + \left\| StC - \sum_{j=m}^{\infty} T_j x_j^{(i)} \right\| < \varepsilon$$

for $i \geq m_0$, and hence the proof is complete. □

Finally, we give the following corollary as a result of the above theorems.

Corollary 4.6. *Let Y be a Banach space and T_i be completely continuous for each $i \in \mathbb{N}$. Then, the following are equivalent:*

(i) $\sum_i T_i$ *is* $\ell_\infty(X)-$*multiplier convergent series.*

(ii) $\sigma : \Gamma^\infty_{StC}(\sum_i T_i) \to Y$ *is sequentially* $\tau(\Gamma^\infty_{StC}(\sum_i T_i), \ell_1(X^*))-$ *continuous.*

(iii) $\mu : \Gamma^\infty_{StwC}(\sum_i T_i) \to Y$ *is sequentially* $\tau(\Gamma^\infty_{StwC}(\sum_i T_i), \ell_1(X^*))-$ *continuous.*

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