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λ-rings, ϕ-λ-rings, and ϕ-Δ-rings

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Abstract. Let R be a commutative ring with unity. The notion of λ-rings, ϕ-λ-rings, and ϕ-Δ-rings is introduced which generalize the concept of λ-domains and Δ-domains. A ring R is said to be a λ-ring if the set of all overrings of R is linearly ordered under inclusion. A ring $R \in \mathcal{H}$ is said to be a ϕ-λ-ring if $\phi(R)$ is a λ-ring, and a ϕ-Δ-ring if $\phi(R)$ is a Δ-ring, where $\mathcal{H}$ is the set of all rings such that $Nil(R)$ is a divided prime ideal of R and $\phi : T(R) \to R_{Nil(R)}$ is a ring homomorphism defined as $\phi(x) = x$ for all $x \in T(R)$. The equivalence of ϕ-λ-rings, ϕ-Δ-rings with the latest trending rings in the literature, namely, ϕ-chained rings and ϕ-Prüfer rings is established under some conditions. Using the idealization theory of Nagata, examples are also given to strengthen the concept.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity; all ring extensions and ring homomorphisms are unital. By an overring of R, we mean a subring of the total quotient ring of R containing R. By a local ring, we mean a ring with a unique maximal ideal. We use $T(R)$ to denote the total quotient ring of R, R' to denote the integral closure of a ring R in $T(R)$, $Nil(R)$ to denote the set of nilpotent elements of R, and $Z(R)$ to denote the set of zero-divisors of R. All the elements of $R \setminus Z(R)$ are said to be regular elements of R and an ideal is said to be regular if it contains a regular element. A divided prime ideal [14] is a prime ideal Q of a ring R such that $QR_Q = Q$. Badawi characterized the divided prime ideal of R in [3], as a prime ideal which is comparable to every ideal of R. In [4], Badawi introduced the class $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal$\}$. In [1], Anderson and Badawi defined $\mathcal{H}_0$ to denote the subset of $\mathcal{H}$ such that $Nil(R) = Z(R)$. These classes of rings were studied in [1], [2], [4], [5], [6], [7], [8], [9], [12]. The further study of ϕ-rings is given in [10], [11], [13].

For $R \in \mathcal{H}$, Badawi [4] considered the ring homomorphism ϕ from $T(R)$ to $R_{Nil(R)}$ given by $\phi(r/s) = r/s$ for $r \in R$ and $s \in R \setminus Z(R)$. Note that the restriction of ϕ to R is also a ring homomorphism given by $\phi(r) = r/1$ for $r \in R$. A ring R is said to be ϕ-integrally closed [2] if $\phi(R)$ is integrally closed. Griffin [23] introduced Prüfer rings as the rings in which each finitely generated regular ideal is invertible. A ring $R \in \mathcal{H}$ is said to be a ϕ-Prüfer ring [1] if $\phi(R)$ is a Prüfer ring. Recall from [18] that a ring R is said to be a *quasi-valuation ring* if either x divides y or y divides x in R for all $x, y \in R \setminus Z(R)$.

Recall from [6] that a ring $R \in \mathcal{H}$ is said to be a ϕ-chained ring if for each $x \in R_{Nil(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$. An integral domain R is called a Dedekind domain if every nonzero ideal of R is invertible, that

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is, if I is a nonzero ideal of R, then $II^{-1} = R$, where $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. Recall from [17] that an integral domain R is a Krull-domain if $R = \cap V_i$, where each V_i is a discrete valuation overring of R, and every nonzero element of R is a unit in all but finitely many V_i. An ideal I of a ring R is said to be a non-nil ideal if $I \nsubseteq Nil(R)$. A non-nil ideal I of R is ϕ-invertible [2] if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every non-nil ideal of R is ϕ-invertible, then R is said to be a ϕ-Dedekind ring [2]. Also, a ring R is said to be a ϕ-Krull ring [2] if $\phi(R) = \cap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$, and for every non-nilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i.

In this paper, we generalize and study the concept of λ-domains to the rings in $\mathcal{H}$. A ring extension $R \subseteq S$ is said to be a λ-extension [22] if the set of all subrings of S containing R is linearly ordered under inclusion. Note that an integral domain R is said to be a λ-domain [22] if the set of all overrings of R is linearly ordered under inclusion. Motivated by this, we define two new classes of rings, namely, λ-rings and ϕ-λ-rings. A ring R is said to be a λ-ring if the set of all overrings of R is linearly ordered under inclusion. We study λ-rings in class $\mathcal{H}_0$. A ring $R \in \mathcal{H}$ is said to be a ϕ-λ-ring if $\phi(R)$ is a λ-ring. We explore ϕ-λ-rings in $\mathcal{H}$. Recall that a ring extension $R \subseteq S$ is said to be a Δ-extension [20] if sum of any two subrings of S which contains R is again a ring. Moreover, R is a Δ-ring if sum of any two overrings of R is again an overring of R. In this paper, we introduce another new class of rings, namely, ϕ-Δ-ring. A ring $R \in \mathcal{H}$ is said to be a ϕ-Δ-ring if $\phi(R)$ is a Δ-ring. We study the properties of ϕ-λ-rings, ϕ-Δ-rings and characterize the same. In section 2, we study λ-rings and establish the equivalence of an integrally closed λ-ring and a *quasi-valuation ring*. We discuss ϕ-λ-rings and ϕ-Δ-rings in Section 3 and 4, respectively. The equivalence of ϕ-λ-rings and ϕ-Δ-rings with the latest trending rings in the literature, namely, ϕ-chained rings and ϕ-Prüfer rings is established under some conditions. Using the idealization theory of Nagata, examples are also given to strengthen the concept.

Recall from [25, cf. Nagata, 1962, p.2] that if R is a ring and M is an R-module, then the idealization $R(+)M$ is the ring defined as follows: Its additive structure is that of the abelian group $R \oplus M$, and its multiplication is defined by $(r_1, x_1)(r_2, x_2) := (r_1r_2, r_1x_2 + r_2x_1)$ for all $r_1, r_2 \in R$ and $x_1, x_2 \in M$. Given rings $R \subseteq S$, the conductor $(R : S) := \{s \in S \mid sS \subseteq R\}$. Also, dimension(al) refers to Krull dimension.

We now list some results on ϕ-rings which are already in literature and are frequently used in this paper. Note that the first five results are proved in [4] whereas the last one is proved in [2]. Let $R \in \mathcal{H}$. Then

(a) $\phi(R) \in \mathcal{H}_0$.
(b) $Ker(\phi) \subseteq Nil(R)$.
(c) $Nil(T(R)) = Nil(R)$.
(d) $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$.
(e) $T(\phi(R)) = R_{Nil(R)}$ is a local ring with maximal ideal $Nil(\phi(R))$, and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R)) = T(\phi(R)/Nil(\phi(R)))$.
(f) $(R/Nil(R))' = R'/Nil(R)$ provided $R \in \mathcal{H}_0$.

2. λ-rings

In this section, we introduce the concept of λ-rings which can be seen as a generalization of λ-domains [22]. First, we define λ-rings formally.

Definition 2.1. *A ring R is said to be a λ-ring if the set of all overrings of R is linearly ordered under inclusion.*

In Proposition 2.3, we show that the λ-rings in $\mathcal{H}_0$ are precisely the rings for which $R/Nil(R)$ is a λ-domain with quotient field $T(R)/Nil(R)$. First, we observe the following lemma whose proof is routine.

Lemma 2.2. *Let $R \subseteq S$ be a ring extension such that $Nil(R) = Nil(S)$. Then $R \subseteq S$ is a λ-extension if and only if $R/Nil(R) \subseteq S/Nil(R)$ is a λ-extension.*

Proposition 2.3. *Let $R \in \mathcal{H}_0$. Then the following are equivalent:*

(i) R is a λ-ring;

(ii) $R/Nil(R)$ *is a* λ*-domain with quotient field* $T(R)/Nil(R)$.

Proof. First, suppose that R is a λ-ring. As $R \in \mathcal{H}_0$, $Nil(T(R)) = Nil(R)$, by (*c*). It follows that $R/Nil(R) \subseteq T(R)/Nil(R)$ is a λ-extension, by Lemma 2.2. Note that $T(R/Nil(R)) = T(R)/Nil(R)$, by (*e*) as $\phi(R) = R$. Thus, $R/Nil(R)$ is a λ-domain with quotient field $T(R)/Nil(R)$. Converse follows by Lemma 2.2. □

In our first theorem, we show that every λ-ring in $\mathcal{H}_0$ is local and its integral closure is a *quasi-valuation ring*. Moreover, this can be seen as a generalization of [22, Proposition 1.3].

Theorem 2.4. *Let* $R \in \mathcal{H}_0$*. If* R *is a* λ*-ring, then* R *is local and* R' *is a quasi-valuation ring.*

Proof. Note that $R/Nil(R)$ is a λ-domain with quotient field $T(R)/Nil(R)$, by Proposition 2.3. It follows that by [22, Proposition 1.3], either $R/Nil(R)$ is a field or $R/Nil(R)$ is local with $(R/Nil(R))'$ is a valuation domain. If former holds, then R is local, $R = T(R)$ and so $R' = T(R)$ is *quasi-valuation*. If later holds, then R is local. Also, by (*f*), we have $(R/Nil(R))' = R'/Nil(R)$. Now, to show that R' is *quasi-valuation*, let x, y be regular elements in R'. Then $x/y \in T(R)$. Since $R'/Nil(R)$ is a valuation domain, we have either $x/y \in R'$ or $y/x \in R'$ and hence R' is a *quasi-valuation ring*. □

In [22, Corollary 1.5], it was shown that an integrally closed domain is a λ-domain if and only if it is a valuation domain. The next corollary generalize this to rings in $\mathcal{H}_0$.

Corollary 2.5. *Let* $R \in \mathcal{H}_0$*. Then* R *is an integrally closed* λ*-ring if and only if* R *is a quasi-valuation ring.*

Proof. If R is an integrally closed λ-ring, then R is a *quasi-valuation ring*, by Theorem 2.4. Conversely, assume that R is a *quasi-valuation ring*. Let $x = r/s \in T(R) \setminus R$. If $r \in Nil(R)$, then $x \in Nil(T(R)) = Nil(R)$, by (*c*), a contradiction. Therefore, $r \notin Nil(R)$. Thus, $x^{-1} \in R$ and hence R is an integrally closed λ-ring. □

Now, we observe the following result that classifies the integral closure of any λ-ring R in $\mathcal{H}_0$ as a Prüfer ring.

Proposition 2.6. *Let* $R \in \mathcal{H}_0$*. If* R *is a* λ*-ring, then* R' *is a Prüfer ring. In particular, an integrally closed* λ*-ring in* $\mathcal{H}_0$ *is a Prüfer ring.*

Proof. By [22, Lemma 1.1] and [12, Theorem 2.2], we get the desired conclusion. □

In the following example, we observe that the converse of above proposition does not hold.

Example 2.7. *Let* D *be a non-local Prüfer domain of dimension* n*. Take* $R = D(+)qf(D)$*. Then* $R \in \mathcal{H}$ *and is a* ϕ*-Prüfer ring of dimension* n*, by [1, Example 2.18]. It follows that* R *is a Prüfer ring, by [1, Theorem 2.14]. Also,* $Z(R) = Nil(R) = \{0\}(+)qf(D)$ *and so* $R \in \mathcal{H}_0$*. As* D *is not local, we conclude that* R *is not local and hence* R *is not a* λ*-ring, by Theorem 2.4.*

In the next theorem, we present a characterization of a λ-ring in $\mathcal{H}_0$ which generalizes [22, Theorem 1.9].

Theorem 2.8. *Let* $R \in \mathcal{H}_0$*. Then* R *is a* λ*-ring if and only if the following hold:*

(i) R *is a local ring and* R' *is a quasi-valuation ring.*
(ii) All the overrings of R *are comparable to* R' *under inclusion.*
(iii) The set of rings between R *and* R' *is linearly ordered by inclusion.*

Proof. If R is a λ-ring, then (*ii*), (*iii*) hold trivially and (*i*) holds by Theorem 2.4. The proof of sufficient part, follows mutatis mutandis from the proof of [22, Theorem 1.9]. □

Remark 2.9. *It is natural to think if any of the conditions* (*i*), (*ii*), *and* (*iii*) *in Theorem 2.8 is redundant. Unfortunately, we could not find any ring satisfying* (*i*) *and* (*iii*) *but not* (*ii*). *Note that the question of finding domain [22, Remarks 1.14] satisfying* (*i*) *and* (*iii*) *but not* (*ii*) *is still open. However, for the other cases we have the following examples:*

(i) Take R to be an integrally closed ring which is not quasi-valuation, then (*ii*) *and* (*iii*) *hold trivially but* (*i*) *does not hold.*

(ii) Let K be a field and $D_n = K + X^n K[[X]]$ *for all non negative integers n. Take* $R_n = D_n(+)qf(D_n)$*. Then* $Z(R_n) = Nil(R_n) = \{0\}(+)qf(D_n)$ *is a divided prime ideal of* R_n*. For if we take* $(0, r) \in Nil(R_n)$ *and* $(a, s) \in R_n \setminus Nil(R_n)$*, then* $(0, r) = (a, s)(0, r/a)$*. So,* $R_n \in \mathcal{H}_0$*. Now, by [22, Remarks 1.14], for all* $n \geq 4$*,* D_n *is not a* λ*-domain and* D_n *satisfies* $(*)$ *and* $(**)$ *conditions of [22, Theorem 1.9] but do not satisfy* $(***)$*. Note that* $R_n/Nil(R_n) \cong D_n$ *and by* (*e*), (*f*) *we have* $T(R_n/Nil(R_n)) = T(R_n)/Nil(R_n)$*,* $(R_n/Nil(R_n))' = (R_n)'/Nil(R_n)$*, respectively. It is trivial to see that for all* $n \geq 4$*,* R_n *satisfies* (*ii*) *but not* (*iii*)*. Now, by the same argument as in Theorem 2.4, we conclude that* R_n *satisfies* (*i*)*.*

In [19, Theorem 2.4], Gilmer and Heinzer proved that for a local domain R, if $R \subseteq R'$ has no intermediate ring and R' is a Prüfer domain, then R' is contained in every proper overring of R. In Proposition 2.11, we generalize this for rings in $\mathcal{H}_0$. First, we need the following lemma.

Lemma 2.10. *If* $R \in \mathcal{H}_0$*, then* $T(R) \in \mathcal{H}_0$*.*

Proof. Note that $R = \phi(R)$ as $R \in \mathcal{H}_0$. Thus, by (*e*) and (*c*), we have $Nil(T(R))$ (= $Nil(R)$) is the unique maximal ideal of $T(R)$ and hence the result follows. □

Proposition 2.11. *Let* $R \in \mathcal{H}_0$ *be a local ring. If* $R \subseteq R'$ *has no intermediate ring and* R' *is a Prüfer ring, then* R' *is contained in every proper overring of R.*

Proof. Note that $Nil(R') = Nil(R)$ by (*c*). So, $R/Nil(R) \subseteq R'/Nil(R)$ has no intermediate ring. On the other hand, since $\phi(R) = R$ as $R \in \mathcal{H}_0$, we conclude that $Nil(R)$ is a prime ideal of R', by (*e*). Now, by Lemma 2.10, $R' \in \mathcal{H}_0$ and hence by [1, Theorem 2.16], R' is a ϕ-Prüfer ring. Thus, $R'/Nil(R)$ is a Prüfer domain, by [1, Theorem 2.6]. But $(R/Nil(R))' = R'/Nil(R)$, by (*f*). Hence, by [19, Theorem 2.4], $R'/Nil(R)$ is contained in every proper overring of $R/Nil(R)$. Since $T(R/Nil(R)) = T(R)/Nil(R)$ by (*e*), R' is contained in every proper overring of R. □

For the sufficiency of a ring $R \in \mathcal{H}_0$ to be a λ-ring, the conditions (*ii*) and (*iii*) of Theorem 2.8 can be replaced by the condition that $R \subseteq R'$ has no intermediate ring. This we show in the next corollary which generalizes [22, Corollary 1.12].

Corollary 2.12. *Let* $R \in \mathcal{H}_0$*. If R is a local ring,* $R \subseteq R'$ *has no intermediate ring, and* R' *is a quasi-valuation ring, then R is a* λ*-ring.*

Proof. Note that $R' \in \mathcal{H}_0$, by Lemma 2.10. It follows that by Corollary 2.5 and Proposition 2.6, R' is a Prüfer ring. Now, by Proposition 2.11, we have R' is contained in every proper overring of R. Thus, R is a λ-ring, by Theorem 2.8. □

The next proposition is a generalization of [22, Proposition 1.15] to rings in $\mathcal{H}_0$.

Proposition 2.13. *Let* $R \in \mathcal{H}_0$ *be such that* $R \neq R'$*. If all the overrings of R are comparable to* R' *under inclusion, then any overring of R which is properly contained in* R' *is local.*

Proof. First observe that $(R/Nil(R))' = R'/Nil(R)$ by (*f*). Moreover, by (*e*), $T(R/Nil(R)) = T(R)/Nil(R)$. Consequently, by hypothesis, we have $R/Nil(R) \neq (R/Nil(R))'$ and all the overrings of $R/Nil(R)$ are comparable to $R'/Nil(R)$ under inclusion. Thus, any overring of $R/Nil(R)$ which is properly contained in $R'/Nil(R)$ is local, by [22, Proposition 1.15]. Since $R \in \mathcal{H}_0$, $Nil(T(R)) = Nil(R)$ by (*c*) and hence any overring of R which is properly contained in R' is local. □

We end this section with the generalization of [22, Proposition 4.3]. Though the arguments in the proof can be followed mutatis mutandis from the proof of [22, Proposition 4.3], but is given for the sake of completeness. First, we need to recall some necessary definitions. A ring extension $R \subseteq S$ is called a *P*-extension [21] if each $s \in S$ is a root of some $f(X) \in R[X]$ such that at least one of coefficients of f is a unit of R. A ring extension $R \subseteq S$ is said to be an *INC*-pair [24] if for any two distinct prime ideals Q_1, Q_2 of S such that $Q_1 \cap R = Q_2 \cap R$, we have Q_1, Q_2 are incomparable.

Theorem 2.14. *Let $R \in \mathcal{H}_0$. If $R \subseteq S$ is a λ-extension such that R is integrally closed in S, then S is an overring of R.*

Proof. Let $u \in S$. Then $R \subseteq R[u]$ is a P-extension, by [22, Lemma 1.1]. Therefore, $R \subseteq R[u]$ is an INC-pair, by [15, Corollary 4]. Now, if Q is any prime ideal of $R[u]$ and $P = Q \cap R$, then there exists $s \in R \setminus P$ such that $R[u]_s = R_s$, by Zariski's main theorem in [16]. As $s \notin Nil(R)$, it follows that $R[u]_s = R_s \subseteq T(R)$. Thus, $u \in T(R)$ and hence $S \subseteq T(R)$. □

3. ϕ-λ-rings

In this section, we define a new class of rings, namely, ϕ-λ-rings and explore its properties.

Definition 3.1. *Let $R \in \mathcal{H}$. Then R is said to be a ϕ-λ-ring if $\phi(R)$ is a λ-ring.*

We start this section with a necessary and sufficient condition for a ring in $\mathcal{H}$ to be a ϕ-λ-ring.

Proposition 3.2. *Let $R \in \mathcal{H}$. Then the following are equivalent:*

(i) R is a ϕ-λ-ring;
(ii) $R/Nil(R)$ is a λ-domain.

Proof. Let R be a ϕ-λ-ring. Then $\phi(R)$ is a λ-ring. As by (*a*), $\phi(R) \in \mathcal{H}_0$, it follows that by Proposition 2.3, $\phi(R)/Nil(\phi(R))$ is a λ-domain. Now, by (*d*), $\phi(Nil(R)) = Nil(\phi(R))$. Therefore, by [1, Lemma 2.5], we conclude that $\phi(R)/Nil(\phi(R))$ is isomorphic to $R/Nil(R)$ and hence $R/Nil(R)$ is a λ-domain. Conversely, assume that $R/Nil(R)$ is a λ-domain. Then $\phi(R)/Nil(\phi(R))$ is a λ-domain, by [1, Lemma 2.5]. Thus, by Proposition 2.3, $\phi(R)$ is a λ-ring and hence R is a ϕ-λ-ring. □

We now discuss an example that gives a class of ϕ-λ-rings.

Example 3.3. *Consider $R = A(+)qf(A)$, where A is a λ-domain. Note that $Nil(R) = \{0\}(+)qf(A)$. We claim that $Nil(R)$ is a divided prime ideal of R. Let $(0, x) \in Nil(R)$ and $(a, y) \in R \setminus Nil(R)$. Then $(0, x) = (a, y)(0, x/a)$. This settles the claim and hence $R \in \mathcal{H}$. Now, as $R/Nil(R) \cong A$ is a λ-domain, we conclude that R is a ϕ-λ-ring, by Proposition 3.2.*

In Proposition 3.5, we show that a ring is a ϕ-integrally closed ϕ-λ-ring if and only if it is local ϕ-Prüfer. First, we need the following lemma.

Lemma 3.4. *Let R be a ϕ-λ-ring. Then R is local and $\phi(R)'$ is a Prüfer ring.*

Proof. By Proposition 3.2, we have $R/Nil(R)$ is a λ-domain. It follows that $R/Nil(R)$ is local, by [22, Proposition 1.3]. Thus, R is local. As $\phi(R) \in \mathcal{H}_0$ by (*a*), we conclude that $\phi(R)'$ is a Prüfer ring, by Proposition 2.6. □

Proposition 3.5. *Let R be a ring. Then R is a ϕ-integrally closed ϕ-λ ring if and only if R is a local ϕ-Prüfer ring.*

Proof. The "only if" assertion is clear from Lemma 3.4. We turn to converse. Assume that R is local ϕ-Prüfer. Then $\phi(R)$ is a Prüfer ring and hence is integrally closed. Thus, R is ϕ-integrally closed. Also, $R/Nil(R)$ is a Prüfer domain, by [1, Theorem 2.6] and so it is a valuation domain as $R/Nil(R)$ is local. It follows that $R/Nil(R)$ is a λ-domain, by [22, Corollary 1.5]. Thus, by Proposition 3.2, R is a ϕ-λ ring. □

In the above proposition, the condition that R is local is necessary for R to be a ϕ-integrally closed ϕ-λ ring as we have the following example in support.

Example 3.6. *Let D be a non-local Prüfer domain of dimension n and $E = qf(D)/D$. Set $R = D(+)E$. Then $Nil(R) = \{0\}(+)E$. To show that $R \in \mathcal{H}$, let $(0, \bar{x}) \in Nil(R)$ and $(y, \bar{z}) \in R \setminus Nil(R)$. Then $(0, \bar{x}) = (y, \bar{z})(0, \overline{x/y})$ and so $R \in \mathcal{H}$. As $R/Nil(R) \cong D$, we conclude that R is a ϕ-Prüfer ring of dimension n, by [1, Theorem 2.6]. Since D is not local, R is not local and hence R is not a ϕ-λ-ring, by Lemma 3.4.*

In the following theorem, we show the equivalence of a ϕ-integrally closed ϕ-λ-ring and a ϕ-chained ring.

Theorem 3.7. *Let R be a ring. Then R is a ϕ-integrally closed ϕ-λ-ring if and only if R is a ϕ-chained ring.*

Proof. Let R be a ϕ-integrally closed ϕ-λ-ring. Then $\phi(R)$ is an integrally closed λ-ring. Let $a, b \in R \setminus Nil(R)$. Then $x = a/b$ is a unit in $R_{Nil(R)}$. As by (*e*), $R_{Nil(R)} = T(\phi(R))$, we conclude that either $x \in \phi(R)$ or $x^{-1} \in \phi(R)$, by [22, Lemma 1.2]. Thus, R is a ϕ-chained ring, by [6, Proposition 2.2].

Conversely, assume that R is a ϕ-chained ring. Then R is ϕ-integrally closed, by [6, Proposition 2.10]. Also, $R/Nil(R)$ is a valuation domain, by [1, Theorem 2.7]. Thus, $R/Nil(R)$ is a λ-domain, by [22, Corollary 1.5] and hence R is a ϕ-λ-ring, by Proposition 3.2. □

The next proposition gives a necessary and sufficient condition for a ring $R \in \mathcal{H}$ to be a ϕ-λ-ring.

Proposition 3.8. *Let $R \in \mathcal{H}$. Then R is a ϕ-λ-ring if and only if the following hold:*

(i) R is a local ring and $\phi(R)'$ is a quasi-valuation ring.
(ii) All the overrings of $\phi(R)$ are comparable to $\phi(R)'$ under inclusion.
(iii) The set of all rings between $\phi(R)$ and $\phi(R)'$ is linearly ordered by inclusion.

Proof. Note that $\phi(R) \in \mathcal{H}_0$, by (*a*). Let R be a ϕ-λ-ring. Then R is local, by Lemma 3.4. Now, the necessity follows from Theorem 2.8. Conversely, assume that (*i*), (*ii*) and (*iii*) holds. Then $\phi(R)$ is local, by [1, Lemma 2.5]. Now, the sufficiency follows from Theorem 2.8. □

Remark 3.9. *Note that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}_0$, by (a). Now, one can apply Remark 2.9 to discuss irredundancy of conditions (i), (ii), and (iii) in the above proposition.*

In the next two propositions, we continue our discussion on ϕ-λ-rings and ϕ-rings.

Proposition 3.10. *Let $R \in \mathcal{H}$ be a local ring such that $\phi(R) \subseteq \phi(R)'$ has no intermediate ring. Then the following hold:*

(i) If R′ is a ϕ-Prüfer ring, then $\phi(R)'$ is contained in every proper overring of $\phi(R)$.
(ii) If R′ is a quasi-valuation ring, then R is a ϕ-λ-ring.

Proof. Note that $\phi(R) \in \mathcal{H}_0$ by (*a*), and $\phi(R)$ is local by [1, Lemma 2.5]. Also, $\phi(R') \subseteq \phi(R)'$. Thus, (*i*) and (*ii*) follows from Proposition 2.11 and Corollary 2.12, respectively, as every overring of Prüfer (resp., *quasi-valuation*) is Prüfer (resp., *quasi-valuation*). □

Proposition 3.11. *Let R be a ϕ-integrally closed ϕ-λ-ring. If S is an overring of R, then there exists a prime ideal P of R containing Z(R) such that $S = R_P$.*

Proof. By Theorem 3.7, R is a ϕ-chained ring and hence S is a ϕ-chained ring. Also, R is a ϕ-Prüfer ring, by Proposition 3.5. Thus, by [1, Theorem 2.11], the result follows. □

Now, we show that every ϕ-λ-ring is a λ-ring.

Proposition 3.12. *Let R be a ϕ-λ-ring. Then R is a λ-ring.*

Proof. Let S and T be any overrings of R. Then either $\phi(S) \subseteq \phi(T)$ or $\phi(T) \subseteq \phi(S)$. If $S \subseteq T$, then we are done. Now, assume that $S \nsubseteq T$. We assert that $\phi(S) \nsubseteq \phi(T)$. Choose $x \in S \setminus T$. If $\phi(x) \in \phi(T)$, then $\phi(x) = \phi(y)$ for some $y \in T$. It follows that $x - y \in Ker(\phi)$. But $Ker(\phi) \subseteq Nil(R)$, by (*b*). Consequently, $x - y \in T$ and so $x \in T$, which is a contradiction. This proves our assertion. Similarly, if $T \nsubseteq S$, then $\phi(T) \nsubseteq \phi(S)$. Thus, we have either $S \subseteq T$ or $T \subseteq S$ and hence R is a λ-ring. □

Note that a λ-ring is not necessarily a ϕ-λ-ring as can be seen in the following example.

Example 3.13. *Consider $R = A(+)E$, where A is a non λ-domain and $E = qf(A)/A$. Then by the same argument as in Example 3.6, we have $R \in \mathcal{H}$. As every nonunit of R is a zero-divisor in R, we have $R = T(R)$. Thus, R is a λ-ring. Note that $R/Nil(R) \cong A$ as $Nil(R) = \{0\}(+)E$. It follows that $R/Nil(R)$ is not a λ-domain. Consequently, R is not a ϕ-λ-ring, by Proposition 3.2.*

Next we present a class of ϕ-λ-rings. Note that it is also a class of λ-rings, by Proposition 3.12.

Example 3.14. *Let R be same as in last example, where A is a λ-domain. Then $R/Nil(R)$ is a λ-domain. Consequently, R is a ϕ-λ-ring, by Proposition 3.2.*

In the next theorem, we show that a ring R is a ϕ-λ-ring if and only if $\phi(R)$ is a pullback of a λ-domain (up to isomorphism).

Theorem 3.15. *Let $R \in \mathcal{H}$. Then R is a ϕ-λ-ring if and only if $\phi(R)$ is isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional local ring with maximal ideal M, A/M is a λ-subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Let R be a ϕ-λ-ring. Take $T = R_{Nil(R)}$ and $A = \phi(R)$. Then by (e), $M = Nil(A)$ and T/M is the quotient field of A/M. Thus, we get the desired pullback diagram.

Conversely, assume that $\phi(R)$ is isomorphic to a ring A obtained from the above pullback diagram. Then $A \in \mathcal{H}_0$, by (a). Note that, by hypothesis, $Nil(A)$ is the prime ideal of T. It follows that $M = Nil(A)$. Since A/M is a λ-domain, A is a λ-ring, by Proposition 2.3. Thus, R is a ϕ-λ-ring. □

4. ϕ-Δ-rings

In this section, we introduce a new class of rings, namely, ϕ-Δ-rings and analyze the same. The definition is as follows.

Definition 4.1. *Let $R \in \mathcal{H}$ be a ring. Then R is said to be a ϕ-Δ-ring if $\phi(R)$ is a Δ-ring.*

In the end of [19, Section 3], Gilmer and Huckaba admitted his inability to produce an example of an integrally closed Δ-ring that is not a Prüfer ring. If such a ring exists, then that can not be in $\mathcal{H}_0$ as we have the next proposition that follows directly from [22, Lemma 1.1] and [12, Theorem 2.2].

Proposition 4.2. *Let $R \in \mathcal{H}_0$. If R is a Δ-ring, then R' is a Prüfer ring. In particular, an integrally closed Δ-ring in $\mathcal{H}_0$ is a Prüfer ring.*

A direct companion to above proposition is the next two corollaries.

Corollary 4.3. *Let $R \in \mathcal{H}_0$ be an integrally closed ring. Then R is a Δ-ring if and only if R is a Prüfer ring.*

Proof. The result follows from Proposition 4.2 and [20, Theorem 4]. □

Corollary 4.4. *Let R be a ring. Then the following hold:*

- *(i) If R is a ϕ-Δ-ring, then $\phi(R)'$ is a Prüfer ring. In particular, a ϕ-integrally closed ϕ-Δ-ring in $\mathcal{H}$ is a ϕ-Prüfer ring.*
- *(ii) R is a ϕ-integrally closed ϕ-Δ-ring if and only if R is a ϕ-Prüfer ring.*

Proof. Note that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}_0$, by ($a$). Now, ($i$) follows from Proposition 4.2 and (ii) follows from Corollary 4.3. □

Let $R \subseteq S$ be a ring extension such that $Nil(R) = Nil(S)$. Then it is easy to verify that $R \subseteq S$ is a Δ-extension if and only if $R/Nil(R) \subseteq S/Nil(R)$ is a Δ-extension. Now, the proof of next proposition follows mutatis mutandis from the proof of Proposition 2.3.

Proposition 4.5. *Let $R \in \mathcal{H}_0$. Then the following are equivalent:*

(i) R is a Δ-ring;
(ii) $R/Nil(R)$ is a Δ-domain with quotient field $T(R)/Nil(R)$.

Also, the proof of next proposition follows mutatis mutandis from the proof of Proposition 3.2.

Proposition 4.6. *Let $R \in \mathcal{H}$. Then the following are equivalent:*

(i) R is a ϕ-Δ-ring;
(ii) $R/Nil(R)$ is a Δ-domain.

In the next theorem, we show that every ϕ-Δ-ring is a Δ-ring. However, not every Δ-ring is a ϕ-Δ-ring as shown in Example 4.8.

Theorem 4.7. *Let R be a ϕ-Δ-ring. Then R is a Δ-ring.*

Proof. Let R_1, R_2 be overrings of R. Then $\phi(R_1), \phi(R_2)$ are overrings of $\phi(R)$. Since $\phi(R)$ is a Δ-ring, $\phi(R_1) + \phi(R_2)$ is a ring. Consider $a, b \in R_1 + R_2$. Then $a = r_1 + r_2$, $b = s_1 + s_2$ for some $r_1, s_1 \in R_1$ and $r_2, s_2 \in R_2$ and so $\phi(r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2) \in \phi(R_1) + \phi(R_2)$. It follows that $\phi(r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2) = \phi(t_1) + \phi(t_2)$ for some $t_1 \in R_1$ and $t_2 \in R_2$. Thus, $r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2 - t_1 - t_2 \in Ker(\phi)$. As by ($b$), $Ker(\phi) \subseteq Nil(R)$, we have $r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2 - t_1 - t_2 \in R \subseteq R_1 + R_2$. Thus, we have $ab = r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2 \in R_1 + R_2$ and hence $R_1 + R_2$ is a ring. Since R_1, R_2 was arbitrary overrings of R, R is a Δ-ring. □

Example 4.8. *Consider $R = A(+)E$, where A is a non Δ-domain and $E = qf(A)/A$. Then by the same argument as in Example 3.6, we conclude that $R \in \mathcal{H}$. Note that $R = T(R)$ as every nonunit of R is a zero-divisor in R. Thus, R is a Δ-ring. Since $R/Nil(R) \cong A$, $R/Nil(R)$ is not a Δ-domain. Thus, R is not a ϕ-Δ-ring, by Proposition 4.6.*

Now, we discuss some examples of a ϕ-Δ-ring.

Example 4.9. *(i) Let R be same as in the above example, where A is a Δ-domain. Then by another appeal to Proposition 4.6, R is a ϕ-Δ-ring.*
(ii) Let R be same as in the Example 3.3, where A is a Δ-domain. Then by applying the same argument as in the Example 3.3, we conclude that R is a ϕ-Δ-ring.

The next theorem shows that a ring $R \in \mathcal{H}$ is a ϕ-Δ-ring if and only if $\phi(R)$ is a pullback of a Δ-domain (up to isomorphism). We omit the proof as it follows mutatis mutandis from the proof of Theorem 3.15.

Theorem 4.10. *Let $R \in \mathcal{H}$. Then R is a ϕ-Δ-ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional local ring with maximal ideal M, A/M is an Δ-subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

In the next theorem, we show that ϕ-Δ property is a local property for any ring $R \in \mathcal{H}$.

Theorem 4.11. *Let $R \in \mathcal{H}$. Then R is a ϕ-Δ-ring if and only if R_P is a ϕ-Δ-ring for each prime ideal P of R.*

Proof. Let R be a ϕ-Δ-ring. Then $R/Nil(R)$ is a Δ-domain, by Proposition 4.6. By [20, Theorem 3], it follows that $(R/Nil(R))_{P/Nil(R)}$ is a Δ-domain for each prime ideal P of R. Note that $(R/Nil(R))_{P/Nil(R)} \cong R_P/Nil(R_P)$ and $R_P \in \mathcal{H}$ for all prime ideals P of R. Thus, by another appeal to Proposition 4.6, R_P is a ϕ-Δ-ring for all prime ideals P of R.

Conversely, assume that R_P is a ϕ-Δ-ring for each prime ideal P of R. Then by Proposition 4.6, $R_P/Nil(R_P)$ is a Δ-domain for each prime ideal P of R. As $(R/Nil(R))_{P/Nil(R)} \cong R_P/Nil(R_P)$, we conclude that $R/Nil(R)$ is a Δ-domain, by [20, Theorem 3]. Thus, by another application of Proposition 4.6, R is a ϕ-Δ-ring. □

In [20, Proposition 12], the equivalence of a Dedekind domain and a Δ-domain is established provided the domain is a Krull domain. In the next theorem, the equivalence of a ϕ-Dedekind ring and a ϕ-Δ-ring is shown provided the ring is a ϕ-Krull ring. Note that this generalizes [20, Proposition 12].

Theorem 4.12. *Let R be a ϕ-Krull ring. Then the following are equivalent:*

(i) R is a ϕ-Δ-ring;
(ii) R is a ϕ-Dedekind ring.

Proof. Note that $R/Nil(R)$ is a Krull domain, by [2, Theorem 3.1]. First, assume that R is a ϕ-Δ-ring. Then $R/Nil(R)$ is a Δ-domain, by Proposition 4.6. It follows that $R/Nil(R)$ is a Dedekind domain, by [20, Proposition 12]. Thus, by [2, Theorem 2.5], R is a ϕ-Dedekind ring.

Conversely, assume that R is a ϕ-Dedekind ring. Then $R/Nil(R)$ is a Dedekind domain, by [2, Theorem 2.5]. By [20, Proposition 12], it follows that $R/Nil(R)$ is a Δ-domain. Hence, by Proposition 4.6, R is a ϕ-Δ-ring. □

Note that by definition, every λ-ring is a Δ-ring. Similary, every ϕ-λ-ring is a ϕ-Δ-ring. However, the next two examples endorse that the converse is not true.

Example 4.13. *Let K be a field and $D_n = K+X^nK[[X]]$ for all non negative integers n. Then by [20, Proposition 10,11], D_n is not a λ-domain but a Δ-domain, for $n = 4, 5$. Now, consider $R_n = D_n(+)qf(D_n)$. Then note that $Z(R_n) = Nil(R_n) = \{0\}(+)qf(D_n)$. Clearly, it is a divided prime ideal of R_n. For if $(0, r) \in Nil(R_n)$ and $(a, s) \in R_n \setminus Nil(R_n)$, then $(0, r) = (a, s)(0, r/a)$. So, $R_n \in \mathcal{H}_0$. As $R_n/Nil(R_n) \cong D_n$, we conclude that by Proposition 2.3 and Proposition 4.5, R_n is not a λ-ring but a Δ-ring, for $n = 4, 5$.*

Example 4.14. *Let D be a non-local Prüfer domain of dimension n. Take $R = D(+)E$ where $E = qf(D)/D$. Then $R \in \mathcal{H}$ and is not a ϕ-λ-ring, by Example 3.6. Also, $R/Nil(R) \cong D$ and so R is a ϕ-Prüfer ring of dimension n, by [1, Theorem 2.6]. It follows that R is a ϕ-Δ-ring, by Corollary 4.4.*

We now end this paper with generalization of some results of Δ-domains [20] to Δ-rings. First we observe the following lemma whose proof is routine.

Lemma 4.15. *Let $R \subseteq S$ be a ring extension such that $Nil(R) = Nil(S)$. Then $Rad(R/Nil(R) : S/Nil(R)) = Rad(R : S)/Nil(R)$.*

The next lemma extends [20, Proposition 8] to rings in $\mathcal{H}_0$.

Lemma 4.16. *Let $R \in \mathcal{H}_0$ be a local ring. If R' is a quasi-valuation ring with maximal ideal M' such that $M' = Rad(R : R')$, then each overring of R is comparable with R' under inclusion.*

Proof. Note that $Nil(T(R)) = Nil(R)$, by (c). Since $Rad(R : R') = M'$, $Rad(R/Nil(R) : R'/Nil(R)) = M'/Nil(R)$, by Lemma 4.15. Also, $R'/Nil(R)$ is a valuation domain as R' is a *quasi-valuation ring*. Moreover, by (f), $(R/Nil(R))' = R'/Nil(R)$ and so each overring of $R/Nil(R)$ is comparable with $R'/Nil(R)$ under inclusion, by [20, Proposition 8]. Now, by (e), $T(R/Nil(R)) = T(R)/Nil(R)$ and hence the result follows. □

The following proposition extends [20, Corollary 3] to Δ-rings.

Proposition 4.17. *Let $R \in \mathcal{H}_0$ be a local ring, $Rad(R : R') = M'$, and R' be a quasi-valuation ring with maximal ideal M'. Then R is a Δ-ring if and only if $R \subseteq R'$ is a Δ-extension.*

Proof. Note that if R is a Δ-ring, then $R \subseteq R'$ is a Δ-extension. Conversely, assume that $R \subseteq R'$ is a Δ-extension. Let S, T be overrings of R. Now, by Lemma 4.16, the following three cases arise:

(i) $S \subseteq R', T \subseteq R'$. Then by hypothesis, the result follows.
(ii) $S \subseteq R' \subseteq T$ or $T \subseteq R' \subseteq S$. Clearly, $S + T$ is a ring.
(iii) $R' \subseteq S, R' \subseteq T$. Now, by Lemma 2.10, $R' \in \mathcal{H}_0$. It follows that R' is a λ-ring, by Corollary 2.5. Thus, $S + T$ is a ring.

□

In the last proposition, we extend [20, Proposition 9] to rings in $\mathcal{H}_0$.

Proposition 4.18. *Let $R \in \mathcal{H}_0$ be a local ring with maximal ideal M. Assume that R' is a quasi-valuation ring with maximal ideal M'. If $M = M'$, then the following are equivalent:*

(i) R is a λ-ring;
(ii) $R \subseteq R'$ is a Δ-extension.

Proof. $(i) \Rightarrow (ii)$ follows by definition. Now, assume that (ii) holds. Then $R/Nil(R) \subseteq R'/Nil(R)$ is a Δ-extension. Note that $Nil(T(R)) = Nil(R)$, $T(R/Nil(R)) = T(R)/Nil(R)$, and $(R/Nil(R))' = R'/Nil(R)$, by (c), (e), and (f), respectively. Since $R'/Nil(R)$ is a valuation domain with maximal ideal $M/Nil(R)$, $R/Nil(R)$ is a λ-domain, by [20, Proposition 9]. Thus, R is a λ-ring, by Proposition 2.3. □

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