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Evolution of Some Geometric Quantities along the Cotton Flow

Debabrata Chakraborty[a], Yadab Chandra Mandal[b], Shyamal Kumar Hui[b]

[a]*Department of Mathematics, Jagannath Kishore College, Purulia -723101, West Bengal, India*
[b]*Department of Mathematics, The University of Burdwan, Burdwan, 713104, West Bengal, India.*

Abstract. In this paper we have addressed the behaviour of Yamabe constant along the Cotton flow. We have also studied the evolution of ADM mass along the Cotton flow and it is shown that the ADM mass is conserved along the Cotton flow. Among others evolution of Bach tensor under Cotton flow is derived. It is shown that if the metric of a local conformally flat 3-manifold evolves under the Cotton flow, then the Bach tensor satisfies the heat equation.

1. Introduction

After successful applicability of Ricci flow [8], various geometric flows have been widely studied e.g. the mean curvature flow, Yamabe flow, cross curvature flow etc. We know that local conformally flatness of a manifold is measured by the vanishing of Weyl tensor. Since, the Weyl tensor vanishes identically for all 3-manifolds and this responsibility goes back to the Cotton tensor. This tensor was studied in the context of topologically massive gravity [7]. In [1], Kisisel et al. introduced a new geometric flow, called the Cotton flow. Unlike the Yamabe flow, which preserves the conformal class of the metric, Cotton flow tends to evolve an initial metric to the local conformally flat one [1]. One can also observe that the Yamabe flow and the Cotton flow are orthogonal. One of the most important geometric quantity which comes out from the solution of the Yamabe problem is the Yamabe invariant. The sign of Yamabe invariant carries important topological information for the manifold. In [6], Chang and Lu studied the evolution of Yamabe constant under the Ricci flow and under some technical assumption obtained that the Yamabe constant is non-decreasing along the Ricci flow. Motivated by the above studies, in this paper we have obtained evolution of the Yamabe constant along Cotton flow.

The paper is arranged as follows. Section 2 carries the preliminary discussions to get the results. Section 3 deals with the study of evolution of Yamabe constant. In section 4, we study the evolution of ADM mass under Cotton flow. It is known that the Bach tensor is a trace-free tensor of rank 2 and also it is conformally invariant in 4-dimension [3]. Before 1968, it was the only known conformally invariant tensor that is algebraically independent of the Weyl tensor [3]. Such tensor is significant in the field of differential geometry and general relativity. Here in the last section we consider the evolution of Bach tensor under Cotton flow and it is proved that if the metric of a locally conformally flat 3-manifold evolves under the Cotton flow, then the Bach tensor satisfies the heat equation.

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Corresponding author: Shyamal Kumar Hui
Email addresses: debabratamath@gmail.com (Debabrata Chakraborty), myadab436@gmail.com (Yadab Chandra Mandal), skhui@math.buruniv.ac.in (Shyamal Kumar Hui)

2. Preliminaries

Let g be a Riemannian metric of a n-Riemannian manifold with Riemannian connection ∇. The $(0,2)$ type Cotton-York tensor C of 3-manifold is

$$C^{ij} = \frac{\epsilon^{imn}}{\sqrt{g}}\nabla_m(R^j_n - \frac{1}{4}\delta^j_n R), \tag{1}$$

where R_{ij} is the Ricci tensor and R is the scalar curvature [12]. The important fact about the tensor C^{ij} is that, it is covariantly conserved, symmetric and traceless [12]. This property of the Cotton tensor motivated the authors in [1] to introduce the Cotton flow given by

$$\frac{\partial g_{ij}}{\partial t} = KC_{ij}, \text{ where } C^{ij} = g^{il}g^{jk}C_{kl} \tag{2}$$

and K is a positive constant.

One may scale the parameter t to set $K = 1$. In [5], Chow studied Yamabe flow for local conformally flat metrics and proved that the metric converges to a round metric in C^∞ topology. Keeping this fact in mind the constant K is choosen to be positive, so that the metric on the 3-sphere converge to the round metric rather than diverge from it. The fixed point of (2) are necessarily locally conformally flat metrics. Another interesting fact about the flow (2) is that, it preserves the volume and hence unlike the Ricci flow, we do not need any normalization. The above facts will be used frequently in our analysis. The evolution equations for the standard geometric quantities are given by

$$\frac{\partial}{\partial t}\Gamma^i_{jk} = \frac{1}{2}g^{il}(\nabla_j C_{kl} + \nabla_k C_{jl} - \nabla_l C_{jk}), \tag{3}$$

$$\frac{\partial}{\partial t}R_{ij} = 3R_{li}C^l_j - R^{lm}C_{lm}g_{ij} - \frac{1}{2}RC_{ij} - \frac{1}{2}\nabla^2 C_{ij}, \tag{4}$$

$$\frac{\partial R}{\partial t} = -C^{ij}R_{ij}. \tag{5}$$

Being a third order partial differential equation, there is no known technique available in the literature for the existence and uniqueness of (2). But in [1], the authors have been able to formalize (2) as a grade flow of some functional.

Now the Yamabe constant of g on a closed manifold M^n is given by

$$\mathcal{Y}(g) = \inf_{\substack{u\in C^\infty(M)\\ u>0}} \frac{\int_M(\frac{4(n-1)}{n-2}|\nabla u|^2 + R_g u^2)d\mu}{(\int_M u^{\frac{2n}{n-2}}d\mu)^{\frac{n-2}{n}}}, \tag{6}$$

where $d\mu$ is the volume form of g. The classical Euler-Lagrange equation for a minimizer u is

$$-\frac{4(n-1)}{n-2}\Delta u + R_g u = \mathcal{Y}(g)u^{\frac{n+2}{n-2}}, \tag{7}$$

$$\int_M u^{\frac{2n}{n-2}}d\mu = 1, \tag{8}$$

where Δ is a Laplace-Beltrami operator. Now from the solution of the Yamabe problem [10], one can note that there always exist a minimizer for (6). Thus having a solution u, the metric $u^{\frac{4}{n-2}}g$ is called the Yamabe metric and has constant scalar curvature $\mathcal{Y}(g)$.

Now the familiar subcritical regularization of (7) and (8) is defined by

$$-\frac{4(n-1)}{n-2}\Delta u + R_g u = \mathcal{Y}_p(g)u^p, \tag{9}$$

$$\int_M u^{p+1} d\mu = 1, \tag{10}$$

where $p \in (1, \frac{n+2}{n-2}]$ and $\mathcal{Y}_p(g)$ is a constant. Also (9) and (10) is nothing but the Euler-Lagrange equation for the minimizer of the functional

$$\mathcal{Y}_p(g) = \inf_{\substack{u \in C^\infty(M) \\ u>0}} \frac{\int_M (\frac{4(n-1)}{n-2}|\nabla u|^2 + R_g u^2) d\mu}{(\int_M u^{p+1} d\mu)^{\frac{2}{p+1}}}.$$

3. Evolution of Yamabe Constant

First we prove a lemma which will be used for later purpose.

Lemma 3.1. *Let M be a closed manifold and g be the Riemannian metric evolving by the Cotton flow. If $u \in C^\infty(M)$, then*

$$\frac{\partial}{\partial t}|\nabla u|^2 = -\langle C, |\nabla u|^2 \rangle + 2\langle \nabla u, \nabla h \rangle \tag{11}$$

and

$$\frac{\partial}{\partial t}(\Delta u) = -\langle C, Hess\ u \rangle + \Delta h, \tag{12}$$

where $h = \frac{\partial u}{\partial t}$ and 'Hess' is the Hessian operator.

Proof. For a local coordinate system (x^i), we compute that

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla u|^2 &= \frac{\partial}{\partial t}(g^{ij}\nabla_i u \nabla_j u) \\ &= (\frac{\partial}{\partial t} g^{ij})\nabla_i u \nabla_j u + 2g^{ij}\nabla_i u \nabla_j h \\ &= -g^{ik}g^{jl}C_{kl}\nabla_i u \nabla_j u + 2g^{ij}\nabla_i u \nabla_j h \\ &= -\langle C, |\nabla u|^2 \rangle + 2\langle \nabla u, \nabla h \rangle.\end{aligned}$$

This proves (11). Also,

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta u) &= \frac{\partial}{\partial t}(g^{ij}\nabla_i \nabla_j u) \\ &= \frac{\partial}{\partial t}(g^{ij})\nabla_i \nabla_j u + g^{ij}\frac{\partial}{\partial t}(\nabla_i \nabla_j u) \\ &= -g^{ik}g^{jl}C_{kl}\nabla_i \nabla_j u + \Delta h \\ &= -\langle C, Hess\ u \rangle + \Delta h.\end{aligned}$$

Now, we are ready to state our main result following the same technique used in [6].

Theorem 3.2. *Let $g(t), t \in [0, T)$, be a solution of the Cotton flow on a closed manifold M^3, where T is the maximal time of existence of the solution $g(t)$. Given $p \in (1, 5]$, assume that there is a C^1-family of positive functions $u(t), t \in [0, T)$ which satisfy*

$$-8\Delta_{g(t)} u(t) + R_{g(t)} u(t) = \widetilde{\mathcal{Y}}_p(t)\{u(t)\}^p,$$

$$\int_M \{u(t)\}^{p+1} d\mu_{g(t)} = 1,$$

where $\widetilde{\mathcal{Y}}_p$ is a function of t only. Then

$$\frac{\partial}{\partial t}\widetilde{\mathcal{Y}}_p(t) = -2\int_M u^2 C^{ij} R_{ij} d\mu.$$

Proof. We compute

$$\frac{\partial}{\partial t}\widetilde{\mathcal{Y}}_p(t) = \int_M 8\frac{\partial}{\partial t}|\nabla u|^2 d\mu + \int_M \frac{\partial R}{\partial t}u^2 d\mu + \int_M 2R_u h\, d\mu.$$

Using (11), (12) and the fact that volume element is preserved along the Cotton flow we have

$$\frac{\partial}{\partial t}\widetilde{\mathcal{Y}}_p(t) = \int_M (-8\langle C, |\nabla u|^2\rangle + 16\langle \nabla u, \nabla h\rangle) d\mu - \int_M C^{ij}R_{ij}u^2 d\mu + 2\int_M R_u h\, d\mu. \tag{13}$$

Now, taking derivative with respect to t we get from (9) that

$$-8\frac{\partial}{\partial t}(\Delta u(t)) + \frac{\partial R}{\partial t}u(t) + Rh = \frac{d}{dt}(\widetilde{\mathcal{Y}}_p(t))\{u(t)\}^p + p\widetilde{\mathcal{Y}}_p(t)\{u(t)\}^{p-1}h.$$

Using (12) in above equation, we get

$$8\langle C, Hess\ u\rangle - 8\Delta h - C^{ij}R_{ij}u + Rh = \frac{d}{dt}(\widetilde{\mathcal{Y}}_p(t))u^p + p\widetilde{\mathcal{Y}}_p(t)u^{p-1}h.$$

Multiplying both sides of above equation by $2u$ we get

$$\begin{aligned} -16u\Delta h + 2R_u h &= -16u\langle C, Hess\ u\rangle \\ &= 2u^2C^{ij}R_{ij} + 2\frac{d}{dt}(\widetilde{\mathcal{Y}}_p(t))u^{p+1} \\ &= 2p\widetilde{\mathcal{Y}}_p(t)u^p h. \end{aligned} \tag{14}$$

Now applying integration by parts from (13) and (14) we have,

$$\begin{aligned} \frac{\partial}{\partial t}(\widetilde{\mathcal{Y}}_p(t)) &= -8\int_M \langle C, |\nabla u|^2\rangle d\mu + \int_M -16(u\Delta h + 2R_u h) d\mu - \int_M C^{ij}R_{ij}u^2 d\mu \\ &= -8\int_M \langle C, |\nabla u|^2\rangle d\mu - 16\int_M u\langle C, Hess\ u\rangle d\mu + \int_M 2u^2C^{ij}R_{ij} d\mu + 2\frac{d}{dt}(\widetilde{\mathcal{Y}}_p(t)) + 2p\widetilde{\mathcal{Y}}_p(t)\int_M u^p h d\mu. \end{aligned} \tag{15}$$

From (10) we have

$$\frac{d}{dt}\int_M u^{p+1} d\mu = 0,$$

which implies that

$$\int_M u^p h\, d\mu = 0. \tag{16}$$

Now we have

$$\begin{aligned} \int_M \langle C, |\nabla u|^2\rangle d\mu &= \int_M g^{ik}g^{jl}C_{kl}\nabla_i u \nabla_j u\, d\mu \\ &= -\int_M g^{ik}g^{jl}u\nabla_i C_{kl}\nabla_j u\, d\mu \\ &= -\int_M u g^{jl}\nabla^k C_{kl}\nabla_j u\, d\mu \\ &= 0. \end{aligned} \tag{17}$$

Finally,

$$\begin{aligned}\int_M u\langle C, Hess\ u\rangle d\mu &= \int_M ug^{ik}g^{jl}C_{kl}\nabla_i\nabla_j u\ d\mu \\ &= -\int_M ug^{ik}g^{jl}\nabla_i C_{kl}\nabla_j u\ d\mu \\ &= -\int_M ug^{jl}\nabla^k C_{kl}\nabla_j u\ d\mu \\ &= 0.\end{aligned} \tag{18}$$

Using (16), (17) and (18) in (15) we have

$$\frac{d}{dt}(\widetilde{\mathcal{Y}}_p(t)) = -2\int u^2 C^{ij}R_{ij}d\mu.$$

Thus proof of Theorem 3.1 is complete.

Corollary 3.3. *Let M^3 be a closed manifold with non-negative Cotton tensor and positive Ricci curvature. If the metric evolves under Cotton flow, then the Yamabe constant is non-increasing along the flow.*

4. Evolution of ADM mass under Cotton flow

In this section, we discuss the evolution of an important physical and geometrical invariant, called ADM mass, under Cotton flow.

Definition 4.1. *A Riemannian manifold (M^n, g) is said to be asymptotically flat or asymptotically Euclidean if $M = M_0 \cap M_\infty$ with M_0 compact and $M_\infty \approx \mathbb{R}^n - B_R(0)$ for some $R > 0$ so that in the induced Euclidean coordinate system, the metric satisfies the asymptotic conditions*

$$g_{ij} = \delta_{ij} + o(r^{-\alpha}),\ \ \partial_k g_{ij} = o(r^{-\alpha-1}),\ \ \partial_k\partial_l g_{ij} = o(r^{-\alpha-2}),$$

where $\alpha > 0$ is the asymptotic order and r is the distance from a fixed base point with respect to the induced Euclidean metric.

Asymptotically Euclidean manifold has been intensively studied on general relativity as the spatial slice for the Minkowski space-time are asymptotically Euclidean.

In this setup, the total mass or ADM mass of the gravitational system can be defined by [10]

$$m(g) = \lim_{R\to\infty}\frac{1}{4w_n}\int_{S_R}(\partial_i g_{ij} - \partial_j g_{ii})\,dS_R, \tag{19}$$

where w_n denoted the volume of the $(n-1)$-sphere and S_R is the Euclidean sphere of radius R centered at the base point. Here dS_R is the volume element of S_R.

When the scalar curvature is integrable and $\alpha > \frac{n-2}{2}$, $m(g)$ is well defined and independent of the coordinate at infinity [4] and therefore is a metric invariant. The famous positive mass theorem, proved firstly by Schoen and Yau [11], states that $m(g) \geq 0$ if the scalar curvature is non-negative. Moreover, $m(g) = 0$ if and only if M is the Euclidean space. For such a manifold, one can define the asymptotic volume ratio by

$$\mu = \lim_{r\to\infty}\frac{V(B_r, g)}{w_n r^n},$$

where $V(B_r, g)$ is the volume of a metric ball of radius r and w_n is the volume of unit n-ball in Euclidean space. If $\mu = 1$, an almost Euclidean manifold is said to be almost locally Euclidean. For an almost locally Euclidean manifold, the mass is also defined by (19), except that S_R should be taken as the distant sphere. We now prove the following:

Theorem 4.2. *Under the Cotton flow, the asymptotic volume ratio remains invariant.*

Proof. The result follows from the evolution of the volume element under Cotton flow.

Corollary 4.3. *The almost locally Euclidean and almost Euclidean properties are preserved under Cotton flow.*

Theorem 4.4. *The ADM mass is conserved along the Cotton flow.*

Proof: Taking the manifold to be almost locally Euclidean, we compute

$$\begin{aligned}\frac{\partial}{\partial t}m(g(t)) &= \int_{S_R\to\infty}\left(\frac{\partial}{\partial t}g_{ij,i}-\frac{\partial}{\partial t}g_{ii,j}\right)ds^i\\ &= \int_{S_R\to\infty}(C_{ij,i}-C_{ii,j})ds^i\\ &= -\int_{S_R\to\infty}C_{ii,j}\,ds^i.\end{aligned}$$

As, the Cotton-York tensor is trace-free, we have

$$\frac{\partial}{\partial t}m(g(t)) = 0.$$

This proves the Theorem.

5. The Bach Tensor and the Cotton flow

The Bach tensor in dimension three is given by [9]

$$B_{ik} = \nabla_j C_{ijk}.$$

The Schouten tensor is given by

$$S_{ij} = R_{ij} - \frac{1}{2(n-1)}Rg_{ij}.$$

Then in terms of Schouten tensor, the Bach tensor can be written as [9]

$$B_{ik} = \nabla_j\nabla_k S_{ij} - \Delta S_{ik}. \tag{20}$$

In dimension three, the covariant derivative of the Cotton tensor is given by [9]

$$\nabla_j C_{ijk} = 3R_{ij}R_{kj} - \frac{3}{2}RR_{ik} - |Ric|^2 g_{ik} + \frac{R^2}{2}g_{ik} + \frac{1}{4}\nabla_k\nabla_i R - \Delta S_{ik}. \tag{21}$$

We need the following Schur lemma

$$\nabla_j S_{ij} = \frac{1}{4}\nabla_i R. \tag{22}$$

Using (22) in (21), we have [9]

$$\nabla_k\nabla_j C_{ijk} = R_{jl}C_{lji} \tag{23}$$

and hence the divergence of the Bach tensor is given by

$$\nabla_k B_{ik} = \nabla_k B_{ki} = R_{jl}C_{lji}. \tag{24}$$

To calculate the evolution of the Bach tensor along Cotton flow, we use the following formula:

$$(\partial_t - \Delta)B_{ik} = \nabla_j(\partial_t - \Delta)C_{ijk} - [\Delta, \nabla_j]C_{ijk} + 2R_{pj}\nabla_p C_{ijk} + [\partial_t, \nabla_j]C_{ijk}. \tag{25}$$

Using commutation for covariant derivatives we find [9]

$$\begin{aligned}[\Delta, \nabla_j]C_{ijk} &= \nabla_i R_{lp}C_{plk} - \nabla_p R_{li}C_{plk} + \nabla_k R_{lp}C_{ilp} \\ &\quad -\nabla_p R_{lk}C_{ilp} + R_{lp}\nabla_p C_{ilk} + 2R_{lp}\nabla_k C_{ilp} - 2R_{lp}\nabla_l C_{ipk} + \frac{1}{2}\nabla_p R C_{ikp} + 2R_{lk}B_{il} - RB_{ik}.\end{aligned} \tag{26}$$

To move further, we calculate evolution of Cotton tensor under Cotton flow. We calculate the time derivative directly from the expression of Cotton tensor in dimension 3, viz.

$$C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{4}(\nabla_k R g_{ij} - \nabla_j R g_{ik}),$$

$$\frac{\partial}{\partial t}C_{ijk} = \frac{\partial}{\partial t}\nabla_k R_{ij} - \frac{\partial}{\partial t}\nabla_j R_{ik} - \frac{1}{4}[(\nabla_k \frac{\partial R}{\partial t}g_{ij} + \nabla_k R\frac{\partial g_{ij}}{\partial t}) - (\nabla_j \frac{\partial R}{\partial t}g_{ik} + \nabla_j R\frac{\partial g_{ij}}{\partial t})]. \tag{27}$$

Using (4) we have,

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_k R_{ij} &= 3\nabla_k(R_{li}C^l_j) - g_{ij}\nabla_k(R^{lm}C_{lm}) - \frac{1}{2}\nabla_k(RC_{ij}) - \frac{1}{2}\nabla^3 C_{ij} \\ &\quad +(\nabla_l C_{ik} - \nabla_k C_{il}\nabla_i C_{kl})R_{jl} + (\nabla_l C_{jk} - \nabla_k C_{jl}\nabla_j C_{kl})R_{il} \\ &= 3\nabla_k(g^{lm}C_{jm}R_{li}) - g_{ij}\nabla_k(g^{il}g^{jm}R_{ij}C_{lm}) - \frac{1}{2}\nabla_k RC_{ij} \\ &\quad -\frac{1}{2}R\nabla_k C_{ij} - \frac{1}{2}\nabla^3 C_{ij} + (\nabla_l C_{ik} - \nabla_k C_{il} - \nabla_i C_{kl})R_{jl} + (\nabla_l C_{jk} - \nabla_k C_{jl} - \nabla_j C_{kl})R_{il} \\ &= 3g^{lm}R_{li}\nabla_k C_{jm} + 3g^{lm}C_{jm}\nabla_k R_{li} - \frac{1}{2}\nabla_k RC_{ij} - \frac{1}{2}R\nabla_k C_{ij} \\ &\quad -\frac{1}{2}\nabla^3 C_{ij} + (\nabla_l C_{ik} - \nabla_k C_{il} - \nabla_i C_{kl})R_{jl} + (\nabla_l C_{jk} - \nabla_k C_{jl} - \nabla_j C_{kl})R_{il} \\ &= 3R^m_i\nabla_k C_{jm} + 3C^l_j\nabla_k R_{li} - \frac{1}{2}\nabla_k RC_{ij} \\ &\quad -\frac{1}{2}R\nabla_k C_{ij} - \frac{1}{2}\nabla^3 C_{ij} + (\nabla_l C_{ik} - \nabla_k C_{il} - \nabla_i C_{kl})R_{jl} + (\nabla_l C_{jk} - \nabla_k C_{jl} - \nabla_j C_{kl})R_{il}.\end{aligned} \tag{28}$$

Since the Cotton tensor is traceless and covariantly conserved, we have from (27) and (28) that

$$\begin{aligned}\frac{\partial}{\partial t}C_{ijk} &= 3R^m_i\nabla_k C_{jm} + 3C^l_j\nabla_k R_{li} - \frac{1}{2}\nabla_k RC_{ij} - \frac{1}{2}R\nabla_k C_{ij} - \frac{1}{2}\nabla^3 C_{ij} \\ &\quad +(\nabla_l C_{ik} - \nabla_k C_{il} - \nabla_i C_{kl})R_{jl} + (\nabla_l C_{jk} - \nabla_k C_{jl} - \nabla_j C_{kl})R_{il} \\ &\quad -3R^m_i\nabla_j C_{km} - 3C^l_k\nabla_j R_{li} + \frac{1}{2}\nabla_j RC_{ik} + \frac{1}{2}R\nabla_j C_{ik} + \frac{1}{2}\nabla^3 C_{ik} \\ &\quad -(\nabla_l C_{ij} - \nabla_j C_{il} - \nabla_i C_{jl})R_{kl} - (\nabla_l C_{kj} - \nabla_j C_{kl} - \nabla_k C_{jl})R_{il} - \frac{1}{4}(\nabla_k RC_{ij} - \nabla_j RC_{ik}).\end{aligned} \tag{29}$$

Again we have

$$\Delta C_{ijk} = \Delta\nabla_k R_{ij} - \Delta\nabla_j R_{ik} - \frac{1}{4}\Delta\nabla_k R g_{ij} + \frac{1}{4}\Delta\nabla_j R g_{ik}. \tag{30}$$

Combining (29) and (30) we have the following:

Proposition 5.1. *If the metric of a* 3*-manifold* $(M^3, g(t))$ *evolves under Cotton flow, then the* $(3,0)$ *Cotton tensor satisfies the following PDE*

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)C_{ijk} &= 3(C^l_j\nabla_k R_{li}-C^l_k\nabla_j R_{li})-\frac{3}{4}(\nabla_k RC_{ij}-\nabla_j RC_{ik})+3R^m_i(\nabla_k C_{jm}-\nabla_j C_{km})\\ &+\frac{R}{2}(\nabla_j C_{ik}-\nabla_k C_{ij})+(\nabla_l C_{ik}-\nabla_k C_{il}-\nabla_i C_{kl})R_{jl}-(\nabla_l C_{ij}-\nabla_j C_{il}-\nabla_i C_{jl})R_{kl}\\ &-\frac{1}{2}(\nabla^3 C_{ij}-\nabla^3 C_{ik})-(\Delta\nabla_k R_{ij}-\Delta\nabla_j R_{ik})+\frac{1}{4}(\Delta\nabla_k Rg_{ij}-\Delta\nabla_j Rg_{ik}).\end{aligned} \tag{31}$$

Using the second Bianchi identity we get from (31) that

$$\begin{aligned}\nabla_j(\frac{\partial}{\partial t}-\Delta)C_{ijk} &= 3(C^l_j\nabla_j\nabla_k R_{li}-C^l_k\Delta R_{li})-\frac{3}{4}(C_{lj}\nabla_j\nabla_k R-\Delta RC_{ik}-\nabla_j R\nabla_j C_{ik})\\ &+\frac{R}{2}(\Delta C_{ik}-\nabla_j\nabla_k C_{ij})+(\nabla_j\nabla_l C_{ik}-\nabla_j\nabla_k C_{il}-\nabla_j\nabla_i C_{kl}R_{jl})\\ &-(\nabla_j\nabla_l C_{ij}-\Delta C_{il}-\nabla_j\nabla_i C_{jl})R_{kl}+(\nabla_l C_{ik}-\nabla_k C_{il}-\nabla_i C_{kl})\nabla_j R_{jl}\\ &-(\nabla_l C_{ij}-\nabla_j C_{il}-\nabla_i C_{jl})\nabla_j R_{kl}-\frac{1}{2}(\nabla^4 C_{ij}-\nabla^4 C_{ik})-\nabla_j\Delta C_{ijk},\end{aligned} \tag{32}$$

since the Cotton-York tensor is trace-free and covariantly conserved. Finally, we calculate

$$\begin{aligned}[\partial_t,\nabla_j]C_{ijk} &= -\partial_t\Gamma^p_{ij}C_{pjk}-\partial_t\Gamma^p_{jk}C_{ijp}\\ &= -\frac{1}{2}g^{pl}(\nabla_i C_{jl}+\nabla_j C_{il}-\nabla_l C_{ij})C_{pjk}-\frac{1}{2}g^{pl}(\nabla_j C_{kl}+\nabla_k C_{jl}-\nabla_l C_{jk})C_{ijp}\\ &= \frac{1}{2}\nabla^p C_{ij}C_{pjk}+\frac{1}{2}\nabla^p C_{jk}C_{ijp},\end{aligned} \tag{33}$$

where we have used the property that the Cotton-York tensor is covariantly conserved.
Combining (21), (25), (26), (31), (32) and (33) we have

$$\begin{aligned}\frac{\partial}{\partial t}B_{ik} &= \Delta B_{ik}+3(C^l_j\nabla_j\nabla_k R_{li}-C^l_k\Delta R_{li})-\frac{3}{4}(C_{lj}\nabla_j\nabla_k R-\Delta RC_{ik}-\nabla_j R\nabla_j C_{ik})\\ &+\frac{R}{2}(\Delta C_{ik}-\nabla_j\nabla_k C_{ij})+(\nabla_j\nabla_l C_{ik}-\nabla_j\nabla_k C_{il}-\nabla_j\nabla_i C_{kl})R_{jl}\\ &-(\nabla_j\nabla_l C_{ij}-\Delta C_{il}-\nabla_j\nabla_i C_{jl})R_{kl}+(\nabla_l C_{ik}-\nabla_k C_{il}-\nabla_i C_{kl})\nabla_j R_{jl}\\ &-(\nabla_l C_{ij}-\nabla_j C_{il}-\nabla_i C_{jl})\nabla_j R_{kl}-\frac{1}{2}(\nabla^4 C_{ij}-\nabla^4 C_{ik})-\nabla_j\Delta C_{ijk}\\ &-\nabla_i R_{lp}C_{plk}+\nabla_p R_{il}C_{plk}-\nabla_k R_{lp}C_{ilp}+\nabla_p R_{lk}C_{ilp}-\nabla_p R_{lp}C_{ilk}-2\nabla_k R_{lp}C_{ilp}\\ &+2R_{lp}\nabla_l C_{ipk}-\frac{1}{2}\nabla_p RC_{ikp}-2R_{lk}B_{il}+RB_{ik}+2R_{pj}\nabla_p C_{ijk}+\frac{1}{2}\nabla^p C_{ij}C_{pjk}+\frac{1}{2}\nabla^p C_{jk}C_{ijp}.\end{aligned} \tag{34}$$

This leads to the following:

Theorem 5.2. *Let* $(M^3, g(t))$ *be a complete* 3*-manifold evolving under Cotton flow. Then the Bach tensor evolves by (34).*

Corollary 5.3. *If the metric of a local conformally flat* 3*-manifold evolves under the Cotton flow, then the Bach tensor satisfies the heat equation given by* $\frac{\partial}{\partial t}B_{ik}=\Delta B_{ik}$.

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