Filomat 33:15 (2019), 5023–5035
https://doi.org/10.2298/FIL1915023L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Global Existence and Boundedness of Solutions in a Lotka-Volterra Reaction-Diffusion System of Predator-Prey Model with Nonlinear Prey-Taxis

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Abstract. In this paper, we investigate a diffusive Lotka-Volterra predator-prey model with nonlinear prey-taxis under Neumann boundary conditions. This system describes a prey-taxis mechanism that is an immediate movement of the predator u in response to a change of the prey v (which lead to the collection of u). We apply some methods to overcome the substantial difficulty of the existence of nonlinear prey-taxis term and prove that the unique global classical solutions of Lotka-Volterra predator-prey model are globally bounded.

1. Introduction

In this article, we consider the following Lotka-Volterra reaction-diffusion system of predator-prey model with prey-taxis:

$$\begin{cases} u_t - d_1\Delta u + \nabla\cdot(\chi(u)u\nabla v) = (a_1 - b_1u - c_1v)u, & x\in\Omega, t\in(0,T),\\ v_t - d_2\Delta v = (a_2 - b_2u - c_2v)v, & x\in\Omega, t\in(0,T),\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t\in(0,T),\\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \geq 0, & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N(N=1,2,3)$ with smooth boundary $\partial\Omega\in C^{2+\alpha}(\overline{\Omega})$, where $0<\alpha<1$, $0<T\leq+\infty$, initial condition $u_0(x), v_0(x)\in C^{2+\alpha}(\overline{\Omega})$ compatible on $\partial\Omega$, the constants a_i, b_i, c_i, d_i, $i=1,2$ are nonnegative and ecological which means that they are positive constants and represent some parameters in ecology, and ν is the outward directional derivative normal to $\partial\Omega$. a_1 and a_2 reflect the intrinsic growth rates of the species, b_1 and c_2 measure the levels of intraspecific crowding, while b_2 and c_1 interpret the intensities of interspecific competition. As is well-known, there are more than one relationship between two species in many cases, such as snake and hawk, spider and frog and so on. Therefore, the investigation of this model is useful and meaningful.

2010 *Mathematics Subject Classification*. Primary 35A01; Secondary 35K57
Keywords. Boundedness, Lotka-Volterra-type, predator-prey model, reaction-diffusion-taxis system
Received: 02 August 2018; Revised 10 September 2018; Accepted: 10 September 2018
Communicated by Maria Alessandra Ragusa
This research was supported by the NNSF of China (Nos. 11271379 and 11671406).
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There is the Lotka-Volterra functional response contained in the model (1.1), where u and v represent the population density of two species at time t with diffusion rates d_1 and d_2 (the tendency of random walks of the species), respectively. As a matter of fact, there are many famous reaction-diffusion systems such as Keller-Segel systems [5, 6], Holling-type systems [7], Holling-type II systems [8], Ivlev-type systems [9], Lotka-Volterra-type systems [10, 16] and so on. The model (1.1) was described by Lotka [3] and Volterra [4]. In recent two decades, it is of great interests to investigate the Lotka-Volterra predator-prey system. In 2003, Liu and Chen [15] discussed the complex dynamics of Holling type II Lotka-Volterra predator-prey system with impulsive perturbations on the predator. In 2005, Zhang et al. [16] study the dynamical behaviors of a Lotka-Volterra predator-prey model concerning integrated pest management. On the other hand, the researchers in [17] investigates the existence and stability of periodic solution of a Lotka-Volterra predator-prey model with state dependent impulsive effects. With the rise of biological mathematics, many biologists, ecologists and mathematicians apply their efforts to the studies of Partial Differential Equations (PDEs), especially in Nonlinear Parabolic Partial Differential Equations (NPPDEs) [18, 19, 23, 24]. In addition, PDEs are supposed to be sufficient in modeling of the countless processes in all fields of science. Many phenomena in physical sciences, chemistry and biology are naturally described by NPPDEs, such as competition systems, chemotaxis systems, predator-prey models and so on.

Under some certain conditions, Gai, Wang and Yan [2] considered global existence and boundedness, bifurcation analysis, as well as the transition later solution to the following model, similar to (1.1):

$$
\begin{cases}
u_t - d_1\Delta u + \chi\nabla\cdot(u\nabla v) = (a_1 - b_1 u - c_1 v)u, & x\in\Omega, t\in(0,T),\\
v_t - d_2\Delta v = (a_2 - b_2 u - c_2 v)v, & x\in\Omega, t\in(0,T),\\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t\in(0,T),\\
(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \geq 0, & x\in\Omega,
\end{cases}
$$

which is a Lotka-Volterra competition system with advection. In the present work, motivated by [13] and [14], we consider the global boundedness of classical solutions to (1.1) under the predator-prey-taxis mechanism with simplified conditions on $\chi(u)$, which is weaker than that supposed in [1].

The following theorem is the main result of this paper.

Theorem 1.1. *Suppose that $\chi(u)$ satisfies*

(i) $\chi(u) \in C^1([0,+\infty))$;

(ii) $\chi(u) \equiv 0$ *for* $u \geq M$, *with* $M > 0$;

(iii) $|\chi'(u_1) - \chi'(u_2)| \leq L|u_1 - u_2|$ *for* $u_1, u_2 \in [0,+\infty)$, *with* $L > 0$,

then we have that the solutions to (1.1) are global and uniformly bounded in time.

The nonlinear prey-taxis mechanism contained in the system means a immediate movement of the predator u in response to a change of the prey v which lead to the collection of u. Here we assume that $\chi(u) \equiv 0$ for $u \geq M$ means that there exists a marginal value M for the cumulation of predator u, over which the prey-tactic cross-diffusion $\chi(u)$ vanishes. In addition, it is necessary for the existence of classical solutions of the system (1.1) to suppose that $\chi'(u)$ satisfies $|\chi'(u_1) - \chi'(u_2)| \leq L|u_1 - u_2|$ for $u_1, u_2 \in [0,+\infty)$, with $L > 0$. Refer to Remark 2.1 in [13] for a detailed explanation. Throughout this paper we also denote that $\omega(u) = u\chi(u)$, then it follows from the assumptions of Theorem 1.1 that $\omega(u)$ and $\omega'(u)$ are bounded, and $\omega'(u)$ is Lipschitz continuous.

The remainder of this article is organized as follows. In Section 2, we propose some preliminary results which are essential to the proof of Theorem 1.1. Section 3 illustrates the proof of Theorem 1.1. In Section 4, we will discuss how to generalize our results to more general setting.

2. Preliminaries

In this section we state the following lemmas which are essential in the proofs of our main theorem. The first is on the boundedness of v.

Lemma 2.1. *Suppose that*

$$(u,v) \in (C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times (0,T)))^2$$

is a solution of (1.1). Then

$$u \geq 0$$

and

$$0 \leq v \leq K_0 = \max\left\{\max_{\overline{\Omega}} v_0(x), K\right\}.$$

The proof of Lemma 2.1 is similar as the proof of Lemma 3.1 in [13]. We only need to set

$$K = \frac{a_2}{c_2}$$

in Lemma 3.1, and hence it is omitted. Now, we need to establish a priori estimate of u.

Lemma 2.2. *Suppose that*

$$(u,v) \in C^{2,1}(\Omega \times (0,T))$$

is a solution of (1.1), then there holds

$$\|u\|_{L^{p+1}(\Omega\times(0,T))} \leq C$$

for any $p > 1$.

Proof. Multiplying

$$u_t - d_1\Delta u + \nabla\cdot(\chi(u)u\nabla v) = (a_1 - b_1 u - c_1 v)u$$

by u^p, integrating over $\Omega \times (0,t)$, applying the no-flux boundary condition

$$\frac{\partial u}{\partial \nu} = 0,$$

and noting

$$u \geq 0,$$

$$0 \leq v \leq K_0$$

and

$$a_1, b_1, c_1 \geq 0,$$

one gets

$$
\begin{aligned}
&\int_0^t \int_\Omega \frac{d}{dt} u^{p+1} dt - d_1 \int_0^t \int_\Omega \Delta u \cdot u^p dt \\
&= \int_\Omega u^{p+1}(t) - \int_\Omega u^{p+1}(0) + (p+1)pd_1 \int_0^t \int_\Omega u^{p-1} |\nabla u|^2 dt \\
&= -\int_0^t \int_\Omega \nabla \cdot (\chi(u) u \nabla v) \cdot u^p dt + \int_0^t \int_\Omega (a_1 - b_1 u - c_1 v) u^{p+1} dt \\
&\leq (p+1)p \int_0^t \int_\Omega u^p \chi(u) |\nabla u \cdot \nabla v| dt + a_1 \int_0^t \int_\Omega u^{p+1} dt.
\end{aligned} \tag{2.1}
$$

Combining the assumption of $\chi(u)$ and Young's inequality, we can get that

$$
\begin{aligned}
\left| \int_0^t \int_\Omega u^p \chi(u) |\nabla u \cdot \nabla v| dt \right| \leq & M^{\frac{p+1}{2}} \left| \int_0^t \int_\Omega u^{\frac{p-1}{2}} \chi(u) |\nabla u \cdot \nabla v| dt \right| \\
\leq & M^{\frac{p+1}{2}} \max_{0 \leq u \leq M} \chi(u) \int_0^t \int_\Omega \left| u^{\frac{p-1}{2}} \nabla u \cdot \nabla v \right| dt \\
= & M^{\frac{p+1}{2}} \max_{0 \leq u \leq M} \chi(u) \int_0^t \int_\Omega \left| u^{\frac{p-1}{2}} \nabla u \right| \cdot |\nabla v| \, dt \\
\leq & \varepsilon \int_0^t \int_\Omega u^{p-1} |\nabla u|^2 dt + \frac{C_0}{2\varepsilon} \int_0^t \int_\Omega |\nabla v|^2 dt
\end{aligned} \tag{2.2}
$$

for any sufficiently small $\varepsilon > 0$.

Multiplying

$$v_t - d_2 \Delta v = (a_2 - b_2 u - c_2 v) v$$

by v, integrating over $\Omega \times (0, t)$, applying the no-flux boundary condition

$$\frac{\partial v}{\partial \nu} = 0,$$

and using again the nonnegativity of the functions u and v, we get

$$0 \leq (a_2 - c_2 v) v \leq a_2 v$$

and

$$
\begin{aligned}
\int_0^t \int_\Omega \frac{d}{dt} v^2 dt - d_2 \int_0^t \int_\Omega \Delta v \cdot v dt = & \int_\Omega v^2(t) - \int_\Omega v^2(0) + 2d_2 \int_0^t \int_\Omega |\nabla v|^2 dt \\
= & \int_0^t \int_\Omega (a_2 - b_2 u - c_2 v) v^2 dt \\
\leq & a_2 \int_0^t \int_\Omega v^2 dt.
\end{aligned}
$$

According to $0 \leq v \leq K_0$, we obtain

$$\int_0^t \int_\Omega |\nabla v|^2 dt \leq C. \tag{2.3}$$

Thanks to (2.1), (2.2) and (2.3), we have

$$\int_\Omega u^{p+1}(t) + (p+1)p(d_1 - \varepsilon) \int_0^t \int_\Omega u^{p-1} |\nabla u|^2 dt \leq C + C_0 \int_0^t \int_\Omega u^{p+1} dt. \tag{2.4}$$

Setting $0 < \varepsilon < d_1$, we can conclude that

$$\int_\Omega u^{p+1}(t) \leq C + C_0 \int_0^t \int_\Omega u^{p+1} dt.$$

Applying Gronwall's lemma yields

$$\int_0^t \int_\Omega u^{p+1} dt \leq C.$$

The proof is complete. $\square$

Lemma 2.3. *Assume that*

$$(u, v) \in C^{2,1}(\Omega \times (0, T))$$

is a solution of (1.1), then there holds

$$\|u, v\|_{W_p^{2,1}(\Omega \times (0,T))} \leq C$$

for any $p > 5$.

Proof. Assume that

$$(u, v) \in C^{2,1}(\Omega \times (0, T))$$

is a solution of (1.1). Note that

$$v_t - d_2 \Delta v = (a_2 - b_2 u - c_2 v) v$$

can be rewritten as follows:

$$v_t - d_2 \Delta v - (a_2 - b_2 u - c_2 v) v = 0 \tag{2.5}$$

where

$$\|a_2 - b_2 u - c_2 v\|_{L^p(\Omega \times (0,T))} \leq C \tag{2.6}$$

by

$$0 \leq v \leq K_0$$

and

$$\|u\|_{L^{p+1}(\Omega \times (0,T))} \leq C.$$

Based on (2.5), (2.6) and the parabolic L^p-estimate, we obtain

$$\|v\|_{W_p^{2,1}(\Omega \times (0,T))} \leq C. \tag{2.7}$$

This, together with Sobolev embedding theorem, yields

$$\|\nabla v\|_{L^\infty(\Omega \times (0,T))} \leq C. \tag{2.8}$$

Now, we consider the equation of u. It can be rewritten as in non-divergence form:

$$u_t - d_1 \Delta u + \omega'(u) \cdot \nabla v = -\omega(u) \Delta v + (a_1 - b_1 u - c_1 v) u. \tag{2.9}$$

where

$$\|\omega'(u)\nabla v\|_{L^\infty(\Omega\times(0,T))} \le C,$$

$$\|-\omega(u)\Delta v + (a_1 - b_1 u - c_1 v)u\|_{L^p(\Omega\times(0,T))} \le C$$

by (2.7), (2.8),

$$0 \le v \le K_0$$

and

$$\|u\|_{L^{p+1}(\Omega\times(0,T))} \le C.$$

Using the parabolic L^p-estimate, we have

$$\|u\|_{W_p^{2,1}(\Omega\times(0,T))} \le C.$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume that*

$$(u, v) \in C^{2,1}(\Omega \times (0, T))$$

is a solution of (1.1), then there holds

$$\|u, v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))} \le C.$$

Proof. Applying the Sobolev embedding theorem and Lemma 2.3, yields

$$\|u, v\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(0,T))} \le C. \tag{2.10}$$

Then, together with the parabolic

$$v_t - d_2\Delta v = (a_2 - b_2 u - c_2 v)v,$$

$$\partial\Omega \in C^{2+\alpha},$$

$$u_0(x), v_0(x) \in C^{2+\alpha}(\overline{\Omega}),$$

where $0 < \alpha < 1$, $0 \le v \le K_0$ and (2.10), we obtain

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))} \le C. \tag{2.11}$$

Using the same method to the equation of u, we have

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))} \le C. \tag{2.12}$$

The proof is complete. □

Lemma 2.5. *Under the assumptions for initial data in the paper, there exist a unique solution*

$$(u, v) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times (0, T))$$

of (1.1) for any given $T > 0$.

The proof of the lemma is based on

$$\|u, v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(0,T))} \leq C.$$

Refer to Theorem 3.5 in [13] for the details.

The following lemma is the well-known classical $L^p - L^q$ estimate for the Neumann heat semigroup on bounded domains.

Lemma 2.6. *Suppose $(e^{t\Delta})_{t>0}$ is the Neumann heat semigroup in Ω, and $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then the following $L^p - L^q$ estimates hold with $C_1, C_2 > 0$ only depending on Ω:*

(i) If $1 \leq q \leq p \leq +\infty$, then

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \leq C_1(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^p(\Omega)}, \quad t > 0$$

for all $w \in L^q(\Omega)$;

(ii) If $2 \leq q \leq p < +\infty$, then

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \leq C_2(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|\nabla w\|_{L^p(\Omega)}, \quad t > 0$$

for all $w \in W^{1,q}(\Omega)$.

3. Proof of main result

In this section, we will prove the global boundedness of classical solutions to (1.1).

Proof. [Proof of Theorem 1.1] The proof consists of four parts.

Part 1: *Boundedness of $\|u\|_{L^1(\Omega)}$.*

The proof of this part is available in [13], and hence we omit it.

Part 2: *Boundedness of $\|u\|_{L^p(\Omega)}$ with $p > 2$.*

Multiply the equation of u in (1.1) by u^{p-1} and integrate on Ω by parts, then we have

$$\int_\Omega u_t \cdot u^{p-1} - \int_\Omega d_1 \Delta u \cdot u^{p-1} + \int_\Omega \nabla \cdot (\chi(u) u \nabla v) \cdot u^{p-1} = \int_\Omega (a_1 - b_1 u - c_1 v) u^p.$$

Next, we need to prove an important inequality

$$(p-1)\int_\Omega \chi(u) u^{p-1} \nabla u \cdot \nabla v \leq \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi(u)^2 u^p |\nabla v|^2.$$

By simplifying the problem, we only need to prove

$$\chi(u) u^{p-1} \nabla u \cdot \nabla v \leq \frac{d_1}{2} u^{p-2}|\nabla u|^2 + \frac{1}{2d_1}\chi(u)^2 u^p |\nabla v|^2.$$

Applying Young's inequality with ε $\left(ab \leq \frac{\varepsilon}{p}a^p + \frac{\varepsilon^{-\frac{q}{p}}}{q}b^q\right)$ and setting $p = q = 2$, $\varepsilon = d_1$, $a = u^{\frac{p-2}{2}}\nabla u$ and $b = \chi(u) u^{\frac{p}{2}} \nabla v$, we obtain

$$\begin{aligned}\chi(u) u^{p-1} \nabla u \cdot \nabla v =& \chi(u) u^{\frac{p-2}{2}+\frac{p}{2}} \nabla u \cdot \nabla v \\ =& (u^{\frac{p-2}{2}} \nabla u) \cdot (\chi(u) u^{\frac{p}{2}} \nabla v) \\ \leq& \frac{d_1}{2} u^{p-2}|\nabla u|^2 + \frac{1}{2d_1}\chi(u)^2 u^p |\nabla v|^2.\end{aligned}$$

Multiply the inequality by $(p-1)$ and integrate on Ω by parts yielding

$$(p-1)\int_\Omega \chi(u)u^{p-1}\nabla u\cdot\nabla v \le \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi(u)^2u^p|\nabla v|^2.$$

According to

$$\int_\Omega u_t\cdot u^{p-1} = \frac{1}{p}\int_\Omega pu^{p-1}\cdot u_t = \frac{1}{p}\int_\Omega \frac{d}{dt}u^p = \frac{1}{p}\frac{d}{dt}\int_\Omega u^p,$$

$$\int_\Omega d_1\cdot\nabla\cdot(\nabla u\cdot u^{p-1}) = d_1\int_\Omega \Delta u\cdot u^{p-1} + d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2 = 0$$

and

$$\int_\Omega \nabla\cdot(\chi(u)u\nabla v)\cdot u^{p-1} + (p-1)\int_\Omega \chi(u)u^{p-1}\nabla u\cdot\nabla v = 0,$$

we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2\\
=&\int_\Omega (a_1-b_1u-c_1v)u^p + (p-1)\int_\Omega \chi(u)u^{p-1}\nabla u\cdot\nabla v\\
\le& a_1\int_\Omega u^p + \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi(u)^2u^p|\nabla v|^2.
\end{aligned}$$

Consequently, together with $\chi(u)\le M_1$ due to $\chi(u)\in C^1$ and $\chi(u)\equiv 0$ for $u\ge M$, we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 \le& a_1\int_\Omega u^p + \frac{p-1}{2d_1}\int_\Omega \chi(u)^2u^p|\nabla v|^2\\
\le& a_1\int_\Omega u^p + \frac{(p-1)M_1^2M^p}{2d_1}\int_\Omega |\nabla v|^2.
\end{aligned}\tag{3.1}$$

Multiply the equation of v in (1.1) by $-\Delta v$, and integrate on Ω by parts to get

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^2 + 2d_2\int_\Omega |\Delta v|^2 =& 2a_2\int_\Omega |\nabla v|^2 - 4c_2\int_\Omega v|\nabla v|^2 + 2b_2\int_\Omega uv\Delta v\\
\le& 2a_2\int_\Omega |\nabla v|^2 + 2b_2\int_\Omega uv\Delta v\\
\le& 2a_2\int_\Omega |\nabla v|^2 + 2b_2K_0\int_\Omega u\Delta v\\
\le& 2a_2\int_\Omega |\nabla v|^2 + 2b_2K_0\int_\Omega u|\Delta v|.
\end{aligned}$$

Employing Young's inequality, we have

$$2b_2K_0\int_\Omega u|\Delta v| \le \frac{\varepsilon}{2}\int_\Omega |\Delta v|^2 + \frac{2b_2^2K_0^2}{\varepsilon}\int_\Omega u^2.$$

Setting $\varepsilon = 2d_2$, we can obtain that

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + d_2\int_\Omega |\Delta v|^2 \le 2a_2\int_\Omega |\nabla v|^2 + \frac{b_2^2K_0^2}{d_2}\int_\Omega u^2.\tag{3.2}$$

According to

$$\begin{aligned} d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2 =& d_1(p-1)\int_\Omega u^{\frac{p-2}{2}\cdot 2}|\nabla u|^2 \\ =&\frac{4d_1(p-1)}{p^2}\left[\int_\Omega \left(\frac{p}{2}\right)^2 u^{(\frac{p}{2}-1)\cdot 2}|\nabla u|^2\right] \\ =&\frac{4d_1(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \end{aligned}$$

for $p > 2$, we know from (3.1) and (3.2) by Young's inequality that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega |\nabla v|^2 + \frac{2d_1(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + d_2\int_\Omega |\Delta v|^2 \\ \le& a_1\int_\Omega u^p + \frac{(p-1)\chi(u)^2u_m^p}{2d_1}\int_\Omega |\nabla v|^2 + 2a_2\int_\Omega |\nabla v|^2 + \frac{b_2^2K_0^2}{d_2}\int_\Omega u^2 \\ =& a_1\int_\Omega u^p + \left(\frac{(p-1)\chi(u)^2u_m^p}{2d_1} + 2a_2\right)\int_\Omega |\nabla v|^2 + \frac{b_2^2K_0^2}{d_2}\int_\Omega u^2 \\ \le& (a_1+1)\int_\Omega u^p + \left(\frac{(p-1)M_1^2u_m^p}{2d_1} + 2a_2\right)\int_\Omega |\nabla v|^2 + k_3 \end{aligned} \tag{3.3}$$

with $k_3 = \frac{b_2^2K_0^2M^2|\Omega|}{d_2} > 0$.

For $\int_\Omega |\nabla v|^2$, applying the Sobolev interpolation inequality

$$\|D^jv\|_{p,\Omega} \le \varepsilon\|D^kv\|_{p,\Omega} + C\|v\|_{p,\Omega},$$

setting $j = 1$, $k = 2$, $p = 2$, and integrating on Ω by parts, it is easy to check that

$$\begin{aligned} \int_\Omega |\nabla v|^2 \le& \varepsilon_1\int_\Omega |\Delta v|^2 + k_4\int_\Omega |v|^2 \\ \le& \varepsilon_1\int_\Omega |\Delta v|^2 + k_4K_0^2|\Omega| \\ =& \varepsilon_1\int_\Omega |\Delta v|^2 + k_5 \end{aligned} \tag{3.4}$$

for any ε_1, k_4 and $k_5 = k_4K_0^2|\Omega| > 0$ depending on ε_1.

For $\int_\Omega u^p$, by the Gagliardo-Nirenberg inequality with $u \ge 0$, one gets

$$\begin{aligned} \int_\Omega u^p = \int_\Omega |u^{\frac{p}{2}}|^2 \le& k_6\left(\|\nabla u^{\frac{p}{2}}\|_2^{\frac{2Np-2N}{Np-N+2}} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{\frac{4}{Np-N+2}} + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2\right) \\ =& k_6\left(\|\nabla u^{\frac{p}{2}}\|_2^{\frac{2Np-2N}{Np-N+2}} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{2-\frac{2Np-2N}{Np-N+2}} + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2\right) \\ =& k_6\left(\|\nabla u^{\frac{p}{2}}\|_2^{2\theta} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\theta)} + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2\right) \end{aligned} \tag{3.5}$$

with $k_6 > 0$ and $0 < \theta = \frac{Np-N}{Np-N+2} < 1$. Applying Young's inequality yields

$$\|\nabla u^{\frac{p}{2}}\|_2^{2\theta} \cdot \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\theta)} \le \epsilon\theta\|\nabla u^{\frac{p}{2}}\|_2^2 + \epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2$$

with $\epsilon > 0$. Setting the last estimate into (3.5), we see that

$$\begin{aligned}\int_\Omega u^p = \int_\Omega |u^{\frac{p}{2}}|^2 \leq & k_6\left(\|\nabla u^{\frac{p}{2}}\|_2^{2\theta}\cdot\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\theta)} + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2\right)\\ \leq & k_6\left(\epsilon\theta\|\nabla u^{\frac{p}{2}}\|_2^2 + \epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 + \|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2\right)\\ = & k_6\epsilon\theta\|\nabla u^{\frac{p}{2}}\|_2^2 + k_6\epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2 + k_6\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2\\ = & k_6\epsilon\theta\|\nabla u^{\frac{p}{2}}\|_2^2 + k_6\left[\epsilon^{\frac{\theta}{\theta-1}}(1-\theta)+1\right]\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2\\ = & \varepsilon_2\|\nabla u^{\frac{p}{2}}\|_2^2 + k_7\|u^{\frac{p}{2}}\|_{\frac{2}{p}}^2\\ = & \varepsilon_2\|\nabla u^{\frac{p}{2}}\|_2^2 + k_7\|u\|_1^p\end{aligned}$$

for any $\varepsilon_2 = k_6\epsilon\theta > 0$, with $k_7 = k_6\left[\epsilon^{\frac{\theta}{\theta-1}}(1-\theta)+1\right] > 0$ depending on ε_2. Because of $\|u\|_1 \leq A_1$ by Part 1, we know that

$$\int_\Omega u^p \leq \varepsilon_2\|\nabla u^{\frac{p}{2}}\|_2^2 + k_7A_1^p = \varepsilon_2\|\nabla u^{\frac{p}{2}}\|_2^2 + k_8 \tag{3.6}$$

with $k_8 = k_7A_1^p > 0$.

Now, we need to consider the value of ε_1 and ε_2. Fix them with

$$\left(\frac{(p-1)M_1^2M^p}{2d_1} + 2a_2\right)\varepsilon_1 = \frac{d_2}{2}$$

and

$$(2a_1+1)\varepsilon_2 = \frac{2d_1(p-1)}{p^2}.$$

We have from (3.3), (3.4) and (3.6) that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega|\nabla v|^2 + (2a_1+1)\varepsilon_2\int_\Omega|\nabla u^{\frac{p}{2}}|^2 + 2\left(\frac{(p-1)\chi(u)^2M^p}{2d_1}+2a_2\right)\varepsilon_1\int_\Omega|\Delta v|^2\\ \leq&(a_1+1)\int_\Omega u^p + \left(\frac{(p-1)\chi(u)^2M^p}{2d_1}+2a_2\right)\int_\Omega|\nabla v|^2 + k_3\\ \leq&-a_1\int_\Omega u^p + (2a_1+1)\varepsilon_2\|\nabla u^{\frac{p}{2}}\|_2^2 + (2a_1+1)k_8 + \left(\frac{(p-1)M_1^2M^p}{2d_1}+2a_2\right)\varepsilon_1\int_\Omega|\Delta v|^2\\ &+\left(\frac{(p-1)M_1^2M^p}{2d_1}+2a_2\right)k_5 + k_3.\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega|\nabla v|^2\\ \leq&-a_1\int_\Omega u^p - \left(\frac{(p-1)\chi(u)^2M^p}{2d_1}+2a_2\right)\varepsilon_1\int_\Omega|\Delta v|^2 + \left[(2a_1+1)k_8 + \left(\frac{(p-1)\chi(u)^2M^p}{2d_1}+2a_2\right)k_5 + k_3\right]\\ =&-a_1\int_\Omega u^p - \left(\frac{(p-1)\chi(u)^2M^p}{2d_1}+2a_2\right)\left(\varepsilon_1\int_\Omega|\Delta v|^2 + k_5\right) + \left[(2a_1+1)k_8 + 2\left(\frac{(p-1)\chi(u)^2M^p}{2d_1}+2a_2\right)k_5 + k_3\right]\\ \leq&-a_1\int_\Omega u^p - \left(\frac{(p-1)M_1^2M^p}{2d_1}+2a_2\right)\int_\Omega|\nabla v|^2 + k_9\end{aligned}$$

with $k_9 > 0$. Thus, we can define a function

$$y_2(t) = \frac{1}{p}\int_\Omega u^p + \int_\Omega |\nabla v|^2, \quad t > 0$$

satisfies $y_2'(t) + k_{10} y_2(t) \le k_9$ for all $t > 0$ with

$$k_{10} = \min\left\{\frac{(p-1)M_1^2 M^p}{2d_1} + 2a_2, a_1 p\right\}.$$

This also guarantees

$$y_2(t) \le C_2 = \max\left\{y_2(0), \frac{k_9}{k_{10}}\right\}$$

for all $t > 0$ by the comparison principle of ordinary differential equations (ODEs).
Part 3: *Boundedness of* $\| \nabla v \|_{L^\infty(\Omega)}$.

For notational simplicity, we denote by $f(u, v) = (a_2 - b_2 u - c_2 v)v$. It follows from Part 2 and Lemma 2.3 that there is $C_3 > 0$ such that

$$\sup_{t>0} \|f(u, v)\|_{L^p(\Omega)} \le C_3 < +\infty.$$

Thanks to the variation-of-constants formula for v, one gets

$$v(\cdot, t) = e^{d_2 t\Delta} v_0 + \int_0^t e^{d_2(t-s)\Delta} f(u(s), v(s))ds, \quad t > 0.$$

Due to Lemma 2.3, it follows that

$$\begin{aligned}
\|\nabla v\|_{L^p(\Omega)} &= \left\| \nabla e^{d_2 t\Delta} v_0 + \int_0^t \nabla e^{d_2(t-s)\Delta} f(u(s), v(s))ds \right\|_{L^p(\Omega)} \\
&\le \left\|\nabla e^{d_2 t\Delta} v_0\right\|_{L^p(\Omega)} + \left\| \int_0^t \nabla e^{d_2(t-s)\Delta} f(u(s), v(s))ds \right\|_{L^p(\Omega)} \\
&\le C_2(1 + d_2 t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}) e^{-\lambda_1 d_2 t} \|\nabla v_0\|_{L^p(\Omega)} + \int_0^t \left\|\nabla e^{d_2(t-s)\Delta} f(u(s), v(s))\right\|_{L^p(\Omega)} ds \\
&\le 2C_2 e^{-\lambda_1' t} \|\nabla v_0\|_{L^p(\Omega)} + C_1 \int_0^t (1 + d_2^{-\frac{1}{2}} (t-s)^{-\frac{1}{2}}) e^{-\lambda_1'(t-s)} \|f(u(s), v(s))\|_{L^p(\Omega)} ds \\
&\le 2C_2 e^{-\lambda_1' t} \|\nabla v_0\|_{L^p(\Omega)} + C_1 C_3 \int_0^t (1 + d_2^{-\frac{1}{2}} s^{-\frac{1}{2}}) e^{-\lambda_1' s} ds \\
&\le 2C_2 \|\nabla v_0\|_{L^p(\Omega)} + C_1 C_3 \left(\frac{1}{\lambda_1'} + d_2^{-\frac{1}{2}}\left(2 + \frac{1}{\lambda_1'}\right)\right)
\end{aligned}$$

for all $t > 0$. Thus, $\|\nabla v\|_{L^p(\Omega)}$ is global bounded. We can apply

$$\frac{d\bar{v}(t)}{dt} = a_2 \bar{v}(t) - c_2 \bar{v}(t)^2$$

and the Moser iteration to obtain the boundedness of $\|\nabla v\|_{L^\infty(\Omega)}$, since $\|u\|_p$ for any $p > N$ is bounded.
Part 4: *Global boundedness*.

Thanks to Part 2, Part 3 and Lemma A.1 in [12], the global boundedness of solutions can be proved by applying of the standard Moser iterative method. This finishes the proof. □

Remark 3.1. *It is not hard to see that without nonlinear prey-taxis, the global boundedness of solutions is an obvious result to the corresponding predator-prey model [20, 21]. The existence of prey-taxis in (1.1) makes stupendous difficulty to obtain the global boundedness, and even the global existence of solutions. On the other hand, the nonlinear prey-taxis term $\nabla \cdot (\chi(u)u\nabla v)$ contained in the system is supposed that $\chi(u) \equiv 0$ whenever $u \geq M$, where the maximal density M acts as a switch to repulsion at high densities of the predator population, very similar to the volume-filling effect or prevention of overcrowding for chemotaxis [22]. Therefore, the global boundedness of solutions established by Theorem 1.1 should be reasonable and natural.*

Remark 3.2. *To investigate the qualitative behavior of the class of reaction-diffusion equations, in which the global bounded argument is incorporated together with the prey-taxis term $\nabla \cdot (\chi(u)u\nabla v)$, a standard technique have been applied. According to the boundedness of $\|u\|_{L^1(\Omega)}$, $\|u\|_{L^p(\Omega)}$ with $p > 2$ and $\|\nabla v\|_{L^p(\Omega)}$ with $p > 2$, using the standard Moser's iterative technique of parabolic partial differential equations, we obtain a sufficient condition to verify whether the unique nonnegative solution of (1.1) is global bounded.*

4. Generalization and future works

The method we propose in this paper can be applied to many interesting reaction-diffusion systems with nonlinear prey-taxis. The existence of solution is an important problem to be considered. For instance, the famous predator-prey model with Holling type II functional response

$$\begin{cases} u_t - d_1\Delta u + \nabla \cdot (u\chi(u)\nabla v) = -au + \beta\dfrac{cuv}{m+bv}, & x \in \Omega, t \in (0,T), \\ v_t - d_2\Delta v = rv - \dfrac{r}{K}v^2 - \dfrac{cuv}{m+bv}, & x \in \Omega, t \in (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \geq 0, & x \in \Omega, \end{cases}$$

is an interesting model worth of investigation. In addition, the coefficients a, c, d_1, d_2 could be functions belonging to the vanishing mean oscillation class of Sarason.

Acknowledgments

The authors are grateful to the referee for careful reading of the paper and for his or her useful reports which help them to improve the paper.

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