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On Rough Convergence in 2-Normed Spaces and Some Properties

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Abstract. In this study, we investigated relationships between rough convergence and classical convergence and studied some properties about the notion of rough convergence, the set of rough limit points and rough cluster points of a sequence in 2-normed space. Also, we examined the dependence of r-limit $\mathrm{LIM}_2^r x_n$ of a fixed sequence (x_n) on varying parameter r in 2-normed space.

1. Introduction and Background

The concept of 2-normed spaces was initially introduced by Gähler [10, 11] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [15] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Gürdal and Açık [16] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [24] studied statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. Arslan and Dündar [1, 2] investigated the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [6, 17, 19, 23, 25, 27–29]).

The idea of rough convergence was first introduced by Phu [20] in finite-dimensional normed spaces. In [20], he showed that the set $\mathrm{LIM}^r x_i$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\mathrm{LIM}^r x_i$ on the roughness degree r. In another paper [21] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r-continuous at every point $x \in X$ under the assumption $dim Y < \infty$ and $r > 0$ where X and Y are normed spaces. In [22], he extended the results given in [20] to infinite-dimensional normed spaces. Aytar [4] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [5] studied that the r-limit set of the sequence is equal to the intersection of these sets and that r-core of the sequence is equal to the union of these sets. Recently, Dündar and Çakan [7–9] introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and studied the notions of rough convergence, $\mathcal{I}_2$-convergence and the sets of rough limit points and rough $\mathcal{I}_2$-limit points of a double sequence. Also, Arslan and Dündar [3] introduced rough convergence in 2-normed spaces.

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In this paper, we investigated relationships between rough convergence and classical convergence and studied some properties about the notion of rough convergence, the set of rough limit points and rough cluster points of a sequence in 2-normed space. Also, we examined the dependence of r-limit $\mathrm{LIM}_2^r x_n$ of a fixed sequence (x_n) on varying parameter r in 2-normed space. We note that our results and proof techniques presented in this paper are analogues of those in Phu's [20] paper. Namely, the actual origin of most of these results and proof techniques is them papers. The following our theorems and results are the extension of theorems and results in [20].

Now, we recall the concepts of 2-normed space, rough convergence and some fundamental definitions and notations (See [3, 4, 1218, 20-24]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x, y\| = \|y, x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim_{n\to\infty} \|x_n - L, y\| = 0$, for every $y \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

Example 1.1. *Let $x = (x_n) = (\frac{n}{n+1}, \frac{1}{n})$, $L = (1, 0)$ and $z = (z_1, z_2)$. It is clear that (x_n) convergent to $L = (1, 0)$ in 2-normed space.*

Let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_n) \subset \mathbb{R}^n$.

The sequence $x = (x_n)$ is said to be r-convergent to L, denoted by $x_n \xrightarrow{r} L$ provided that

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} : \;\; n \geq n_\varepsilon \Rightarrow \|x_n - L\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x_n := \{L \in \mathbb{R}^n : x_n \xrightarrow{r} L\}$$

is called the r-limit set of the sequence $x = (x_n)$. A sequence $x = (x_n)$ is said to be r-convergent if $\mathrm{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_n)$. For $r = 0$, we get the ordinary convergence.

Let (x_n) be a sequence in $(X, \|.,.\|)$ 2-normed linear space and r be a non-negative real number. (x_n) is said to be rough convergent (r-convergent) to L denoted by $x_n \xrightarrow{\|.,.\|}_r L$ if

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow \|x_n - L, z\| < r + \varepsilon \tag{1}$$

or equivalently, if

$$\limsup \|x_n - L, z\| \leq r, \tag{2}$$

for every $z \in X$.

If (1) holds L is an r-limit point of (x_n), which is usually no more unique (for $r > 0$). So, we have to consider the so-called r-limit set (or shortly r-limit) of (x_n) defined by

$$\mathrm{LIM}_2^r x_n := \{L \in X : x_n \xrightarrow{\|.,.\|}_r L\}. \tag{3}$$

The sequence (x_n) is said to be rough convergent if $\mathrm{LIM}_2^r x_n \neq \emptyset$. In this case, r is called a convergence degree of (x_n). For $r = 0$ we have the classical convergence in 2-normed space again. But our proper interest is case $r > 0$. There are several reasons for this interest. For instance, since an orginally convergent sequence (y_n) (with $y_n \to L$) in 2-normed space often cannot be determined (i.e., measured or calculated) exactly, one has to do with an approximated sequence (x_n) satisfying $\|x_n - y_n, z\| \leq r$ for all n and for every $z \in X$, where $r > 0$ is an upper bound of approximation error. Then, (x_n) is no more convergent in the classical sense, but for every $z \in X$,

$$\|x_n - L, z\| \leq \|x_n - y_n, z\| + \|y_n - L, z\| \leq r + \|y_n - L, z\|$$

implies that is r-convergent in the sense of (1).

Example 1.2. *The sequence $x = (x_n) = ((-1)^n, (-1)^n)$ is not convergent in 2-normed space $(X, \|.,.\|)$ but it is rough convergent to $L = (0, 0)$, for every $z \in X$. It is clear that*

$$LIM_2^r x_n = \begin{cases} \emptyset & , \ if \ \ r < 1 \\ [(-r, -r), (r, r)] & , \ otherwise. \end{cases}$$

Lemma 1.3 ([3], Theorem 2.2). *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. The sequence (x_n) is bounded if and only if there exist an $r \geq 0$ such that $\mathrm{LIM}_2^r x_n \neq \emptyset$. For all $r > 0$, a bounded sequence (x_n) is always contains a subsequence x_{n_k} with*

$$\mathrm{LIM}_2^{(x_{n_k}), r} x_{n_k} \neq \emptyset.$$

Lemma 1.4 ([3], Theorem 2.3). *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. For all $r \geq 0$, the r-limit set $\mathrm{LIM}_2^r x_n$ of an arbitrary sequence (x_n) is closed.*

Lemma 1.5 ([3], Theorem 2.4). *Let $(X, \|.,.\|)$ be a 2-normed space and consider a sequence $x = (x_n) \in X$. If $y_0 \in \mathrm{LIM}_2^{r_0} x_n$ and $y_1 \in \mathrm{LIM}_2^{r_1} x_n$, then*

$$y_\alpha := (1 - \alpha) y_0 + \alpha y_1 \in \mathrm{LIM}_2^{(1-\alpha) r_0 + \alpha r_1} x_n, \ \ for \ \ \alpha \in [0, 1].$$

2. MAIN RESULTS

Theorem 2.1. *Let $(X, \|.,.\|)$ be a 2-normed space, $x = (x_n)$ be a sequence in X, $r_1 \geq 0$ and $r_2 > 0$. $x = (x_n)$ is $(r_1 + r_2)$-convergent to L in X if and only if there exists a sequence $(y_n) \in X$ such that for every $z \in X$*

$$y_n \xrightarrow{\|.,.\|}_{r_1} L \ \ and \ \ \|x_n - y_n, z\| \leq r_2, \ \ n = 1, 2, \ldots . \tag{4}$$

Proof. Suppose that (4) is true. $y_n \xrightarrow{\|.,.\|}_{r_1} L$ means that for all $\varepsilon > 0$ and every $z \in X$ there exists an n_ε such that

$$\|y_n - L, z\| < r_1 + \varepsilon,$$

whenever $n \geq n_\varepsilon$. Since

$$\|x_n - y_n, z\| \leq r_2$$

then, for $n \geq n_\varepsilon$ and every $z \in X$ we have

$$\|x_n - L, z\| \leq \|x_n - y_n, z\| + |y_n - L, z\| < r_1 + r_2 + \varepsilon.$$

Hence, (x_n) is $(r_1 + r_2)$-convergent to L.

Now we let (x_n) is $(r_1 + r_2)$-convergent to L. We define a sequence (y_n) for every $z \in X$ as following:

$$y_n = \begin{cases} L & , \ if \ \|x_n - L, z\| \leq r_2, \\ x_n + r_2 \dfrac{(L - x_n)}{\|L - x_n, z\|} & , \ if \ \|x_n - L, z\| > r_2. \end{cases}$$

Then, we have

$$\|y_n - L, z\| \le \begin{cases} 0 & , \quad if\ \|x_n - L, z\| \le r_2 \\ \|x_n - L, z\| - r_2 & , \quad if\ \|x_n - L, z\| > r_2 \end{cases}$$

and so,

$$\|x_n - y_n, z\| \le r_2,$$

for $n = 1, 2, ...$ and every $z \in X$. Since $L \in \text{LIM}_2^{r_1+r_2} x_n$, by (3) we have

$$\limsup \|x_n - L, z\| \le r_1 + r_2,$$

and so,

$$\limsup \|y_n - L, z\| \le r_1,$$

for every $z \in X$. Hence, we have

$$y_n \xrightarrow{\|.,.\|}_{r_1} L.$$

This completed the proof. □

Theorem 2.2. *Let $(X, \|.,.\|)$ be a 2-normed space and (x_n) be a sequence in X. (x_n) converges to L if and only if $\text{LIM}_2^r x_n = \overline{B}_r(L)$, where $\overline{B}_r(L) := \{x_1 \in X : \|x_1 - L, z\| \le r\}$.*

Proof. It remains to show that $\text{LIM}_2^r x_n = \overline{B}_r(L)$ implies $x_n \to L$. Suppose that (x_n) has a cluster point L' different from L. Then, the point

$$\overline{L} := L + \frac{r}{\|L - L', z\|}(L - L')$$

satisfies

$$\|\overline{L} - L', z\| = r + \|L - L', z\| > r, \tag{5}$$

for every $z \in X$. Since L' is a cluster point, by definition inequality (5) implies that

$$\overline{L} \notin \text{LIM}_2^r x_n,$$

a contradiction to $\|\overline{L} - L, z\| = r$ and $\text{LIM}_2^r x_n = \overline{B}_r(L)$. Therefore, L is only cluster point of (x_n) as a bounded sequence (by Lemma 1.3) is some finite-dimensional normed space. Hence, (x_n) converges to L in X. □

Theorem 2.3. *If $(X, \|.,.\|)$ is a finite-dimensional strictly convex 2-normed space (that is, the closed unit ball is strictly convex) then, $\text{LIM}_2^r x_n$ is strictly convex, that is, $t_0, t_1 \in \text{LIM}_2^r x_n$ and $t_0 \ne t_1$ imply*

$$t_\lambda \in int(\text{LIM}_2^r x_n), \textit{ for all } \lambda \in (0, 1).$$

Proof. Suppose that 2-normed space $(X, \|.,.\|)$ is strictly convex. In order to prove the strict convexity of $\text{LIM}_2^r x_n$ it suffices to show that $t_0, t_1 \in \text{LIM}_2^r x_n$ and $t_o \ne t_1$ imply

$$t_{0.5} = \frac{1}{2}(t_0 + t_1) \in \text{int}(\text{LIM}_2^r x_n),$$

because for each $t_\lambda, 0 < \lambda < 1$, there exist $t_0', t_1' \in \text{LIM}_2^r x_n$ satisfying $t_o' \ne t_1'$ and

$$t_\lambda = \frac{1}{2}(t_0' + t_1').$$

This completed the proof. □

Theorem 2.4. *Assume (x_n) be a sequence in some (finite-dimensional) strictly convex 2-normed space. If there are $t_1, t_2 \in \text{LIM}_2^r x_n$ satisfying $\|t_1 - t_2, z\| = 2r$, for every $z \in X$ then, (x_n) converges to $\frac{1}{2}(t_1 + t_2)$.*

Proof. Let t_3 be an arbitrary cluster point of (x_n). Then $t_1, t_2 \in LIM_2^r x_n$ implies

$$\|t_1 - t_3, z\| \leq r \text{ and } \|t_2 - t_3, z\| \leq r, \tag{6}$$

for every $z \in X$. By assumption and (6), for every $z \in X$, we have

$$2r = \|t_1 - t_2, z\| \leq \|t_1 - t_3, z\| + \|t_2 - t_3, z\|$$

and so

$$\|t_1 - t_3, z\| = \|t_2 - t_3, z\| = r.$$

Since

$$\frac{1}{2}(t_2 - t_1) = \frac{1}{2}((t_3 - t_1) + (t_2 - t_3)) \text{ and } \|\frac{1}{2}(t_2 - t_1), z\| = r,$$

the strict convexity of the 2-normed space considered implies

$$\frac{1}{2}(t_2 - t_1) = t_3 - t_1 = t_2 - t_3.$$

Hence, we have $t_3 = \frac{1}{2}(t_1 + t_2)$. That means $\frac{1}{2}(t_1 + t_2)$ is the only cluster point of (x_n) as a bounded sequence(by Lemma 1.3) in some finite-dimensional 2-normed space. Hence, (x_n) must converge to $\frac{1}{2}(t_1 + t_2)$. □

The previous two assertions are concerned with the relation between a convergent sequence and its r-limit set in 2-normed space. In general, we do not expect sequences considered to be convergent, but to have several cluster points.

Theorem 2.5. *Let $(X, \|.,.\|)$ be a 2-normed space. Then,*

(a) If c is a cluster point of the sequence (x_n) then

$$\mathrm{LIM}_2^r x_n = \overline{B}_r(c). \tag{7}$$

(b) Let C be the set of all cluster points of $(x_n) \subset X$. Then,

$$\mathrm{LIM}_2^r x_n = \bigcap_{c \in C} \overline{B}_r(c) = \{L \in X : C \subseteq \overline{B}_r(L)\}. \tag{8}$$

Proof. (a) For an arbitrary cluster point c of (x_n) and for every $z \in X$, we have

$$\|L - c, z\| \leq r, \tag{9}$$

for all $L \in \mathrm{LIM}_2^r x_n$. Otherwise, since c is a cluster point of (x_n), there are infinite x_n such that for every $z \in X$,

$$\|L - x_n, z\| > r + \varepsilon,$$

where

$$\varepsilon := (\|L - c, z\| - r)/2 > 0,$$

which contradicts (1). Therefore, we have

$$\mathrm{LIM}_2^r x_n = \overline{B}_r(c).$$

(b) From (a), it is clear that

$$\mathrm{LIM}_2^r x_n \subseteq \bigcap_{c \in C} \overline{B}_r(c). \tag{10}$$

Now, we let $y \in \bigcap_{c \in C} \overline{B}_r(c)$, then for all $c \in C$ and every $z \in X$ we have

$$\|y - c, z\| \leq r,$$

which is equivalent to $C \subseteq \overline{B}_r(y)$, that is, we have

$$\bigcap_{c\in C} \overline{B}_r(c) \subseteq \{L \in X : C \subseteq \overline{B}_r(L)\}. \tag{11}$$

Conversely, we let $y \notin LIM_2^r x_n$ then, by definition, there is an $\varepsilon > 0$ such that there exist infinite x_n such that for every $z \in X$,

$$\|x_n - y, z\| \geq r + \varepsilon,$$

which implies the existence of a cluster point c of (x_n) with $\|y - c, z\| \geq r + \varepsilon$, i.e.,

$$C \nsubseteq \overline{B}_r(y) \text{ and } y \notin \{L \in X : C \subseteq \overline{B}_r(L)\}.$$

Therefore, if

$$y \in \mathrm{LIM}_2^r x_n$$

then,

$$y \in \{L \in X : C \subseteq \overline{B}_r(L)\},$$

that is, we have

$$\{L \in X : C \subseteq \overline{B}_r(L)\} \subseteq \mathrm{LIM}_2^r x_n. \tag{12}$$

Hence, from (10)-(12) we have

$$\mathrm{LIM}_2^r x_n = \bigcap_{c\in C} \overline{B}_r(c) = \{L \in X : C \subseteq \overline{B}_r(L)\}.$$

This completed the proof. □

By definition, $\mathrm{Lim\,sup}\{x_n\}$ is the set of cluster points of (x_n). Hence, by (9) we have

$$\mathrm{Lim\ sup}\{x_n\} \subset \overline{B}_r(L),$$

for all $L \in \mathrm{LIM}_2^r x_n$ and by (8)

$$\mathrm{LIM}_2^r x_n = \bigcap_{c\in \mathrm{Lim\,sup}\{x_n\}} \overline{B}_r(c).$$

Theorem 2.6. *Let $(X, \|.,.\|)$ be a 2-normed space. We have*

$$\mathrm{LIM}_2^r x_n = \mathrm{Lim\,inf}\, \overline{B}_r(x_n).$$

Proof. First let $y \in \mathrm{LIM}_2^r x_n$. For every $z \in X$, we define $y = (y_n)$ as following:

$$y_n = \begin{cases} x_n + \frac{r}{\|y-x_n,z\|}(y - x_n) & , \ if \|y - x_n, z\| > r \\ y & , \ otherwise. \end{cases}$$

Since for every $z \in X$,

$$\|y_n - y, z\| = \left| \frac{r}{\|y - x_n, z\|} - 1 \right| \|y - x_n, z\| = \big|\|y - x_n, z\| - r\big|,$$

then, we have

$$\|y_n - y, z\| = \begin{cases} \|y - x_n, z\| - r & , \ if \|y - x_n, z\| > r \\ 0 & , \ otherwise. \end{cases}$$

Therefore, $y \in \mathrm{LIM}_2^r x_n$ yields that y_n tends to y as $n \to \infty$. But $\|x_n - y_n, z\| \leq r$, i.e., $y_n \in \overline{B}_r(x_n)$. Consequently,

$$\lim_{n\to\infty} d(y, \overline{B}_r(x_n)) = 0,$$

which means by definition that $y \in \operatorname{Lim\,inf} \overline{B}_r(x_n)$. Hence,

$$\mathrm{LIM}_2^r x_n \subset \operatorname{Lim\,inf} \overline{B}_r(x_n).$$

Now, let $y \in \operatorname{Lim\,inf} \overline{B}_r(x_n)$. By definition, there exists a sequence (y_n) satisfying $y_n \to y$ and $y_n \in \overline{B}_r(x_n)$, i.e., for every $z \in X$, $\|x_n - y_n, z\| \leq r$. Therefore, Theorem 2.1 implies $y \in \mathrm{LIM}_2^r x_n$. Hence, we have

$$\operatorname{Lim\,inf} \overline{B}_r(x_n) \subset \mathrm{LIM}_2^r x_n$$

and so,

$$\mathrm{LIM}_2^r x_n = \operatorname{Lim\,inf} \overline{B}_r(x_n).$$

This completed the proof. □

The prior theorems are related to the r-limit properties determined for a constant degree of roughness r. Let us now investigate the dependence of r-limit $\mathrm{LIM}_2^r x_n$ of a fixed sequence (x_n) on a varying parameter r.

It follows from definition

$$\mathrm{LIM}_2^{r_1} x_n \subseteq \mathrm{LIM}_2^{r_2} x_n, \text{ if } r_1 < r_2. \tag{13}$$

This monotonicity is included in the following.

Theorem 2.7. *Let $(X, \|.,.\|)$ be a 2-normed space and suppose $r \geq 0$ and $\rho > 0$. Then,*

(a) $\mathrm{LIM}_2^r x_n + \overline{B}_\rho(0) \subseteq \mathrm{LIM}_2^{r+\rho} x_n$.

(b) $\overline{B}_\rho(y) \subseteq \mathrm{LIM}_2^r x_n$ *implies* $y \in \mathrm{LIM}_2^{r-\rho} x_n$.

Proof. (a) Let $y \in \mathrm{LIM}_2^r x_n$ and $t \in \overline{B}_\rho(0)$. By definition for all $\varepsilon > 0$ there exists an n_ε such that for every $z \in X$,

$$\|x_n - y, z\| < r + \varepsilon, \text{ if } n \geq n_\varepsilon$$

which implies by $\|t, z\| < \rho$ that

$$\|x_n - y - t, z\| \leq r + \rho + \varepsilon, \text{ if } n \geq n_\varepsilon.$$

Hence, $y + t \in LIM_2^{r+\rho} x_n$.

(b) Let c be an arbitrary cluster point of (x_n). If for every $z \in X$,

$$\|y - c, z\| > r - \rho,$$

then the point

$$L := y + \frac{\rho}{\|y - c, z\|}(y - c)$$

satisfies

$$\|L - c, z\| = \rho + \|y - c, z\| > \rho + (r - \rho) = r.$$

By (7), this yields $L \notin \mathrm{LIM}_2^r x_n$, a contradiction to $\|L - y, z\| = \rho$ and $\overline{B}_\rho(y) \subseteq \mathrm{LIM}_2^r x_n$. Hence, for every $z \in X$ $\|y - c, z\| \leq r - \rho$, for all cluster points $c \in C$. Consequently, it follows from (8)

$$y \in \bigcap_{c \in C} \overline{B}_{r-\rho}(c) = \mathrm{LIM}_2^{r-\rho} x_n.$$

This completed the proof. □

Now, define

$$\overline{r} := \inf\{r > 0 : LIM_2^r x_n \neq \emptyset\}. \tag{14}$$

By the monotonicity given in (13), we have

$$\mathrm{LIM}_2^r x_n \begin{cases} = \emptyset & , \quad r < \overline{r} \\ \neq \emptyset & , \quad r > \overline{r}. \end{cases} \tag{15}$$

Moreover, by Theorem 2.7, for all $r > \overline{r}$ and $\rho \in (0, r - \overline{r})$, $\mathrm{LIM}_2^r x_n$ always contains some ball with radius ρ, that means at least

$$\mathrm{int}(\mathrm{LIM}_2^r x_n) \neq \emptyset \text{ for } r > \overline{r}. \tag{16}$$

Therefore,

$$\mathrm{int}(\mathrm{LIM}_2^r x_n) = \emptyset, \text{ implies } r \leq \overline{r} \text{ and } \mathrm{LIM}_2^{r'} x_n = \emptyset, \text{ for } r' \in [0, r). \tag{17}$$

Theorem 2.8. *Let $(X, \|.,.\|)$ be a 2-normed space.*

(a) $r = \overline{r}$ if and only if

$$\mathrm{LIM}_2^r x_n \neq \emptyset \textit{ and } \mathrm{int}(\mathrm{LIM}_2^r x_n) = \emptyset. \tag{18}$$

(b) If $(X, \|.,.\|)$ is a finite-dimensional strictly convex space then, $r = \overline{r}$ if and only if $\mathrm{LIM}_2^r x_n$ is a singleton.

Proof. (a) Let $r = \overline{r}$. We have to show (18). It follows from Theorem 2.9 proved later that $\mathrm{LIM}_2^r x_n = \bigcap_{r' > \overline{r}} \mathrm{LIM}_2^{r'} x_n$. For $r' > \overline{r}$, $\mathrm{LIM}_2^{r'} x_n$ is nonempty by (15) and closed (by Lemma 1.4). And by (13), we have

$$\bigcap_{r' > \overline{r}} \mathrm{LIM}_2^{r'} x_n = \bigcap_{\overline{r} < r' \leq \overline{r}+1} \mathrm{LIM}_2^{r'} x_n$$

and $\mathrm{LIM}_2^{r'} x_n, r' \in (\overline{r}, \overline{r}+1]$ is a family of nonempty closed subsets in the compact set $\mathrm{LIM}_2^{\overline{r}+1} x_n$ having the finite intersection property. Hence, their intersection is nonempty and so,

$$\mathrm{LIM}_2^{\overline{r}} x_n \neq \emptyset.$$

If $\mathrm{int}(\mathrm{LIM}_2^r x_n) \neq \emptyset$, then it contains some ball $\overline{B}_\rho(y)$ with $\rho > 0$, and by Theorem(2.7) we have $\mathrm{LIM}_2^{r-\rho} x_n \neq \emptyset$, that is, $r > \overline{r}$. Thus, $r = \overline{r}$ yields $\mathrm{int}(\mathrm{LIM}_2^r x_n) = \emptyset$.

Suppose (18) holds. Since $\mathrm{LIM}_2^r x_n \neq \emptyset$, we have $r \geq \overline{r}$. On the other hand, by (17), $r \leq \overline{r}$ follows from $\mathrm{int}(\mathrm{LIM}_2^r x_n) = \emptyset$. Consequently, $r = \overline{r}$.

(b) If $LIM_2^r x_n$ is singleton then (18) is fulfilled. Therefore, by (a), $r = \overline{r}$. It remains to show that $\mathrm{LIM}_2^{\overline{r}} x_n$ is a singleton. This follows directly from its strict convexity (by Theorem 2.3), $\mathrm{LIM}_2^{\overline{r}} x_n \neq \emptyset$ and so,

$$\mathrm{int}(\mathrm{LIM}_2^{\overline{r}} x_n) = \emptyset.$$

This completed the proof. □

Theorem 2.9. *Let $(X, \|.,.\|)$ be a 2-normed space. The following holds:*

$$cl\left(\bigcup_{0 \leq r' < r} \mathrm{LIM}_2^{r'} x_n \right) \subseteq \mathrm{LIM}_2^r x_n = \bigcap_{r' > r} \mathrm{LIM}_2^{r'} x_n.$$

If $r \neq \overline{r}$, then

$$cl\left(\bigcup_{0 \leq r' < r} \mathrm{LIM}_2^{r'} x_n \right) = \mathrm{LIM}_2^r x_n.$$

Proof. By the monotonicity given in (13) and the closedness of r-limit (by Lemma 1.4) we have

$$cl\left(\bigcup_{0\leq r'<r} LIM_2^{r'} x_n\right) \subseteq \mathrm{LIM}_2^{r} x_n \subseteq \bigcap_{r'>r} \mathrm{LIM}_2^{r'} x_n.$$

Now, let an arbitrary $y \in X \setminus LIM_2^r x_n$. By definition, there is an $\varepsilon > 0$ such that for every $z \in X$,

$$\forall k \in \mathbb{N},\ \exists n \geq k : \ \|x_n - y, z\| \geq r + \varepsilon.$$

This implies for $r' < r + \varepsilon$ that $\varepsilon' := r + \varepsilon - r' > 0$ and

$$\forall k \in \mathbb{N},\ \exists n \geq k : \ \|x_n - y, z\| \geq r' + \varepsilon',$$

for every $z \in X$. Thus, $y \notin \mathrm{LIM}_2^{r'} x_n$ for $r' < r + \varepsilon$, which implies $y \notin \bigcap_{r'>r} \mathrm{LIM}_2^{r'} x_n$. Hence,

$$\mathrm{LIM}_2^{r} x_n = \bigcap_{r'>r} \mathrm{LIM}_2^{r'} x_n.$$

For $r < \overline{r}$, we have clearly

$$cl\left(\bigcup_{0\leq r'<r} \mathrm{LIM}_2^{r'} x_n\right) = \mathrm{LIM}_2^{r} x_n = \emptyset.$$

Let $r = r_1 > \overline{r}$ and $r_0 = (\overline{r} + r_1)/2$. Since $r_0 > \overline{r}$, we can choose a $y_0 \in \mathrm{LIM}_2^{r_0} x_n \neq \emptyset$. Select an arbitrary $y_1 \in \mathrm{LIM}_2^{r_1} x_n$. By Lemma 1.5, we have

$$y_\alpha = (1-\alpha)y_0 + \alpha y_1 \in \mathrm{LIM}_2^{(1-\alpha)r_0+\alpha r_1} x_n, \ \text{for } \alpha \in [0,1].$$

Therefore, as a result we have

$$y_\alpha \in \bigcup_{0\leq r'<r} LIM_2^{r'} x_n \text{ for } \alpha \in [0,1).$$

Since for every $z \in X$,

$$\|y_\alpha - y_1, z\| = (1-\alpha)\|y_0 - y_1, z\| \to 0, \ \text{as } \alpha \to 1,$$

then

$$y_1 \in cl\Big(\bigcup_{0\leq r'<r} LIM_2^{r'} x_n\Big).$$

Hence,

$$cl\Big(\bigcup_{0\leq r'<r} LIM_2^{r'} x_n\Big) = LIM_2^{r} x_n$$

holds true for $r > \overline{r}$, too. □

References

[1] M. Arslan, E. Dündar, $\mathcal{I}$-Convergence and $\mathcal{I}$-Cauchy Sequence of Functions In 2-Normed Spaces, Konuralp Journal of Mathematics **6**(1) (2018) 57–62.

[2] M. Arslan, E. Dündar, On $\mathcal{I}$-Convergence of sequences of functions in 2-normed spaces, Southeast Asian Bulletin of Mathematics **42** (2018) 491–502.

[3] M. Arslan, E. Dündar, Rough convergence in 2-normed spaces, Bulletin of Mathematical Analysis and Applications **10**(3) (2018) 1–9.

[4] S. Aytar, Rough statistical convergence, Numer. Funct. Anal. and Optimiz. **29**(3-4) (2008) 291–303.

[5] S. Aytar, The rough limit set and the core of a real requence, Numer. Funct. Anal. and Optimiz. **29**(3-4) (2008) 283–290.

[6] H. Çakalli, S. Ersan, New types of continuity in 2-normed spaces, Filomat **30**(3) (2016) 525–532.

[7] E. Dündar, C. Çakan, Rough $\mathcal{I}$-convergence, Gulf Journal of Mathematics **2**(1) (2014) 45–51.

[8] E. Dündar, C. Çakan, Rough convergence of double sequences, Demonstratio Mathematica **47**(3) (2014) 638–651.

[9] E. Dündar, On Rough $\mathcal{I}_2$-convergence, Numer. Funct. Anal. and Optimiz. **37**(4) (2016) 480–491.

[10] S. Gähler, 2-metrische Räume und ihre topologische struktur, Math. Nachr. **26** (1963) 115–148.

[11] S. Gähler, 2-normed spaces, Math. Nachr. **28** (1964) 1–43.

[12] H. Gunawan, M. Mashadi, On n-normed spaces, Int. J. Math. Math. Sci. **27**(10) (2001) 631–639.

[13] H. Gunawan, M. Mashadi, On finite dimensional 2-normed spaces, Soochow J. Math. **27**(3) (2001) 321–329.

[14] M. Gürdal, S. Pehlivan, The statistical convergence in 2-Banach spaces, Thai J. Math. **2**(1) (2004) 107–113.

[15] M. Gürdal, S. Pehlivan, Statistical convergence in 2-normed spaces, Southeast Asian Bull. Math. **33** (2009) 257–264.

[16] M. Gürdal, I. Açık, On $\mathcal{I}$-Cauchy sequences in 2-normed spaces, Math. Inequal. Appl. **11**(2) (2008) 349–354.

[17] M. Gürdal, On ideal convergent sequences in 2-normed spaces, Thai J. Math. **4**(1) (2006) 85–91.

[18] Ö. Kişi, E. Dündar, Rough $\mathcal{I}_2$-lacunary statistical convergence of double sequences, Journal of Inequalities and Applications **2018:230** (2018) 16 pages, https://doi.org/10.1186/s13660-018-1831-7

[19] M. Mursaleen, A. Alotaibi, On $\mathcal{I}$-convergence in random 2-normed spaces, Math. Slovaca **61**(6) (2011) 933–940.

[20] H. X. Phu, Rough convergence in normed linear spaces, Numer. Funct. Anal. and Optimiz. **22** (2001) 199–222.

[21] H. X. Phu, Rough continuity of linear operators, Numer. Funct. Anal. and Optimiz. **23** (2002) 139–146.

[22] H. X. Phu, Rough convergence in infinite dimensional normed spaces, Numer. Funct. Anal. and Optimiz. **24** (2003) 285–301.

[23] A. Şahiner, M. Gürdal, S. Saltan, H. Gunawan, Ideal convergence in 2-normed spaces, Taiwanese J. Math. **11**(5) (2007) 1477–1484.

[24] S. Sarabadan, S. Talebi, Statistical convergence and ideal convergence of sequences of functions in 2-normed spaces, Int. J. Math. Math. Sci. **2011** (2011) 10 pages, doi:10.1155/2011/517841.

[25] E. Savaş, M. Gürdal, Ideal Convergent Function Sequences in Random 2-Normed Spaces, Filomat **30**(3) (2016) 557–567.

[26] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly **66** (1959) 361–375.

[27] A. Sharma, K. Kumar, Statistical convergence in probabilistic 2-normed spaces, Mathematical Sciences **2**(4) (2008) 373–390.

[28] S. Yegül, E. Dündar, On Statistical Convergence of Sequences of Functions In 2-Normed Spaces, Journal of Classical Analysis **10**(1) (2017) 49–57.

[29] S. Yegül, E. Dündar, Statistical Convergence of Double Sequences of Functions and Some Properties In 2-Normed Spaces, Facta Universitatis Series Mathematics and Informatics **33**(5) (2018) 705–719.