Filomat 33:16 (2019), 5061–5075
https://doi.org/10.2298/FIL1916061C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Remarks on an Eigenvalue Problem for an Anisotropic Elliptic Equation with Indefinite Weight

Nguyen Thanh Chung[a]

[a]*Department of Mathematics, Quang Binh University, 312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam*

Abstract. In this paper, we consider an eigenvalue problem for an anisotropic elliptic equation with indefinite weight, in which the differential operator involves partial derivatives with different variable exponents. Under some suitable conditions on the growth rates of the anisotropic coefficients involved in the problem, we prove some results on the existence and non-existence of a continuous family of eigenvalues by using variational methods.

1. Introduction

In this paper, we are interested in the eigenvalue problem for the following anisotropic elliptic equation

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\right) = \lambda V(x)|u|^{q(x)-2}u \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, and p_i, $i = 1, 2, ..., N$ are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$, $q(x) > 1$ for all $x \in \overline{\Omega}$, $V : \Omega \to \mathbb{R}$ is an indefinite weight function in the sense that V can change sign in Ω, λ is a positive parameter.

In the particular case when $p_i(x) = p(x)$ for each i the differential operator involved in (1) becomes $\sum\limits_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p(x)-2}\partial_{x_i} u\right)$ and has similar properties with the $p(x)$-Laplace operator $\Delta_{p(x)}u = \text{div}(|\nabla u|^{p(x)}\nabla u)$. We know that the $p(x)$-Laplace operator helps us to model some materials involving nonhomogenities, such as electrorheological fluids, see [18], but if we want to seek for the model of an inhomogeneous material which has a different behavior on each direction we note it is not adequate. For this case, we need to use the differential operator of anisotropic type as in (1) and we then call (1) an anisotropic partial differential equation with variable exponent. Problems of this type were firstly studied by Mihailescu et al. in the celebrated paper [13], in which the authors developed some previous results on anisotropic elliptic

2010 *Mathematics Subject Classification*. Primary 35J60; Secondary 35J62, 35J70
Keywords. Eigenvalue problems; Anisotropic elliptic equations; Indefinite weight functions; Variable exponents; Variational methods
Received: 07 June 2019; Accepted: 11 July 2019
Communicated by Marko Nedeljkov
This research is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) (Grant N.101.02.2017.04).
Email address: ntchung82@yahoo.com (Nguyen Thanh Chung)

equations with constant exponents of Fragalà et al. [8] by considering eigenvalue problem (1) in the case when the weight function $V \equiv 1$ and obtained some interesting results. After that, many authors studied the existence of solutions for anisotropic partial differential equations by various methods, we refer the readers to some papers [1–4, 6, 9, 17, 20].

In [10], Kefi studied the nonlinear eigenvalue problem for the $p(x)$-Laplace operator of the form

$$\begin{cases} -\Delta_{p(x)} = \lambda V(x)|u|^{q(x)-2}u \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, $V : \Omega \to \mathbb{R}$ is a indefinite weight function belonging to $L^{s(x)}(\Omega)$ in the sense that V may change sign in Ω and $p, q, s \in C(\overline{\Omega})$. Using variational arguments based on the Ekeland variational principle [7], the author proved that any $\lambda > 0$ sufficient small is an eigenvalue of problem (2) under the following sublinear condition at infinify

$$1 < q(x) < p(x) \leq N < s(x), \quad \forall x \in \overline{\Omega}.$$

In spired by the ideas introduced by Kefi [10] and some results on anisotropic elliptic equations mentioned above, in this paper we study a class of nonlinear eigenvalue problems for anisotropic elliptic equations with indefinite weight of the form (1). Our goal is to complement and extend the previous ones in [13] and [10] in the sense that Mihailescu et al. [13] considered problem (1) in the special case $V \equiv 1$ while Kefi [10] considered $p(x)$-Laplacian problem (2) only in the sublinear case. We also find that our results are better than those presented in [15, 16] since we consider the problem in the anisotropic case. By considering two different situations concerning the growth rates involved in the problem, we prove the existence of a continuous family of eigenvalues by using variational methods. It should be noticed that our situations here are different from ones considered in [6, 14, 17].

The remainder of the paper is organized as follows. In Section 2, we will recall the definitions and some properties of generalized Lebesgue-Sobolev spaces and anisotropic variable exponent Sobolev spaces. The readers can consult the papers [5, 13] for details on this class of functional spaces. In Section 3 we will state and prove the main results of the paper.

2. Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the books [5, 18], the papers of Kováčik et al. [11] and Mihailescu et al. [13]. Set

$$C_+(\overline{\Omega}) := \{h;\ h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \inf_{x\in\overline{\Omega}} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \text{a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf\left\{ \mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \tag{3}$$

holds true. Moreover, if $p_1, p_2, p_3 \in C_+(\overline{\Omega})$ and $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1$, then for any $u \in L^{p_1(x)}(\Omega)$, $v \in L^{p_2(x)}(\Omega)$ and $w \in L^{p_3(x)}(\Omega)$ we have

$$\left|\int_\Omega uvw\,dx\right| \leq \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\right)|u|_{p_1(x)}|v|_{p_2(x)}|w|_{p_3(x)}. \tag{4}$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold

$$|u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} \tag{5}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} \tag{6}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \;\Leftrightarrow\; \rho_{p(x)}(u_n - u) \to 0. \tag{7}$$

Proposition 2.1. *Let $p(x)$ and $q(x)$ be measurable functions such that $p \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq +\infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$ and $u \neq 0$. Then we have*

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|_{p(x)q(x)}^{p^+} \leq \||u|^{p(x)}|_{q(x)} \leq |u|_{p(x)q(x)}^{p^-},$$

$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|_{p(x)q(x)}^{p^-} \leq \||u|^{p(x)}|_{q(x)} \leq |u|_{p(x)q(x)}^{p^+}.$$

In particular, if $p(x) = p$ is a constant then $\||u|^p|_{q(x)} = |u|_{pq(x)}^p$.

If $p \in C_+(\overline{\Omega})$ the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, consisting of functions $u \in L^{p(x)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $[L^{p(x)}(\Omega)]^N$, endowed with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)},$$

is a separable and reflexive Banach space. The space of smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$, but if the exponent $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is,

$$|p(x) - p(y)| \leq -\frac{M}{\log(|x-y|)}, \quad \forall x, y \in \Omega, \quad |x-y| \leq \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$ and so we can define the space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}$$

by the $p(x)$-Poincaré inequality. We point out that the above norm is equivalent with the following norm

$$\|u\|_{p(x)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p(x)},$$

provied that $p(x) \geq 2$ for all $x \in \overline{\Omega}$. The space $\left(W_0^{1,p(x)}(\Omega), \|.\|_{p(x)}\right)$ is a separable and Banach space.

Proposition 2.2. *If $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding*

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = \infty$ if $p(x) > N$.

We introduce a natural generalization of the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ that will enable us to study problem (1) with sufficient accuracy. Define $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\overrightarrow{p}(x) = (p_1(x), p_2(x), .., p_N(x))$, the components $p_i \in C(\overline{\Omega})$, $i \in \{1, 2, ..., N\}$ are logarithmic Hölder continuous, that is, there exists $M > 0$ such that $|p_i(x) - p_i(y)| \leq -\frac{M}{\log(|x-y|)}$ for any $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ and $i \in \{1, 2, ..., N\}$. We introduce the anisotropic variable exponent Sobolev space, $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, as the colsure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(x)} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(x)}.$$

Then $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is a reflexive and separable Banach space, see [13]. In the case when p_i are all constant functions the resulting anisotropic space is denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1, p_2, ..., p_N)$. The theory of such spaces has been developed in [8]. Let us introduce $\overrightarrow{P}_+, \overrightarrow{P}_- \in \mathbb{R}^N$ and $P_+^+, P_-^+, P_+^-, P_-^- \in \mathbb{R}^+$ as

$$\overrightarrow{P}_+ = (p_1^+, p_2^+, ..., p_N^+), \quad \overrightarrow{P}_- = (p_1^-, p_2^-, ..., p_N^-)$$

and

$$P_+^+ = \max\{p_1^+, p_2^+, ..., p_N^+\}, \quad P_-^+ = \max\{p_1^-, p_2^-, ..., p_N^-\},$$
$$P_+^- = \min\{p_1^+, p_2^+, ..., p_N^+\}, \quad P_-^- = \min\{p_1^-, p_2^-, ..., p_N^-\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \tag{8}$$

and define $P_-^* \in \mathbb{R}$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \quad P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

Proposition 2.3 (see [13, Theorem 1]). *If $s \in C_+(\overline{\Omega})$ satisfies $1 < s(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous.*

3. Main results

In this section, we will state and prove the main results of the paper. Let us denote by X the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ and denote by c_i a general positive constant whose value may change from line to line. Problem (1) will be considered in two different situations when the nonlinearity is sublinear or superlinear at infinify in the sense that $q^+ < P_-^-$ or $P_+^+ < q^-$, respectively.

Definition 3.1. *We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1) if there exists $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)\backslash\{0\}$ such that*

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \partial_{x_i} v\,dx - \lambda \int_\Omega V(x)|u|^{q(x)-2}uv\,dx = 0$$

for all $v \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$. If λ is an eigenvalue of problem (1) then the corresponding eigenfunction $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)\backslash\{0\}$ is a weak solution of (1).

Our first result concerns the existence of a continuous family of eigenvalues for problem (1) in a neighborhood of the origin.

Theorem 3.2. *Assume that the following conditions hold:*

(H1) $1 < q(x) < P_-^- \leq \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-}} < N < s(x)$ *for all* $x \in \overline{\Omega}$;

(H2) $V \in L^{s(x)}(\Omega)$ *and there exists a measurable set* $\Omega_0 \subset \Omega$, $|\Omega_0| > 0$ *such that* $V(x) > 0$ *for all* $x \in \overline{\Omega}_0$.

Then there exists $\overline{\lambda} > 0$ such that any $\lambda \in (0, \overline{\lambda})$ is an eigenvalue of problem (1).

Proof. Let us consider the energy functional $J_\lambda : X \to \mathbb{R}$ given by the formula

$$J_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) = \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}|\partial_{x_i} u|^{p_i(x)}\,dx, \quad \Psi(u) = \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx. \tag{9}$$

From (H1)-(H2), using Proposition 2.1, it is clear that for all $u \in X$,

$$\begin{aligned}|\Psi(u)| &\leq \frac{2}{q^-}|V|_{s(x)}||u|^{q(x)}|_{\frac{s(x)}{s(x)-1}}\\ &\leq \begin{cases} \frac{2}{q^-}|V|_{s(x)}|u|^{q^-}_{\frac{s(x)q(x)}{s(x)-1}} & \text{if } |u|_{\frac{s(x)q(x)}{s(x)-1}} \leq 1,\\ \frac{2}{q^-}|V|_{s(x)}|u|^{q^+}_{\frac{s(x)q(x)}{s(x)-1}} & \text{if } |u|_{\frac{s(x)q(x)}{s(x)-1}} \geq 1.\end{cases}\end{aligned}$$

On the other hand, by (H1)-(H2), we have $\alpha(x) = \frac{s(x)q(x)}{s(x)-1} < P_-^*$ and $\beta(x) = \frac{s(x)q(x)}{s(x)-q(x)} < P_-^*$ for all $x \in \overline{\Omega}$ and by Proposition 2.3, the embeddings $X \hookrightarrow L^{\alpha(x)}(\Omega)$ and $X \hookrightarrow L^{\beta(x)}(\Omega)$ are continuous and compact. Thus, the functional J_λ is well-defined on X. The proof is divided into the following four steps.

Step 1. We prove that $J_\lambda \in C^1(X, \mathbb{R})$ and its derivative is

$$J'_\lambda(u)(v) = \sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u \cdot \partial_{x_i} v\,dx - \lambda \int_\Omega V(x)|u|^{q(x)-2}uv\,dx$$

for all $u, v \in X$. Hence, we can find weak solutions of problem (1) as the critical points of the functional J_λ in the space X.

We first need to prove that $\Psi \in C^1(X, \mathbb{R})$, that is, for given $u \in X$ we show that for all $v \in X$,

$$\lim_{t\to 0} \frac{\Psi(u+tv)-\Psi(u)}{t} = \Psi'(u)(v),$$

and $\Psi' : X \to X^*$ is continuous, where we denote by X^* the dual space of X.

Indeed, by conditions (H1)-(H2), for $|t| < 1$ using inequality (4) and Proposition 2.1, it implies that

$$\begin{aligned}
\int_\Omega \left| V(x)|u+tv|^{q(x)-2}(u+tv)v \right| dx &\le \int_\Omega |V(x)||u+tv|^{q(x)-1}|v|\,dx \\
&\le \int_\Omega |V(x)|(|u|+|v|)^{q(x)-1}|v|\,dx \\
&\le 3|V(x)|_{s(x)} \left| |u|+|v|^{q(x)-1} \right|_{\frac{q(x)}{q(x)-1}} |v|_{\beta(x)} \\
&\le 3|V|_{s(x)} \, \||u|+|v|\|_{q(x)}^{q^\tau-1} \, |v|_{\beta(x)} \\
&< +\infty,
\end{aligned}$$

where $\tau = +$ if $\||u|+|v|\|_{q(x)} > 1$ and $\tau = -$ if $\||u|+|v|\|_{q(x)} \le 1$ since $X \hookrightarrow L^{\beta(x)}(\Omega)$, $X \hookrightarrow L^{q(x)}(\Omega)$ and $V \in L^{s(x)}(\Omega)$.

For all $v \in X$, using the Lebesgue theorem we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\Psi(u+tv)-\Psi(u)}{t} &= \frac{d}{dt}\Psi(u+tv)\Big|_{t=0} \\
&= \left(\frac{d}{dt}\int_\Omega \frac{V(x)}{q(x)}|u+tv|^{q(x)}\,dx\right)\Bigg|_{t=0} \\
&= \int_\Omega \frac{d}{dt}\left(\int_\Omega \frac{V(x)}{q(x)}|u+tv|^{q(x)}\,dx\right)\Bigg|_{t=0} dx \\
&= \int_\Omega V(x)|u+tv|^{q(x)-2}(u+tv)v|_{t=0}\,dx \\
&= \int_\Omega V(x)|u|^{q(x)-2}uv\,dx \\
&= \Psi'(u)(v).
\end{aligned}$$

Since the embedding $X \hookrightarrow L^{\beta(x)}(\Omega)$ is continuous, there exists $c_1 > 0$ such that $|v|_{\beta(x)} \le c_1\|v\|_{\overrightarrow{p}(x)}$ for all $v \in X$ and by condition (H1)-(H2),

$$\begin{aligned}
|\Psi'(u)(v)| &= \left|\int_\Omega V(x)|u|^{q(x)-2}uv\,dx\right| \\
&\le \int_\Omega |V(x)||u|^{q(x)-1}|v|\,dx \\
&\le 3|V|_{s(x)}\,|u|_{q(x)}^{q^\tau-1}\,|v|_{\beta(x)} \\
&\le 3c_1|V|_{s(x)}\,|u|_{q(x)}^{q^\tau-1}\,\|v\|_{\overrightarrow{p}(x)}
\end{aligned}$$

for any $v \in X$, where $\tau = +$ if $|u|_{q(x)} > 1$ and $\tau = -$ if $|u|_{q(x)} \le 1$. Combining this with the linearity of Ψ' we deduce that $\Psi' \in X^*$. Note that the map $u \mapsto |u|^{q(x)-2}u$ from $L^{q(x)}(\Omega)$ into $L^{\frac{q(x)}{q(x)-1}}(\Omega)$ is continuous. For the Fréchet differentiability, we conclude that Ψ is Fréchet differentiable and

$$\Psi'(u)(v) = \int_\Omega V(x)|u|^{q(x)-2}uv\,dx$$

for all $u, v \in X$. Similarly, we can show that $\Phi \in C^1(X, \mathbb{R})$ and

$$\Phi'(u)(v) = \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \cdot \partial_{x_i} v \, dx$$

for all $u, v \in X$. The step 1 is completed.

Step 2. We prove that there exists $\overline{\lambda} > 0$ such that for any $\lambda \in (0, \overline{\lambda})$, there exist constants $r, \rho > 0$ such that $J_\lambda(u) \geq r$ for all $u \in X$ with $\|u\|_{\vec{p}(x)} = \rho$.

Indeed, since $\alpha(x) = \frac{s(x)q(x)}{s(x)-1} < P_-^*$, the embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$ is continuous, there exists $c_2 > 0$ such that

$$|u|_{\alpha(x)} \leq c_2 \|u\|_{\vec{p}(x)}, \quad \forall u \in X.$$

Now, let us assume that $\|u\|_{\vec{p}(x)} = \rho < \min\left\{1, \frac{1}{c_2}\right\}$ sufficiently small, where c_2 is given in the above inequality. Then we have $|u|_{\alpha(x)} < 1$. For such an element u we get $|\partial_{x_i} u|_{p_i(x)} < 1$ for all $i = 1, 2, ..., N$. Using (6) and some simple computations, we obtain

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)} \, dx &\geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(x)}^{p_i^+} \\ &\geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(x)}^{P_+^+} \\ &\geq N \left(\frac{\sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(x)}}{N} \right)^{P_+^+} \\ &= \frac{\|u\|_{\vec{p}(x)}^{P_+^+}}{N^{P_+^+ - 1}}. \end{aligned} \tag{10}$$

Thus, by (10) for any $u \in X$ with $\|u\|_{\vec{p}(x)} = \rho$ small enough,

$$\begin{aligned} J_\lambda(u) &= \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} \, dx - \lambda \int_\Omega \frac{V(x)}{q(x)} |u|^{q(x)} \, dx \\ &\geq \frac{\|u\|_{\vec{p}(x)}^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} - \lambda \frac{2c_2^{q^-}}{q^-} |V|_{s(x)} \|u\|_{\vec{p}(x)}^{q^-} \\ &= \frac{1}{P_+^+ N^{P_+^+ - 1}} \rho^{P_+^+} - \lambda \frac{2c_2^{q^-}}{q^-} |V|_{s(x)} \rho^{q^-} \\ &= \rho^{q^-} \left(\frac{1}{P_+^+ N^{P_+^+ - 1}} \rho^{P_+^+ - q^-} - \lambda \frac{2c_2^{q^-}}{q^-} |V|_{s(x)} \right). \end{aligned}$$

Putting

$$\overline{\lambda} = \frac{\rho^{P_+^+ - q^-}}{2P_+^+ N^{P_+^+ - 1}} . \frac{q^-}{2c_2^{q^-} |V|_{s(x)}},$$

then for any $\lambda \in (0, \overline{\lambda})$ and $u \in X$ with $\|u\|_{\vec{p}(x)} = \rho$, there exists $r = \frac{\rho^{P_+^+}}{2P_+^+ N^{P_+^+ - 1}}$ such that $J_\lambda(u) \geq r > 0$.

Step 3. We prove that there exists $\varphi \in X$ such that $\varphi \geq 0$, $\varphi \neq 0$ and $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough.

Indeed, condition (H1) implies that $q(x) < P_-^-$ for all $x \in \overline{\Omega}_0$. In the sequel, we use the notation $q_0^- = \inf_{x \in \overline{\Omega}_0} q(x)$. Let $\epsilon_0 > 0$ be such that $q_0^- + \epsilon_0 < P_-^-$. We also have since $q \in C(\overline{\Omega}_0)$ that there exists an open subset $\Omega_1 \subset \Omega_0$ such that

$$|q(x) - q_0^-| < \epsilon_0, \quad \forall x \in \Omega_1$$

and thus

$$q(x) \leq q_0^- + \epsilon_0 < P_-^-, \quad \forall x \in \Omega_1.$$

Let $\varphi \in C_0^\infty(\Omega_0)$ such that $\overline{\Omega}_1 \subset \mathrm{supp}(\varphi)$, $\varphi(x) = 1$ for all $x \in \overline{\Omega}_1$ and $0 \leq \varphi \leq 1$ in Ω_0. Then, using the above information, for any $t \in (0,1)$ we have

$$\begin{aligned} J_\lambda(t\varphi) &= \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} t\varphi|^{p_i(x)}\, dx - \lambda \int_\Omega \frac{V(x)}{q(x)} |t\varphi|^{q(x)}\, dx \\ &\leq \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^N \int_{\Omega_0} |\partial_{x_i}\varphi|^{p_i(x)}\, dx - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} |\varphi|^{q(x)}\, dx \\ &\leq \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^N \int_{\Omega_0} |\partial_{x_i}\varphi|^{p_i(x)}\, dx - \frac{\lambda t^{q_0^- + \epsilon_0}}{q_0^+} \int_{\Omega_1} V(x) |\varphi|^{q(x)}\, dx. \end{aligned}$$

Therefore, $J_\lambda(t\varphi) < 0$ for $0 < t < \delta^{\frac{1}{P_-^- - q_0^- - \epsilon_0}}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda P_-^-}{q_0^+} . \frac{\int_{\Omega_1} V(x)|\varphi|^{q(x)} dx}{\sum_{i=1}^N \int_{\Omega_0} |\partial_{x_i}\varphi|^{p_i(x)}\, dx}\right\}.$$

The above fraction is meaningful if we can show that $\sum_{i=1}^N \int_{\Omega_0} |\partial_{x_i}\varphi|^{p_i(x)}\, dx > 0$. Indeed, it is clear that

$$\int_{\Omega_1} |\varphi|^{q(x)}\, dx \leq \int_\Omega |\varphi|^{q(x)}\, dx \leq \int_\Omega |\varphi|^{q^-}\, dx.$$

On the other hand, the space X is continuously embedded in $L^{q^-}(\Omega)$ and thus, there exists $c_3 > 0$ such that $|\varphi|_{q^-} \leq c_3 \|\varphi\|_{\overrightarrow{p}(x)}$, which implies that $\|\varphi\|_{\overrightarrow{p}(x)} > 0$. Combining this with (5) or (6) the claim follows at once.

By Step 2, we have

$$\inf_{u \in \partial B_\rho(0)} J_\lambda(u) > 0.$$

We also have from Step 2 again, the functional J_λ is bounded from below on $B_\rho(0)$. Moreover, by Step 3, there exists $\varphi \in X$ such that $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. It follows from Step 2 that

$$J_\lambda(u) \geq \frac{\|u\|_{\overrightarrow{p}(x)}^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} - \frac{\lambda c_2^{q^-}}{q^-} |V|_{s(x)} \|u\|_{\overrightarrow{p}(x)}^{q^-},$$

which yields

$$-\infty < \underline{c}_\lambda = \inf_{u \in \overline{B}_\rho(0)} J_\lambda(u) < 0.$$

Let us choose $\epsilon > 0$ such that $0 < \epsilon < \inf_{u \in \partial B_\rho(0)} J_\lambda(u) - \inf_{u \in \overline{B}_\rho(0)} J_\lambda(u)$. Applying the Ekeland variational principle in [7] to the functional $J_\lambda : \overline{B}_\rho(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B}_\rho(0)$ such that

$$\begin{aligned} &J_\lambda(u_\epsilon) < \inf_{u \in \overline{B}_\rho(0)} J_\lambda(u) + \epsilon, \\ &J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon \|u - u_\epsilon\|_{\overrightarrow{p}(x)}, \quad u \neq u_\epsilon, \end{aligned}$$

then, we have $J_\lambda(u_\epsilon) < \inf_{u \in \partial B_\rho(0)} J_\lambda(u)$ and thus, $u_\epsilon \in B_\rho(0)$.

Now, we define the functional $I_\lambda : \overline{B}_\rho(0) \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \|u - u_\epsilon\|_{\overrightarrow{p}(x)}$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + \tau v) - I_\lambda(u_\epsilon)}{t} \geq 0$$

for all $\tau > 0$ small enough and all $v \in B_\rho(0)$. The above information shows that

$$\frac{J_\lambda(u_\epsilon + \tau v) - J_\lambda(u_\epsilon)}{\tau} + \epsilon \|v\|_{\overrightarrow{p}(x)} \geq 0.$$

Letting $\tau \to 0^+$, we deduce that

$$\left\langle J'_\lambda(u_\epsilon), v \right\rangle \geq -\epsilon \|v\|_{\overrightarrow{p}(x)}.$$

It should be noticed that $-v$ also belongs to $B_\rho(0)$, so replacing v by $-v$, we get

$$\left\langle J'_\lambda(u_\epsilon), -v \right\rangle \geq -\epsilon \| - v\|_{\overrightarrow{p}(x)}$$

or

$$\left\langle J'_\lambda(u_\epsilon), v \right\rangle \leq \epsilon \|v\|_{\overrightarrow{p}(x)},$$

which helps us to deduce that $\|J'_\lambda(u_\epsilon)\|_{X^*} \leq \epsilon$. Therefore, there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J_\lambda(u_n) \to \underline{c}_\lambda = \inf_{u \in \overline{B}_\rho(0)} J_\lambda(u) < 0 \text{ and } J'_\lambda(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{11}$$

Is is clear that the sequence $\{u_n\}$ is bounded in X. Now, since the Banach space X is reflexive, there exists $u \in X$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u in X.

Step 4. We prove that $\{u_n\}$ which is given by (11) converges strongly to u in X, i.e. $\lim_{n\to\infty} \|u_n - u\|_{\overrightarrow{p}(x)} = 0$. By conditions (H1)-(H2), using (4) and Proposition 2.1 we have

$$\begin{aligned}
\left| \int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n - u)\, dx \right| &\leq 2|V|_{s(x)} \left| |u_n|^{q(x)-2}u_n(u_n - u) \right|_{\frac{s(x)}{s(x)-1}} \\
&\leq 4|V|_{s(x)} \left| |u_n|^{q(x)-2}u_n \right|_{\frac{q(x)}{q(x)-1}} |u_n - u|_{\beta(x)} \\
&\leq 4|V|_{s(x)} \left(1 + |u_n|_{q(x)}^{q^+-1} \right) |u_n - u|_{\beta(x)} \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

since X is continuously and compactly embedded in $L^{\beta(x)}(\Omega)$ with $\beta(x) = \frac{s(x)q(x)}{s(x)-q(x)}$. Moreover, by (11) we have $\lim_{n\to\infty} J_\lambda(u_n)(u_n - u) = 0$, i.e.

$$\lim_{n\to\infty} \left[\sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u)\, dx - \lambda \int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n - u)\, dx \right] = 0,$$

which yields

$$\lim_{n\to\infty} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u)\, dx = 0.$$

Combining this with the fact that $\{u_n\}$ converges weakly to u in X, we get

$$\lim_{m\to\infty} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i} u_m|^{p_i(x)-2} \partial_{x_i} u_m - |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right) (\partial_{x_i} u_m - \partial_{x_i} u)\, dx = 0. \tag{12}$$

Next, we apply the following inequality (see [19])

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta) \cdot (\xi - \eta) \geq 2^{-r}|\xi - \eta|^r, \quad \xi, \eta \in \mathbb{R}^N, \tag{13}$$

valid for all $r \geq 2$. Relations (12) and (13) show actually the sequence $\{u_m\}$ converges strongly to u in X. Thus, in view of (11), we obtain $J_\lambda(u) = \underline{c}_\lambda < 0$ and $J'_\lambda(u) = 0$. This means that u is a non-trivial weak solution of (1), i.e. any $\lambda \in (0, \overline{\lambda})$ is an eigenvalue of problem (1). Theorem 3.2 is completely proved. □

Now we will prove the second main result for problem (1) regarding the superlinear case. For the case of a non-constant sign weight V we define the following:

$$W^+ = \left\{u \in X : \int_\Omega V(x)|u|^{q(x)}\,dx > 0\right\},$$

$$W^- = \left\{u \in X : \int_\Omega V(x)|u|^{q(x)}\,dx < 0\right\},$$

$$\lambda^* = \inf_{u \in W^+} \frac{\Phi(u)}{\Psi(u)}, \quad \lambda_* = \inf_{u \in W^+} \frac{\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx}{\int_\Omega V(x)|u|^{q(x)}\,dx}, \tag{14}$$

$$\mu^* = \sup_{u \in W^-} \frac{\Phi(u)}{\Psi(u)}, \quad \mu_* = \sup_{u \in W^-} \frac{\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx}{\int_\Omega V(x)|u|^{q(x)}\,dx}. \tag{15}$$

Theorem 3.3. *Assume that (8) and the following conditions hold:*

(H3) $P_+^+ < q^- \le q^+ < P_-^*$ *and* $q^+ - \frac{1}{2} < q^-$*, where* $P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1}$*;*

(H4) $V \in L^{s(x)}(\Omega)$ *is a sign-changing function such that* $s \in C(\overline{\Omega})$ *and*

$$s(x) > \max\left\{1, \frac{P_-^*}{P_-^* - q(x)}\right\}$$

for all $x \in \overline{\Omega}$.

Then we have the following assertions:

(i) The numbers λ^* *and* μ^* *are the positive and negative eigenvalues of problem (1) respectively, satisfying* $\mu^* \le \mu_* < 0 < \lambda_* \le \lambda^*$*;*

(ii) Any $\lambda \in (-\mu^*) \cup (\lambda^*, +\infty)$ *is an eigenvalue of problem (1) while any* $\lambda \in (\mu_*, \lambda_*)$ *is not an eigenvalue of (1).*

Proof. It is clear that if λ is an eigenvalue of problem (1) with weight V then $-\lambda$ is an eigenvalue of problem (1) with weight $-V$. For this reason, it is sufficient to prove Theorem 3.3 only for $\lambda > 0$ and we will consider problem (1) only in the set W^+ defined as before. For this case, the proof of Theorem 3.3 is divided into the following four steps.

Step 1. We prove that $\lambda_* > 0$.

By relation (14), it follows that

$$\frac{q^-}{P_+^+}\lambda_* \le \lambda^* \le \frac{q^+}{P_-^-}\lambda_*, \tag{16}$$

and $0 \le \lambda_* \le \lambda^*$ since $P_+^+ < q^-$.

We will prove that

$$\lim_{\|u\|_{\vec{p}(x)} \to 0,\ u \in W^+} \frac{\Phi(u)}{\Psi(u)} = +\infty, \tag{17}$$

$$\lim_{\|u\|_{\vec{p}(x)} \to +\infty,\ u \in W^+} \frac{\Phi(u)}{\Psi(u)} = +\infty. \tag{18}$$

Indeed, using (3) and Proposition 2.1, for all $u \in X$ we have

$$|\Psi(u)| \le \frac{2}{q^-}|V|_{s(x)}||u|^{q(x)}|_{\frac{s(x)}{s(x)-1}} \le \frac{2}{q^-}|V|_{s(x)}|u|^{q^\tau}_{\alpha(x)},$$

where $\tau = -$ if $|u|_{\alpha(x)} \le 1$ and $\tau = +$ if $|u|_{\alpha(x)} \ge 1$ and $\alpha(x) = \frac{s(x)q(x)}{s(x)-1}$. By condition (H4), we have $1 < \alpha(x) < P^*_-$ for all $x \in \overline{\Omega}$, that is, X is continuously embedded in $L^{\alpha(x)}(\Omega)$, so there exists $c_4 > 0$ such that

$$|\Psi(u)| \le \frac{2c_4}{q^-}|V|_{s(x)}\|u\|^{q^\tau}_{\overrightarrow{p}(x)}. \tag{19}$$

For $u \in W^+$ with $\|u\|_{\overrightarrow{p}(x)} \le 1$ by relations (10) and (19) we infer that

$$\frac{\Phi(u)}{\Psi(u)} = \frac{\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}|\partial_{x_i} u|^{p_i(x)}\, dx}{\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\, dx} \ge \frac{q^-}{2c_4 P^+_+ N^{P^+_+-1}|V|_{s(x)}}\|u\|^{P^+_+-q^-}_{\overrightarrow{p}(x)}. \tag{20}$$

which implies that relation (17) holds since $P^+_+ < q^-$.

On the other hand, since $q^+ - \frac{1}{2} < q^-$, it follows that there exists $\theta > 0$ such that $q^+ - \frac{1}{2} < \theta < q^-$, which gives us $q^+ - 1 < q^- - \frac{1}{2} < \theta$ and

$$1 + \theta - q^+ > 0, \quad 2(q^- - \theta) \le 2(q^+ - \theta) < 1. \tag{21}$$

Take $r(x)$ be any measurable function satisfying

$$\max\left\{\frac{s(x)}{1+\theta s(x)}, \frac{P^*_-}{P^*_- + \theta - q(x)}\right\} < r(x) < \min\left\{\frac{P^*_-}{P^*_- + \theta s(x)}, \frac{1}{1+\theta - q(x)}\right\} \tag{22}$$

for all $x \in \overline{\Omega}$ and

$$\theta\left(\frac{r^+}{r^-} + 1\right) < q^-. \tag{23}$$

From relations (21)-(23), it implies that $r \in L^\infty(\Omega)$ and $1 < r(x) < s(x)$ for any $x \in \overline{\Omega}$. Moreover, we have

$$1 < \frac{\theta r(x)s(x)}{s(x) - r(x)} < P^*_-, \quad 1 < \frac{(q(x)-\theta)r(x)}{r(x)-1} < P^*_-$$

for all $x \in \overline{\Omega}$, so there exists $c_5 > 0$ such that

$$|u|_{\frac{\theta r(x)s(x)}{s(x)-r(x)}} \le c_5\|u\|_{\overrightarrow{p}(x)}, \quad |u|_{\frac{(q(x)-\theta)r(x)}{r(x)-1}} \le c_5\|u\|_{\overrightarrow{p}(x)}, \quad \forall u \in X. \tag{24}$$

For $u \in W^+$, by (3) and (5) we have

$$|\Psi(u)| \le \int_\Omega |V||u|^\theta ||u|^{q(x)-\theta}\, dx \le 2\left|V|u|^\theta\right|_{r(x)}\left||u|^{q(x)-\theta}\right|_{\frac{r(x)}{r(x)-1}}. \tag{25}$$

Using (3) and (5) we have

$$\left|V|u|^\theta\right|_{r(x)} \le \left(\int_\Omega |V|^{r(x)}|u|^{\theta r(x)}\, dx\right)^{\frac{1}{r^-}} \le 2\left||V|^{r(x)}\right|^{\frac{1}{r^-}}_{\frac{s(x)}{r(x)}}\left||u|^{\theta r(x)}\right|^{\frac{1}{r^-}}_{\frac{s(x)}{s(x)-r(x)}}$$

for any $u \in W^+$ with $\left|V|u|^\theta\right|_{r(x)} > 1$ and we deduce from (24) that

$$\begin{aligned}\left|V|u|^\theta\right|_{r(x)} &\leq 1 + 2\left||V|^{r(x)}\right|^{\frac{1}{r^-}}_{\frac{s(x)}{r(x)}}\left||u|^{\theta r(x)}\right|^{\frac{1}{r^-}}_{\frac{s(x)}{s(x)-r(x)}}\\ &\leq 1 + 2\left(1 + |V|^{\frac{r^+}{r^-}}_{r(x)}\right)\left(1 + |u|^{\frac{\theta r^+}{r^-}}_{\frac{\theta r(x)s(x)}{s(x)-r(x)}}\right)\\ &\leq c_6\left(1 + \|u\|^{\frac{\theta r^+}{r^-}}_{\overrightarrow{p}(x)}\right)\end{aligned}$$

for any $u \in W^+$. Similarly,

$$\left||u|^{q(x)-\theta}\right|_{\frac{r(x)}{r(x)-1}} \leq 1 + |u|^{q^+-\theta}_{\frac{r(x)(q(x)-\theta)}{r(x)-1}} \leq 1 + c_7\|u\|^{q^+-\theta}_{\overrightarrow{p}(x)}, \quad u \in W^+.$$

From above information, it follows that

$$\begin{aligned}|\Psi(u)| &\leq c_6\left(1 + \|u\|^{\frac{\theta r^+}{r^-}}_{\overrightarrow{p}(x)}\right)\left(1 + c_7\|u\|^{q^+-\theta}_{\overrightarrow{p}(x)}\right)\\ &= c_6 + c_6c_7\|u\|^{q^+-\theta}_{\overrightarrow{p}(x)} + c_6\|u\|^{\frac{\theta r^+}{r^-}}_{\overrightarrow{p}(x)} + c_6c_7\|u\|^{\frac{\theta r^+}{r^-}}_{\overrightarrow{p}(x)}\|u\|^{q^+-\theta}_{\overrightarrow{p}(x)}\\ &\leq c_8\left(1 + \|u\|^{2\frac{\theta r^+}{r^-}}_{\overrightarrow{p}(x)} + \|u\|^{2(q^+-\theta)}_{\overrightarrow{p}(x)}\right)\end{aligned} \tag{26}$$

for all $u \in W^+$ with $\|u\|_{\overrightarrow{p}(x)} > 1$.

For each $i \in \{1, 2, ..., N\}$, we define

$$\alpha_i = \begin{cases} P^+_+, & \text{if } |\partial_{x_i} u|_{p_i(x)} < 1,\\ P^-_-, & \text{if } |\partial_{x_i} u|_{p_i(x)} > 1.\end{cases}$$

Then, for all $u \in W^+$ with $\|u_n\|_{\overrightarrow{p}(x)} > 1$, it holds that

$$\begin{aligned}\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx &= \sum_{i=1}^N |\partial_{x_i} u|^{\alpha_i}_{p_i(x)}\\ &\geq \sum_{i=1}^N |\partial_{x_i} u|^{P^-_-}_{p_i(x)} - \sum_{\{i:\ \alpha_i = P^+_+\}}\left(|\partial_{x_i} u|^{P^-_-}_{p_i(x)} - |\partial_{x_i} u|^{P^+_+}_{p_i(x)}\right)\\ &\geq \frac{1}{N^{P^-_-}}\|u\|^{P^-_-}_{\overrightarrow{p}(x)} - N.\end{aligned} \tag{27}$$

Hence,

$$\begin{aligned}\frac{\Phi(u)}{\Psi(u)} &= \frac{\sum_{i=1}^N \frac{1}{p_i(x)}\int_\Omega |\partial_{x_i} u|^{p_i(x)}\,dx}{\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx}\\ &\geq \frac{\frac{1}{P^+_+N^{P^-_-}}\|u\|^{P^-_-}_{\overrightarrow{p}(x)} - \frac{N}{P^+_+}}{c_8\left(1 + \|u\|^{2\frac{\theta r^+}{r^-}}_{\overrightarrow{p}(x)} + \|u\|^{2(q^+-\theta)}_{\overrightarrow{p}(x)}\right)}\\ &\to +\infty\end{aligned}$$

as $\|u\|_{\overrightarrow{p}(x)} \to +\infty$ since $P^-_- > 1 > 2(q^+ - \theta) \geq 2(q^- - \theta) \geq 2\theta\frac{r^+}{r^-} > 2\theta$. Thus relation (18) holds.

Now, we are in the position to prove that $\lambda_* > 0$. Assume by contradiction that $\lambda_* = 0$, from (16) we get $\lambda^* = 0$. Then, there exists a sequence $\{u_n\} \subset W^+\backslash\{0\}$ such that

$$\lim_{n\to\infty} \frac{\Phi(u_n)}{\Psi(u_n)} = 0. \tag{28}$$

We also obtain from (20) that

$$\frac{\Phi(u_n)}{\Psi(u_n)} \geq \begin{cases} \frac{q^-}{2P_+^+|V|_{s(x)}} \|u_n\|_{\overrightarrow{p}(x)}^{P_-^- - q^+}, & \text{if } \|u_n\|_{\overrightarrow{p}(x)} \geq 1, \\ \frac{q^-}{2P_+^+|V|_{s(x)}} \|u_n\|_{\overrightarrow{p}(x)}^{P_+^+ - q^-}, & \text{if } \|u_n\|_{\overrightarrow{p}(x)} < 1. \end{cases} \tag{29}$$

By (H3), $P_-^- - q^+ < 0$ and $P_+^+ - q^- < 0$, so (29) implies that $\|u_n\|_{\overrightarrow{p}(x)} \to +\infty$ as $n \to \infty$, as $n \to \infty$. Using again (29), we get

$$\lim_{n\to\infty} \frac{\Phi(u_n)}{\Psi(u_n)} = +\infty,$$

which contradicts with (28) and thus, we conclude that $\lambda_* > 0$.

Step 2. We prove that λ_* is an eigenvalue of problem (1).

Indeed, let $\{u_n\} \subset W^+\backslash\{0\}$ be a minimizing sequence for the number λ^*, that is,

$$\lim_{n\to\infty} \frac{\Phi(u_n)}{\Psi(u_n)} = \lambda^* > 0. \tag{30}$$

By (18), we have $\{u_n\}$ is a bounded sequence in X. Since X is reflexive, there exists $u^* \in X$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that $\{u_n\}$ converges weakly to u^* as $n \to \infty$. Since Φ is a convex function, it is weakly lower semi-continuous, so we get

$$\lim_{n\to\infty} \Phi(u_n) \geq \Phi(u^*). \tag{31}$$

On the other hand, since the embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$ is compact, the sequence $\{u_n\}$ converges strongly to u^* in $L^{\alpha(x)}(\Omega)$, where $\alpha(x) = \frac{q(x)s(x)}{s(x)-1}$. It is noticed that $|u_n|_{\alpha(x)} \to |u^*|_{\alpha(x)}$, $\left||u_n|^{q(x)}\right|_{\frac{s(x)}{s(x)-1}} \to \left||u^*|^{q(x)}\right|_{\frac{s(x)}{s(x)-1}}$, the sequence $\left||u_n|^{q(x)}\right|_{\frac{s(x)}{s(x)-1}}$ is bounded and $\{|u_n|^{q(x)}\}$ converges weakly to $|u^*|^{q(x)}$ in $L^{\frac{s(x)}{s(x)-1}}(\Omega)$, so we deduce that $|u_n|^{q(x)} \to |u^*|^{q(x)}$ in $L^{\frac{s(x)}{s(x)-1}}(\Omega)$. From the above information, it implies that

$$\begin{aligned} |\Phi(u_n) - \Phi(u)| &\leq \int_\Omega \frac{|V(x)|}{q(x)} \left(|u_n|^{q(x)} - |u^*|^{q(x)}\right) dx \\ &\leq \frac{2}{q^-} |V|_{s(x)} \left||u_n|^{q(x)} - |u^*|^{q(x)}\right|_{\frac{s(x)}{s(x)-1}} \\ &\to 0, \end{aligned}$$

as $n \to \infty$, that is,

$$\lim_{n\to\infty} \Psi(u_n) = \Psi(u^*) = \int_\Omega \frac{V(x)}{q(x)} |u^*|^{q(x)}\, dx \geq 0. \tag{32}$$

In view of (31) and (32) we obtain $\frac{\Phi(u^*)}{\Psi(u^*)} = \lambda^*$ if $\Psi(u^*) > 0$, i.e. $u^* \in W^+$. We need to show that $\Psi(u^*) > 0$. Assume by contradiction that $\Psi(u^*) = 0$ or

$$\lim_{n\to\infty} \Psi(u_n) = 0. \tag{33}$$

Now, taking $\epsilon \in (0, \lambda^*)$ be fixed, by (9), for n large enough,

$$\left|\frac{\Phi(u_n)}{\Psi(u_n)} - \lambda^*\right| < \epsilon$$

or

$$(\lambda^* - \epsilon)\Psi(u_n) < \Phi(u_n) < (\lambda^* + \epsilon)\Psi(u_n),$$

which follows from (33) that $\lim_{n\to\infty} \Phi(u_n) = 0$. This means that $u_n \to 0$ in X, that is, $\|u_n\|_{\overrightarrow{p}(x)} \to 0$ as $n \to \infty$ and thus,

$$\lim_{n\to\infty} \frac{\Phi(u_n)}{\Psi(u_n)} = +\infty$$

which is a contradiction. Therefore, $\Psi(u^*) > 0$ and u^* is an eigenfunction and λ^* is an eigenvalue of problem (1).

Step 3. We prove that any $\lambda \in (\lambda^*, +\infty)$ is an eigenvalue of problem (1).

Let $\lambda \in (\lambda^*, +\infty)$ be arbitrary but fixed. We know that λ is an eigenvalue of problem (1) if and only if there exists $u_\lambda \in W^+\backslash\{0\}$ a critical point of J_λ. From relations (26) and (27), it implies that

$$\begin{aligned} J_\lambda(u) &= \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx - \lambda \int_\Omega \frac{V(x)}{q(x)} |u|^{q(x)}\, dx \\ &\geq \frac{1}{P_+^+ N^{P_-^-}} \|u\|_{\overrightarrow{p}(x)}^{P_-^-} - \frac{N}{P_+^+} - \lambda c_8 \left(1 + \|u\|_{\overrightarrow{p}(x)}^{2\frac{\theta r^+}{r^-}} + \|u\|_{\overrightarrow{p}(x)}^{2(q^+-\theta)}\right) \\ &\to +\infty, \end{aligned}$$

as $\|u\|_{\overrightarrow{p}(x)} \to +\infty$ since $P_-^- > 1 > 2(q^+ - \theta) \geq 2(q^- - \theta) \geq 2\theta\frac{r^+}{r^-} > 2\theta$. This follows that J_λ is coercive in W^+.

By the of proof in the step 2, the functional Ψ is weakly-strongly continuous in W^+. We also know that Φ is weakly lower semi-continuous, so by Weierstrass theorem, there exists $u_\lambda \in W^+$ a global minimum point of J_λ. We need to prove that u_λ is non-trivial critical point of J_λ. Indeed, since $\lambda^* = \inf_{u\in W^+} \frac{\Phi(u)}{\Psi(u)}$ and $\lambda > \lambda^*$, it follows that there exists $\overline{u}_\lambda \in W^+$ such that $\frac{\Phi(\overline{u}_\lambda)}{\Psi(\overline{u}_\lambda)} < \lambda$, that is,

$$J_\lambda(\overline{u}_\lambda) = \Phi(\overline{u}_\lambda) - \lambda\Psi(\overline{u}_\lambda) < 0.$$

This means that $\inf_{u\in W^+} J_\lambda(u) < 0$ and thus, u_λ is a non-trivial critical point of J_λ or λ is an eigenvalue of problem (1). The proof of step 3 is completed.

Step 4. We prove that any $\lambda \in (0, \lambda_*)$ is not an eigenvalue of problem (1).

Indeed, assume by contradiction that there exists $\lambda \in (0, \lambda^*)$ is an eigenvalue of problem (1), that is, there exists $u_\lambda \in W^+$ such that

$$\Phi'(u_\lambda)(v) = \lambda\Psi'(u_\lambda)(v), \quad \forall v \in W^+.$$

Thus, for $v = u_\lambda \in W^+$ we have

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i} u_\lambda|^{p_i(x)}\, dx = \lambda \int_\Omega V(x)|u_\lambda|^{q(x)}\, dx. \tag{34}$$

By the definition of the set W^+, we have $\int_\Omega V(x)|u_\lambda|^{q(x)}\, dx > 0$. By (34) and the definition of λ_* and the fact that $\lambda < \lambda_*$, we deduce that

$$\begin{aligned} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_\lambda|^{p_i(x)}\, dx &\geq \lambda_* \int_\Omega V(x)|u_\lambda|^{q(x)}\, dx \\ &> \lambda \int_\Omega V(x)|u_\lambda|^{q(x)}\, dx \\ &= \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_\lambda|^{p_i(x)}\, dx, \end{aligned}$$

which is a contradiction. Therefore, the proof of Theorem 3.3 is now complete. □

Acknowledgments

The author would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

References

[1] G.A. Afrouzi, M. Mirzapour, V.D. Radulescu, Qualitative analysis of solutions for a class of anisotropic elliptic equations with variable exponent, Proc. Edinburgh Math. Soc., **59**(3) (2016), 541–557.
[2] M. Avci, R.A. Ayazoglu (Mashiyev), B. Cekic, Solutions of an anisotropic nonlocal problem involving variable exponent, Adv. Nonlinear Anal., **2**(3) (2013), 325–338.
[3] M.M. Boureanu, V. Rădulescu, Anisotropic Neumann problems in Sobolev spaces with variable exponent, Nonlinear Anal. (TMA), **75** (2012), 4471-4482.
[4] N.T. Chung, H.Q. Toan, On a class of anisotropic elliptic equations without Ambrosetti-Rabinowitz type conditions, Nonlinear Anal. (RWA), **16** (2014), 132–145.
[5] L. Diening, P. Harjulehto, P. Hasto and M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes, vol. 2017, Springer-Verlag, Berlin, 2011.
[6] D.S. Dumitru, Two nontrivial solutions for a class of anisotropic variable exponent problems, Taiwanese J. Math., **16**(4) (2012), 1205–1219.
[7] I. Ekeland, On the variational principle, J. Math. Anal. Appl., **47** (1974), 324–353.
[8] I. Fragalà, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear equations, Ann. Inst. H. Poincaré, Analyse Non Linéaire, **21** (2004), 715–734.
[9] C. Ji, An eigenvalue of an anisotropic quasilinear elliptic equation with variable exponent and Neumann boundary condition, Nonlinear Anal. (TMA), **71**, (2009), 4507–4514.
[10] K. Kefi, $p(x)$-Laplacian with indefinite weight, Proc. Amer. Math. Soc., **139**(12), (2011), 4351–4360.
[11] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J., **41** (1991), 592–618.
[12] J.L. Lions, On some questions in boundary value problems of mathematical physics, in: Proceedings of international Symposium on Continuum Mechanics and Partial Differential Equations, Rio de Janeiro 1977.
[13] M. Mihailescu, P. Pucci, V.D. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, J. Math. Anal. Appl., **340** (2008), 687–698.
[14] M. Mihailescu, D.S. Dumitru, Anisotropic quasilinear elliptic equations with variable exponent, J. Korean Math. Soc., **49**(6) (2012), 1123–1138.
[15] M. Mihailescu, V. Radulescu, Continuous spectrum for a class of nonhomogeneous differential operators, Manuscripta Math., **125** (2008), 157–167.
[16] M. Mihailescu, V. Radulescu, Concentration phenomena in nonlinear eigenvalue problems with variable exponents and sign-changing potential, J. d'Analyse Math., **111**(1) (2010), 267–287.
[17] V.D. Radulescu, I.L. Stancut, Combined concave-convex effects in anisotropic elliptic equations with variable exponent, Nonlinear Differ. Equ. Appl. (NoDEA), **22** (2015), 391–410.
[18] M. Ruzicka, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2002.
[19] J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, in: Ph. Bénilan, J. Robert (Eds.), Journées d'Analyse Non Linéaire, in: Lecture Notes in Math., Vol. 665, Springer-Verlag, Berlin, 1978, 205–227.
[20] M. Yu, L. Wang, S. Tang, Existence of solutions for an anisotropic elliptic problem with variable exponent and singularity, Math. Methods Appl. Sci., **39**(10) (2016), 2761–2767.