Filomat 33:15 (2019), 4951–4966
https://doi.org/10.2298/FIL1915951D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Space Curves Defined by Curvature–Torsion Relations and Associated Helices

Sharief Deshmukh[a], **Azeb Alghanemi**[b], **Rida T. Farouki**[c]

[a]*Department of Mathematics, College of Science, King Saud University, P.O. Box 2455 Riyadh-11451, Saudi Arabia*
[b] *Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203 Jeddah-21589, Saudi Arabia*
[c]*Department of Mechanical and Aerospace Engineering, University of California, Davis, CA 95616, USA*

Abstract. The relationships between certain families of special curves, including the general helices, slant helices, rectifying curves, Salkowski curves, spherical curves, and centrodes, are analyzed. First, characterizations of proper slant helices and Salkowski curves are developed, and it is shown that, for any given proper slant helix with principal normal **n**, one may associate a unique general helix whose binormal **b** coincides with **n**. It is also shown that centrodes of Salkowski curves are proper slant helices. Moreover, with each unit–speed non–helical Frenet curve in the Euclidean space $\mathbb{E}^3$, one may associate a unique circular helix, and characterizations of the slant helices, rectifying curves, Salkowski curves, and spherical curves are presented in terms of their associated circular helices. Finally, these families of special curves are studied in the context of general polynomial/rational parameterizations, and it is observed that several of them are intimately related to the families of polynomial/rational Pythagorean–hodograph curves.

1. Introduction

It is a well–known fact [25] that a space curve is uniquely determined, up to a choice of coordinate system, by specifying the curvature κ and torsion τ as functions of its arc length s. The functions $\kappa(s)$ and $\tau(s)$, which describe the deviation of a curve from linearity and planarity, are known as the “natural” or “intrinsic” equations of a curve [25]. In general, the curvature and torsion are independent, but certain “special” curves with distinctive geometrical properties correspond to the existence of relationships between them.

The simplest cases are the *helical curves*, identified by the proportionality condition $\tau(s)/\kappa(s) = c$, a constant. Equivalently [25], the curve tangent **t** maintains a constant angle $\psi = \cot^{-1} c$ with a fixed direction in space, the *axis* of the helical curve. If κ and τ are both constant we have a *circular helix*, while a *general helix* corresponds to non–constant κ and τ. Helical curves are of interest in molecular biology [3, 18, 26]; computer–aided geometric design [1, 9–11]; mechanical engineering [17, 23]; and physics [6, 21].

A *slant helix* [14] may be regarded as a variation on the general helix, in which the curve principal normal **n** (rather than the tangent **t**) maintains a constant angle with a fixed direction in space. This incurs a more complicated relation between κ, τ, and the derivative of the τ/κ ratio. The slant helices encompass the

2010 *Mathematics Subject Classification*. Primary 53A15; Secondary 53C40 53C42.
Keywords. general helix; slant helix; rectifying curve; spherical curve; associated circular helix; Pythagorean–hodograph curve.
Received: 22 May 2019; Accepted: 04 October 2019
Communicated by Mića S. Stanković
Email addresses: shariefd@ksu.edu.sa (Sharief Deshmukh), aalghanemi@kau.edu.sa (Azeb Alghanemi), farouki@ucdavis.edu (Rida T. Farouki)

general helices as the particular case where the τ/κ ratio is a constant; a *proper* slant helix has a non–constant τ/κ ratio.

The *rectifying curves* [4, 5] are identified by a torsion/curvature ratio that is a linear function of the arc length, rather than a constant, i.e., $\tau(s)/\kappa(s) = as + b$ where $a \neq 0$ and b are constants. A rectifying curve $\boldsymbol{\alpha}(s)$ satisfies the condition $\langle \boldsymbol{\alpha}(s), \mathbf{n}(s)\rangle \equiv 0$, where $\mathbf{n}(s)$ is the principal normal, i.e., at each point the position vector lies in the rectifying plane, spanned by tangent and binormal. Rectifying curves are of interest in analyzing joint kinematics, due to their close relationship with the centrode of a curve [4, 5, 7, 13, 27].

The *Salkowski curves* [22] may be viewed as generalizations of the circular helix, since they exhibit a constant curvature but non–constant torsion. The Salkowski curves are proper slant helices, and they have been employed [20] in the context of computer–aided geometric design to construct closed space curves with constant curvature and continuous torsion.

The *spherical curves* (i.e., curves that lie on a sphere) are a further related category. They are closely related to the construction of rectifying curves, and exhibit many interesting geometric properties [16, 19, 24].

The identification of characterizations for helices, rectifying curves, slant helices, and spherical curves, and the study of their inter–relationships, are interesting basic problems in the theory of Frenet curves. Characterizations for spherical curves have been given in [2, 28–30] and for rectifying curves in [4, 5, 7]. An important concept associated with a unit–speed Frenet curve $\boldsymbol{\alpha}(s)$ is its centrode $\boldsymbol{\omega} = \tau\,\mathbf{t} + \kappa\,\mathbf{b}$, i.e., the locus traced by the angular velocity vector, which determines the variation of the Frenet frame along $\boldsymbol{\alpha}(s)$. The centrode has been employed in [4, 5, 7] to characterize rectifying curves.

This paper develops new characterizations for slant helices, and shows that the centrode of a Salkowski curve is a proper slant helix. Moreover, it is shown that one may associate a unique general helix with each proper slant helix, and the general helices associated with Salkowski curves are identified. We also make the interesting observation that every unit–speed Frenet curve is either a general helix, or has a unique circular helix associated with it — these associated circular helices are used to identify novel characterizations of proper slant helices, Salkowski curves, spherical curves, and rectifying curves. Finally, these results are studied in the context of general parameterizations, defined by polynomial/rational functions, and their connections to the theory of Pythagorean–hodograph curves are elucidated.

2. Preliminaries

A unit–speed curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ is said to be a Frenet curve if $\kappa(s) > 0$ at every point, and $\tau(s) \not\equiv 0$. The *Frenet frame* $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ consisting of the curve tangent, principal normal, and binormal satisfies the Frenet–Serret relations

$$\mathbf{t}' = \kappa\,\mathbf{n}, \quad \mathbf{n}' = -\kappa\,\mathbf{t} + \tau\,\mathbf{b}, \quad \mathbf{b}' = -\tau\,\mathbf{n}, \tag{1}$$

where primes denote arc–length derivatives.

A Frenet curve $\boldsymbol{\alpha}(s)$ is a general helix if a fixed unit vector $\mathbf{u}$ exists, such that $\langle \mathbf{t}(s), \mathbf{u}\rangle = \cos\psi$ for some fixed angle ψ (the helix angle). The Lancret characterization [16, 19, 24] states that a space curve $\boldsymbol{\alpha}(s)$ is a general helix if and only if

$$\frac{\tau(s)}{\kappa(s)} = c, \tag{2}$$

where $c = \cot\psi$. When κ and τ are both constant, $\boldsymbol{\alpha}(s)$ is a circular helix.

A curve $\boldsymbol{\alpha}(s)$ is whose principal normal $\mathbf{n}(s)$ makes a constant angle with a fixed unit vector is called a *slant helix*. It is known [14] that $\boldsymbol{\alpha}(s)$ is a slant helix if and only if its curvature and torsion satisfy

$$\frac{\kappa^2(\tau/\kappa)'}{(\kappa^2 + \tau^2)^{3/2}} = c \tag{3}$$

for some constant c. Note that the slant helix degenerates to a general helix if $c = 0$ in (3). Hence, a slant helix with $c \neq 0$ is called a *proper* slant helix. The Salkowski curves, characterized by constant curvature and non–constant torsion, are proper slant helices (see Theorem 1 in [20]).

A *rectifying curve* $\boldsymbol{\alpha}(s)$ satisfies $\langle \boldsymbol{\alpha}(s), \mathbf{n}(s)\rangle = 0$, i.e., the position vector $\boldsymbol{\alpha}(s)$ always lies in the curve rectifying plane [4, 5]. It is known [4] that $\boldsymbol{\alpha}(s)$ is a rectifying curve if and only if its torsion $\tau(s)$ and curvature $\kappa(s)$ satisfy

$$\frac{\tau(s)}{\kappa(s)} = as + b, \tag{4}$$

where $a \neq 0$ and b are constants. This may be considered the simplest non–trivial generalization of the constant torsion/curvature ratio (2) for a general helix to an arc–length–dependent ratio.

A *spherical curve*, i.e., a curve that lies on a sphere of radius r with center at the origin, may be characterized [19] by the relation

$$(\rho'\sigma)' + \frac{\rho}{\sigma} = 0, \quad \text{where } \rho = \frac{1}{\kappa},\ \sigma = \frac{1}{\tau}. \tag{5}$$

It is known [4] that a Frenet curve $\boldsymbol{\alpha}(s)$ is a rectifying curve if and only if a unit–speed spherical curve $\boldsymbol{\gamma}(s) : I \to S^2$ exists, such that

$$\boldsymbol{\alpha}(s) = a \sec(s + s_0)\, \boldsymbol{\gamma}(s),$$

where S^2 is the unit sphere with center at the origin, and $a \neq 0$ and s_0 are constants. If $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is the Frenet–Serret apparatus of the rectifying curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ and κ_γ is the curvature of the unit–speed curve $\boldsymbol{\gamma}(s) : I \to S^2$, then we have [7]:

$$\kappa = \frac{1}{a}\cos^3(s + s_0)\sqrt{\kappa_\gamma^2 - 1}, \quad \tau = \frac{1}{a}\cos^2(s + s_0)\sin(s + s_0)\sqrt{\kappa_\gamma^2 - 1}. \tag{6}$$

The centrode of a unit–speed curve $\boldsymbol{\alpha}(s)$ is defined by

$$\boldsymbol{\omega}(s) = \tau(s)\,\mathbf{t}(s) + \kappa(s)\,\mathbf{b}(s), \tag{7}$$

i.e., it is the locus traced by the angular velocity vector (or *Darboux vector*) of the Frenet frame along $\boldsymbol{\alpha}(s)$, which describes the variation of the frame vectors through the relations

$$\mathbf{t}' = \boldsymbol{\omega} \times \mathbf{t}, \quad \mathbf{n}' = \boldsymbol{\omega} \times \mathbf{n}, \quad \mathbf{b}' = \boldsymbol{\omega} \times \mathbf{b},$$

which are an alternative expression of equations (1). The centrode of a unit speed curve has been used to characterize rectifying curves [4, 5]. Also, the curve defined by

$$\boldsymbol{\omega}_d(s) = \frac{\boldsymbol{\omega}(s)}{\kappa(s)} \tag{8}$$

is called the *dilated centrode*, and for a non–helical unit speed Frenet curve, it is shown in [7] that $\boldsymbol{\omega}_d(s)$ is always a rectifying curve.

3. Characterizations of slant helices

In this section, some properties and characterizations of proper slant helices and Salkowski curves are derived. In particular, we will show that a unique general helix may be associated with each proper slant helix, and that the centrode of a Salkwoski curve is a proper slant helix. Let $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ be a unit–speed slant helix, with Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$. Then a fixed unit vector $\mathbf{u}$ and constant c exist, such that $\langle \mathbf{u}, \mathbf{n}(s)\rangle = c$, $s \in I$ [14].

For a proper slant helix, with $c \neq 0$, we show that no point $s_0 \in I$ exists, such that $\langle \mathbf{u}, \mathbf{b}(s_0)\rangle = 0$. Differentiating $\langle \mathbf{u}, \mathbf{n}(s)\rangle = c$ and using (1) gives

$$\kappa\,\langle \mathbf{u}, \mathbf{t}(s)\rangle = \tau\,\langle \mathbf{u}, \mathbf{b}(s)\rangle. \tag{9}$$

If $\langle \mathbf{u}, \mathbf{b}(s_0) \rangle = 0$, this equation implies that $\langle \mathbf{u}, \mathbf{t}(s_0) \rangle = 0$, and consequently $\mathbf{u} = \pm\, \mathbf{n}(s_0)$ since $\mathbf{u}$ is a unit vector, so $c = \pm 1$. Writing

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{t}(s) \rangle\, \mathbf{t}(s) \pm \mathbf{n}(s) + \langle \mathbf{u}, \mathbf{b}(s) \rangle\, \mathbf{b}(s)$$

and taking the norm of both sides then gives

$$1 = \sqrt{\langle \mathbf{u}, \mathbf{t}(s) \rangle^2 + 1 + \langle \mathbf{u}, \mathbf{b}(s) \rangle^2},$$

which can only be satisfied if $\langle \mathbf{u}, \mathbf{t}(s) \rangle \equiv 0$ and $\langle \mathbf{u}, \mathbf{b}(s) \rangle \equiv 0$, i.e., $\mathbf{u} = \pm \mathbf{n}(s)$. Differentiating this and using equations (1) gives $\kappa(s) \equiv 0$ and $\tau(s) \equiv 0$, in contradiction with the assumption that $\boldsymbol{\alpha}(s)$ is a proper slant helix. Hence, $\langle \mathbf{u}, \mathbf{b}(s) \rangle \neq 0$ for all $s \in I$, and equation (9) gives

$$\frac{\tau(s)}{\kappa(s)} = \frac{\langle \mathbf{u}, \mathbf{t}(s) \rangle}{\langle \mathbf{u}, \mathbf{b}(s) \rangle}. \tag{10}$$

Lemma 3.1. *If $\boldsymbol{\alpha}(s) : I \to E^3$ is a proper slant helix with the Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ its unit axis vector $\mathbf{u}$ is given by*

$$\mathbf{u} = \frac{\sqrt{1-c^2}}{\sqrt{1+(\tau/\kappa)^2}}\,(\tau/\kappa)\,\mathbf{t} + c\,\mathbf{n} + \frac{\sqrt{1-c^2}}{\sqrt{1+(\tau/\kappa)^2}}\,\mathbf{b}. \tag{11}$$

Proof : We have

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{t} \rangle\, \mathbf{t} + c\,\mathbf{n} + \langle \mathbf{u}, \mathbf{b} \rangle\, \mathbf{b}, \tag{12}$$

which gives $\langle \mathbf{u}, \mathbf{t} \rangle^2 + \langle \mathbf{u}, \mathbf{b} \rangle^2 = 1 - c^2$. From equation (10) we obtain

$$\frac{\tau^2 + \kappa^2}{\kappa^2} = \frac{\langle \mathbf{u}, \mathbf{t} \rangle^2 + \langle \mathbf{u}, \mathbf{b} \rangle^2}{\langle \mathbf{u}, \mathbf{b} \rangle^2} = \frac{1-c^2}{\langle \mathbf{u}, \mathbf{b} \rangle^2}.$$

Since $\langle \mathbf{u}, \mathbf{b}(s) \rangle$ does not change sign on the connected interval $s \in I$, we may choose the direction of $\mathbf{u}$ that gives a positive value, and write

$$\langle \mathbf{u}, \mathbf{b} \rangle = \frac{\sqrt{1-c^2}}{\sqrt{1+(\tau/\kappa)^2}}.$$

Substituting this and (10) into (12) yields the stated form (11) of $\mathbf{u}$. ∎

Corollary 3.1. *A unit–speed Frenet curve $\boldsymbol{\alpha}(s) : I \to E^3$ is a proper slant helix if and only if its curvature $\kappa(s)$ and torsion $\tau(s)$ satisfy*

$$\left(\frac{(\tau/\kappa)}{\sqrt{1+(\tau/\kappa)^2}} \right)' = \frac{c}{\sqrt{1-c^2}}\,\kappa, \quad \left(\frac{1}{\sqrt{1+(\tau/\kappa)^2}} \right)' = \frac{c}{\sqrt{1-c^2}}\,\tau, \tag{13}$$

for some non–zero constant c.

Proof : Suppose the curve $\boldsymbol{\alpha}(s)$ is a proper slant helix. Then differentiating (11) and equating components yields the relations (13).

Conversely, suppose that the two relations (13) hold for a unit–speed Frenet curve. Then the first relation gives

$$(\tau/\kappa)\left(\frac{1}{\sqrt{1+(\tau/\kappa)^2}} \right)' + \left(\frac{1}{\sqrt{1+(\tau/\kappa)^2}} \right)(\tau/\kappa)' = \frac{c}{\sqrt{1-c^2}}\,\kappa,$$

and substituting the second relation of (13) into the above yields

$$-\frac{c}{\sqrt{1-c^2}}\frac{\tau^2}{\kappa}+\left(\frac{1}{\sqrt{1+(\tau/\kappa)^2}}\right)(\tau/\kappa)'=\frac{c}{\sqrt{1-c^2}}\kappa,$$

which reduces to

$$\frac{(\tau/\kappa)'}{(1+(\tau/\kappa)^2)^{3/2}}=\frac{c}{\sqrt{1-c^2}}\,\kappa.$$

Since this is equivalent to equation (3), the curve is a proper slant helix. ∎

Theorem 3.1. *A unit–speed Frenet curve $\boldsymbol{\alpha}(s): I \to E^3$ with Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a proper slant helix if and only if*

$$\tau/\kappa=\frac{f}{\sqrt{1-f^2}}, \quad \textit{where } f=c\int \kappa \,\mathrm{d}s \tag{14}$$

and c is a non–zero constant.

Proof : Suppose the Frenet curve $\boldsymbol{\alpha}(s)$ satisfies the condition (14). Then we have

$$(\tau/\kappa)'=\frac{f'}{(1-f^2)^{3/2}}=\frac{c\kappa}{(1-f^2)^{3/2}} \quad \text{and} \quad 1+(\tau/\kappa)^2=\frac{1}{1-f^2}.$$

These equations give

$$\frac{(\tau/\kappa)'}{(1+(\tau/\kappa)^2)^{3/2}}=c\,\kappa,$$

which with $c \neq 0$ is equivalent to the condition (3) for a proper slant helix.

Conversely, suppose $\boldsymbol{\alpha}(s)$ is a proper slant helix. Then by Theorem A in [15], the indefinite integrals of κ and τ satisfy

$$\left(\int \kappa \,\mathrm{d}s\right)^2+\left(\int \tau \,\mathrm{d}s\right)^2 = \tan^2\theta\,, \tag{15}$$

where $0<\theta<\frac{1}{2}\pi$ is the angle between $\mathbf{n}(s)$ and the fixed direction $\mathbf{u}$. From this, one can easily deduce the relations

$$\frac{\cos^2\theta}{\sin^2\theta}\left(\int \kappa \,\mathrm{d}s\right)^2 < 1\,, \quad \frac{\kappa}{\tau} = -\frac{\int \tau \,\mathrm{d}s}{\int \kappa \,\mathrm{d}s}\,. \tag{16}$$

Now from (15) we obtain

$$1+\frac{\left(\int \tau \,\mathrm{d}s\right)^2}{\left(\int \kappa \,\mathrm{d}s\right)^2}=\frac{\sin^2\theta}{\cos^2\theta\left(\int \kappa \,\mathrm{d}s\right)^2}\,,$$

and on using the second relation in (16), this becomes

$$1+(\kappa/\tau)^2=\frac{\sin^2\theta}{\cos^2\theta\left(\int \kappa \,\mathrm{d}s\right)^2}\,,$$

from which we obtain

$$(\tau/\kappa)^2 = \frac{\cos^2\theta\left(\int \kappa\,\mathrm{d}s\right)^2}{\sin^2\theta - \cos^2\theta\left(\int \kappa\,\mathrm{d}s\right)^2}\,.$$

This is equivalent to the stated condition (14) with $c = \pm\cot\theta$, and we note from (16) that $f^2 < 1$. ∎

As a consequence of Theorem 3.1, and the fact that every Salkowski curve is a proper slant helix, we have the following characterization of Salkowski curves — essentially a result in [20].

Corollary 3.2. *A unit–speed Frenet curve $\boldsymbol{\alpha}(s) : I \to E^3$ with curvature*[1)] *$\kappa = 1$ is a Salkowski curve if and only if its torsion is of the form*

$$\tau(s) = \frac{cs}{\sqrt{1 - c^2 s^2}},$$

where c is a non–zero constant.

It is interesting to observe, as the following theorem shows, that a unique general helix may be associated with each proper slant helix, such that the principal normal vector field of the slant helix coincides with the binormal vector field of the general helix.

Theorem 3.2. *Let $\boldsymbol{\alpha}(s) : I \to E^3$ be a proper slant helix with axis vector $\mathbf{u}$ and Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ where $\kappa > 0$ and $\langle \mathbf{u}, \mathbf{n} \rangle = c$. Then a unique general helix $\boldsymbol{\beta}(s) : I \to E^3$ exists with curvature $c\,\sqrt{\tau^2 + \kappa^2}/\sqrt{1 - c^2}$, torsion $\sqrt{\tau^2 + \kappa^2}$, and binormal vector field $\mathbf{n}$.*

Proof : We define the following unit vector fields

$$\mathbf{p} = \frac{(\tau/\kappa)\,\mathbf{t} + \mathbf{b}}{\sqrt{1 + (\tau/\kappa)^2}}\,, \quad \mathbf{q} = \frac{\mathbf{t} - (\tau/\kappa)\,\mathbf{b}}{\sqrt{1 + (\tau/\kappa)^2}} \tag{17}$$

along the curve $\boldsymbol{\alpha}(s)$. Then one can easily verify that $(\mathbf{p}, \mathbf{q}, \mathbf{n})$ is an oriented orthonormal frame along $\boldsymbol{\alpha}(s)$, with

$$\mathbf{p} \times \mathbf{q} = \mathbf{n}, \quad \mathbf{q} \times \mathbf{n} = \mathbf{p}, \quad \mathbf{n} \times \mathbf{p} = \mathbf{q}.$$

Differentiating equations (17), and using the relations (13) for a proper slant helix, we obtain

$$\mathbf{p}' = \frac{c}{\sqrt{1 - c^2}}\,\sqrt{\tau^2 + \kappa^2}\,\mathbf{q}, \quad \mathbf{q}' = \sqrt{\tau^2 + \kappa^2}\left(\mathbf{n} - \frac{c}{\sqrt{1 - c^2}}\,\mathbf{p}\right), \tag{18}$$

and we also have

$$\mathbf{n}' = -\,\kappa\,\mathbf{t} + \tau\,\mathbf{b} = -\,\sqrt{\tau^2 + \kappa^2}\,\mathbf{q}. \tag{19}$$

Equations (18)–(19) indicate, by the existence theorem [19] for curves, that

$$\left(\frac{c}{\sqrt{1 - c^2}}\,\sqrt{\tau^2 + \kappa^2},\ \sqrt{\tau^2 + \kappa^2}, \mathbf{p}, \mathbf{q}, \mathbf{n}\right)$$

is the Frenet–Serret apparatus for a unique unit–speed curve $\boldsymbol{\beta}(s) : I \to \mathbb{E}^3$, and that $\boldsymbol{\beta}(s)$ is a general helix.∎

[1)] The assumption $\kappa = 1$ is conventional in the study of Salkowski curves [20], and can be achieved for any curve of constant curvature by an appropriate scaling.

Remark 3.1. For the example of a proper slant helix on page 161 of Izumiya–Takeuchi [14], we obtain the associated circular helix with constant curvature $\bar{\kappa} = b/\sqrt{a^2-b^2}$ and constant torsion $\bar{\tau} = \sqrt{a^2-b^2}$.

Remark 3.2. The Salkowski curves considered by Monterde [20] are proper slant helices, with curvature $\kappa = 1$ and torsion

$$\tau(s) = \frac{\pm s}{\sqrt{\tan^2\phi - s^2}},$$

where ϕ is the constant angle made by the principal normal $\mathbf{n}$ with a fixed direction $\mathbf{u}$ (see Lemma 1 and Theorem 1 in [20]). Thus, setting $c = \cot\phi$, the curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ of the general helix associated with a Salkowski curve are given by

$$\bar{\kappa} = \frac{c}{\sqrt{1-c^2}\sqrt{1-c^2s^2}}, \quad \bar{\tau} = \frac{1}{\sqrt{1-c^2s^2}}.$$

Every Salkowski curve is a proper slant helix, but there exist proper slant helices that are not Salkowski curves (for instance, the example given in [14]). The centrodes $\boldsymbol{\omega} = \tau\,\mathbf{t} + \kappa\,\mathbf{b}$ of Frenet curves are valuable in analyzing the kinematics of joints [13, 27], and it is of interest to ask whether the centrode of a proper slant helix is always a proper slant helix. The answer is negative, as illustrated by the example

$$\boldsymbol{\alpha}(s) \;=\; -\frac{a^2-b^2}{2a}\left(\frac{\cos((a+b)s)}{(a+b)^2} + \frac{\cos((a-b)s)}{(a-b)^2}, \frac{\sin((a+b)s)}{(a+b)^2} + \frac{\sin((a-b)s)}{(a-b)^2}, \frac{2}{b\sqrt{a^2-b^2}}\cos bs\right),$$

in [14]. For $0 < b < a$, this is a unit–speed proper slant helix, with curvature and torsion

$$\kappa(s) = \sqrt{a^2-b^2}\,\cos bs, \quad \tau(s) = \sqrt{a^2-b^2}\,\sin bs.$$

The centrode $\boldsymbol{\omega} = \tau\,\mathbf{t} + \kappa\,\mathbf{b}$ of this curve has parametric speed $v_\omega = |\boldsymbol{\omega}'(s)|$, curvature κ_ω, and torsion τ_ω given by

$$v_\omega = b\sqrt{a^2-b^2}, \quad \kappa_\omega = \frac{a}{b\sqrt{a^2-b^2}}, \quad \tau_\omega = 0.$$

Thus, the centrode of $\boldsymbol{\alpha}(s)$ is an arc of a circle, and not a proper slant helix. On the other hand, one can show that the centrode of a Salkowski curve is a slant helix, as follows.

Theorem 3.3. *A Salkowski curve $\boldsymbol{\alpha}(s) : I \to E^3$ has a centrode $\boldsymbol{\omega} = \tau\,\mathbf{t} + \kappa\,\mathbf{b}$ that is a proper slant helix, but is not a Salkowski curve.*

Proof : The unit–speed Salkowski curve $\boldsymbol{\alpha}(s)$ has curvature and torsion given [20] by

$$\kappa(s) = 1, \quad \tau(s) = \frac{\pm ms}{\sqrt{1-m^2s^2}}, \tag{20}$$

where $m \neq 0, \pm 1/\sqrt{3}$ is a real number, and the domain of $\boldsymbol{\alpha}(s)$ is given by $|ms| < 1$. Thus, the centrode of the Salkowski curve is

$$\boldsymbol{\omega}(s) = \frac{\pm ms}{\sqrt{1-m^2s^2}}\,\mathbf{t}(s) + \mathbf{b}(s),$$

from which we obtain

$$\boldsymbol{\omega}'(s) = \frac{\pm m}{(1-m^2s^2)^{3/2}}\,\mathbf{t}(s). \tag{21}$$

If s_ω is arc length along the centrode $\boldsymbol{\omega}(s)$, its parametric speed v_ω is

$$v_\omega(s) = \frac{\mathrm{d}s_\omega}{\mathrm{d}s} = |\boldsymbol{\omega}'(s)| = \frac{|m|}{(1-m^2s^2)^{3/2}}, \tag{22}$$

and by the chain rule we have

$$\frac{\mathrm{d}}{\mathrm{d}s_\omega} = \frac{1}{v_\omega}\frac{\mathrm{d}}{\mathrm{d}s}. \tag{23}$$

From (21) we obtain the tangent to the centrode as

$$\mathbf{t}_\omega(s) = \frac{\boldsymbol{\omega}'(s)}{|\boldsymbol{\omega}'(s)|} = \pm\,\mathbf{t}(s). \tag{24}$$

Its curvature κ_ω and principal normal $\mathbf{n}_\omega$ are obtained using (22)–(23) from

$$\frac{\mathrm{d}\mathbf{t}_\omega}{\mathrm{d}s_\omega} = \frac{\pm 1}{v_\omega}\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa_\omega \mathbf{n}_\omega\,,$$

and since $\mathrm{d}\mathbf{t}/\mathrm{d}s = \kappa\,\mathbf{n}$ with $\kappa(s)$ given by (20), we have

$$\kappa_\omega(s) = \frac{\left(1-m^2s^2\right)^{3/2}}{|m|}, \quad \mathbf{n}_\omega(s) = \pm\,\mathbf{n}(s). \tag{25}$$

Equations (24)–(25) give the centrode binormal vector as $\mathbf{b}_\omega = \mathbf{t}_\omega \times \mathbf{n}_\omega = \mathbf{b}$. Since

$$\frac{\mathrm{d}\mathbf{b}_\omega}{\mathrm{d}s_\omega} = \frac{1}{v_\omega}\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\,\tau_\omega \mathbf{n}_\omega\,,$$

and $\mathrm{d}\mathbf{b}/\mathrm{d}s = -\,\tau\,\mathbf{n}$ where $\tau(s)$ is given by (20), we obtain the torsion of the centrode as

$$\tau_\omega(s) = \pm\,(1-m^2s^2)\,s. \tag{26}$$

Since $\mathbf{n}_\omega(s) = \pm\,\mathbf{n}(s)$, the centrode is a slant helix. Moreover, it is a proper slant helix, since the ratio $\tau_\omega(s)/\kappa_\omega(s)$ is non–constant. The constant c in equation (3) can be found as follows. From (23) and (25)–(26), we have

$$\frac{1}{(\kappa_\omega^2+\tau_\omega^2)^{3/2}}\frac{\mathrm{d}}{\mathrm{d}s_\omega}\frac{\tau_\omega}{\kappa_\omega} = \frac{1}{(\kappa_\omega^2+\tau_\omega^2)^{3/2}}\frac{1}{v_\omega}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\tau_\omega}{\kappa_\omega} = \frac{\pm m^3}{(1-m^2s^2)^3} = \frac{\pm m}{\kappa_\omega^2}.$$

Hence, the centrode of a Salkowski curve is a proper slant helix with constant $c = \pm m$ in equation (3), and it is not a Salkowski curve since $\kappa_\omega \neq$ constant.

Remark 3.3. The torsion/curvature ratio properties of general helices and rectifying curves indicate that they are mutually disjoint families of curves. It is not known whether a proper slant helix can also be a rectifying curve. However, Theorem 3.3 and the following Corollary show that the centrode of a Salkowski curve is both a proper slant helix and a rectifying curve.

Corollary 3.3. *The centrode $\boldsymbol{\omega} = \tau\,\mathbf{t} + \kappa\,\mathbf{b}$ of a Salkowski curve $\boldsymbol{\alpha}(s) : I \to E^3$ is a rectifying curve.*

Proof : Using equations (25) and (26), we have

$$\frac{\tau_\omega(s)}{\kappa_\omega(s)} = \frac{|m|\,s}{\sqrt{1-m^2s^2}}.$$

Consequently, if s_ω is arc length along $\boldsymbol{\omega}(s)$, using equation (23) we have

$$\frac{\mathrm{d}}{\mathrm{d}s_\omega}\frac{\tau_\omega}{\kappa_\omega} = \frac{1}{v_\omega}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\tau_\omega}{\kappa_\omega} = 1,$$

so $\boldsymbol{\omega}(s)$ is a general helix, since it satisfies (2) with non–constant τ_ω and κ_ω. Moreover, integrating the above relation with respect to s_ω gives

$$\frac{\tau_\omega}{\kappa_\omega} = s_\omega + b,$$

for some constant b, i.e., the centrode is a rectifying curve satisfying (4). ∎

4. Associated circular helices of Frenet curves

Among all Frenet curves in $\mathbb{E}^3$, the helices have a special stature due to their widespread applications in science and technology. In the present section, we highlight the importance and ubiquity of helices by showing that every Frenet curve is either a general helix, or else has a unique circular helix associated with it. We begin by proving this very general result.

Theorem 4.1. *Let $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ be a unit–speed Frenet curve of class C^k, $k \geq 4$ with Frenet Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$. Then $\boldsymbol{\alpha}(s)$ is either a general helix, or there is a unique circular helix associated with it, defined by*

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+(\tau/\kappa)^2}}, \frac{\tau/\kappa}{\sqrt{1+(\tau/\kappa)^2}}, \tan^{-1}(\tau/\kappa)\right). \tag{27}$$

Proof : Suppose that $\boldsymbol{\alpha}(s)$ is a Frenet curve that is not a general helix, i.e., $(\tau/\kappa)' \neq 0$. Then $\boldsymbol{\beta}(s) : I \to \mathbb{E}^3$ defined by (27) is a regular curve, with parametric speed

$$v_\beta(s) = \frac{\mathrm{d}s_\beta}{\mathrm{d}s} = |\boldsymbol{\beta}'(s)| = \frac{|(\tau/\kappa)'|}{1+(\tau/\kappa)^2},$$

where s_β is arc length along $\boldsymbol{\beta}(s)$. Hence, using the Frenet–Serret relations, the Frenet–Serret apparatus of $\boldsymbol{\beta}(s)$ can be computed as

$$\kappa_\beta = \tau_\beta = \frac{1}{\sqrt{2}}, \quad \mathbf{t}_\beta = \pm\frac{1}{\sqrt{2}}\left(\frac{-\tau/\kappa}{\sqrt{1+(\tau/\kappa)^2}}, \frac{1}{\sqrt{1+(\tau/\kappa)^2}}, 1\right),$$

$$\mathbf{n}_\beta = \frac{-(1, \tau/\kappa, 0)}{\sqrt{1+(\tau/\kappa)^2}}, \quad \mathbf{b}_\beta = \frac{\pm(\tau/\kappa, -1, \sqrt{1+(\tau/\kappa)^2})}{\sqrt{2}\sqrt{1+(\tau/\kappa)^2}}.$$

Thus, $\boldsymbol{\beta}(s)$ is a circular helix, since $\tau_\beta/\kappa_\beta = 1$. Hence, the unit speed Frenet curve $\boldsymbol{\alpha}(s)$ is either a general helix, or there is a unique circular helix $\boldsymbol{\beta}(s)$ defined by (27) associated with it. ∎

Definition 4.1. For a unit speed Frenet curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ of class C^k, $k \geq 4$ that is not a general helix, the unique circular helix $\boldsymbol{\beta}(s)$ identified by (27) is called the associated circular helix of the Frenet curve $\boldsymbol{\alpha}(s)$.

In the remainder of this section, we use the circular helix associated with non–helical Frenet curves to formulate new characterizations for slant helices, Salkowski curves, spherical curves and rectifying curves. Note that a given proper slant helix $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ has two helices associated with it: the general helix identified in Theorem 3.2 and the associated circular helix (27). We now prove the following characterization for a proper slant helix.

Proposition 4.1. *A unit–speed Frenet curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ of class C^k, $k \geq 4$ with Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a proper slant helix if and only if the circular helix associated with it is given by*

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\sqrt{1-f^2}, f, \sin^{-1} f\right), \tag{28}$$

where $f = c\int \kappa\, \mathrm{d}s$ and c is a non–zero constant.

Proof : Let $\boldsymbol{\alpha}(s)$ be a unit–speed proper slant helix, which by the proof of Theorem 3.1 satisfies $f^2 < 1$. Then substituting $\tau/\kappa = \tan\theta$ in equation (3) yields $\pm\,\theta' \cos\theta = c\,\kappa$. Absorbing the sign ambiguity into the constant c and integrating we find

$$\sin\theta = c\int \kappa\, \mathrm{d}s = f.$$

Since $c \neq 0$, $\boldsymbol{\alpha}(s)$ is not a general helix. The circular helix (27) associated with $\boldsymbol{\alpha}(s)$ is thus given by

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \theta) = \frac{1}{\sqrt{2}}\left(\sqrt{1-f^2}, f, \sin^{-1} f\right).$$

Conversely, let the circular helix associated with the unit–speed Frenet curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ be given by (28), where $f = c\int \kappa\, \mathrm{d}s$, $c \neq 0$. Then we have

$$1 + (\tau/\kappa)^2 = \frac{1}{1-f^2} \quad \text{and} \quad f = \frac{\tau/\kappa}{\sqrt{1+(\tau/\kappa)^2}}, \tag{29}$$

that is,

$$\tau/\kappa = \frac{f}{\sqrt{1-f^2}}, \quad \text{where } f = c\int \kappa\, \mathrm{d}s,$$

which by Theorem 3.1 shows that $\boldsymbol{\alpha}(s)$ is a proper slant helix. ∎

Recalling [20] that every Salkowski curve is a proper slant helix, we now find the constant c in equation (3). The curvature and torsion of a Salkowski curve $\boldsymbol{\alpha}(s)$ are given by (20) with $m = \cot\phi$, where ϕ is the constant angle made by principal normal with a fixed direction and s is arc length. Hence, for a unit–speed Salkowski curve, we obtain

$$(\tau/\kappa)' = \frac{\pm m}{(1-m^2s^2)^{3/2}} \quad \text{and} \quad 1 + (\tau/\kappa)^2 = \frac{1}{1-m^2s^2}.$$

Thus, the equation (3) takes the form

$$\frac{\kappa^2\,(\tau/\kappa)'}{(\tau^2+\kappa^2)^{3/2}} = \pm m,$$

and the constant is $c = \pm m$. This leads to the following characterization of Salkowski curves in terms of their associated circular helices.

Proposition 4.2. *A unit–speed Frenet curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ of class C^k, $k \geq 4$ with Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a Salkowski curve if and only if the circular helix associated with it is given by*

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\sqrt{1-m^2s^2}, \pm ms, \pm\sin^{-1}(ms)\right), \tag{30}$$

where $m \neq 0, \pm 1/\sqrt{3}$ is a non–zero constant.

Proof : Let $\boldsymbol{\alpha}(s)$ be a unit–speed Salkowski curve with curvature and torsion given by (20). Since $\boldsymbol{\alpha}(s)$ is a proper slant helix satisfying (3) with $c = \pm m$, its associated circular helix is given by equation (28) in Proposition 4.1, where $f = \pm m \int \kappa \, \mathrm{d}s = \pm ms + b$. By the re–parametrization $s \to s - b/(\pm m)$, we obtain $f = \pm ms$ and then equation (28) reduces to the stated form (30).

Conversely, suppose the unit–speed Frenet curve $\boldsymbol{\alpha}(s)$ has the curve (27) as its associated circular helix. Setting $f = \pm ms = c \int \mathrm{d}s$, this becomes

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}(\sqrt{1-f^2}, f, \sin^{-1} f),$$

which by Proposition 4.1 indicates that $\boldsymbol{\alpha}(s)$ is a proper slant helix with curvature $\kappa = 1$ and torsion τ satisfying (29) so that

$$1 + \left(\frac{\tau}{1}\right)^2 = \frac{1}{1-f^2}, \quad \text{i.e.,} \ \ \tau(s) = \frac{\pm ms}{\sqrt{1-m^2s^2}}.$$

Hence, $\boldsymbol{\alpha}(s)$ is a Salkowski curve [20]. ∎

We consider next the circular helices associated with spherical curves.

Proposition 4.3. *A non–helical unit–speed Frenet curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ of class C^k, $k \geq 4$ with Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a spherical curve on a sphere of radius c if and only if the circular helix associated with it is given by*

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+f^2}}, \frac{f}{\sqrt{1+f^2}}, \tan^{-1} f\right), \tag{31}$$

where $f = c\,\tau \cos\left(\int \tau \,\mathrm{d}s\right)$ and c is a positive constant.

Proof : Suppose that $\boldsymbol{\alpha}(s)$ is a non–helical unit–speed spherical curve that lies on a sphere of radius c. Then by integration of equation (5) we have

$$\frac{1}{\kappa^2} + \frac{\kappa'^2}{\kappa^4\tau^2} = c^2,$$

which gives

$$\frac{\kappa'}{\kappa\sqrt{\kappa^2 - 1/c^2}} = \pm c\,\tau$$

and on integration this yields

$$c\,\kappa = \pm \sec\left(\int \tau \,\mathrm{d}s\right).$$

Absorbing the sign ambiguity into the constant c and setting $f = \tau/\kappa$, this is equivalent to

$$f = c\,\tau\cos\left(\int \tau \,\mathrm{d}s\right).$$

Hence, the circular helix (27) associated with $\boldsymbol{\alpha}(s)$ is given by

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+f^2}}, \frac{f}{\sqrt{1+f^2}}, \tan^{-1} f\right), \quad f = c\,\tau\cos\left(\int \tau \,\mathrm{d}s\right).$$

Conversely, suppose that the circular helix associated with $\boldsymbol{\alpha}(s)$ is given by (31), where $f = c\,\tau \cos\left(\int \tau \; \mathrm{d}s\right)$ with c a non–zero constant. Then the first component of $\boldsymbol{\beta}(s)$ gives $\tau/\kappa = f$, and consequently we have

$$\rho = c\,\cos\left(\int \tau \; \mathrm{d}s\right).$$

Differentiating this twice yields

$$(\rho'\sigma)' = -\,c\,\tau\,\cos\left(\int \tau \; \mathrm{d}s\right),$$

and combining these two relations indicates satisfaction of equation (5), so that $\boldsymbol{\alpha}(s)$ is a spherical curve that lies on the sphere of radius c. ■

Finally, we consider the circular helices associated with rectifying curves. We first obtain the following result, characterizing rectifying curves in terms of their dilated centrodes $\boldsymbol{\omega}_d(s)$ defined by (8).

Proposition 4.4. *A unit–speed Frenet curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ with Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a rectifying curve if and only if its position vector is given by*

$$\boldsymbol{\alpha}(s) = \frac{\boldsymbol{\omega}_d(s)}{(\tau/\kappa)'}, \tag{32}$$

where $\boldsymbol{\omega}_d(s)$ is the dilated centrode of $\boldsymbol{\alpha}(s)$.

Proof : Suppose that the unit–speed curve $\boldsymbol{\alpha}(s)$ is a rectifying curve. Then its position vector is given [4] by

$$\boldsymbol{\alpha}(s) = (s+a)\,\mathbf{t} + c\,\mathbf{b}, \tag{33}$$

where a and $c \neq 0$ are constants. Differentiating this relation yields $\boldsymbol{\alpha}'(s) = \mathbf{t} + ((s+a)\kappa - c\tau)\,\mathbf{n} = \mathbf{t}$, since $\boldsymbol{\alpha}(s)$ is unit speed. Hence, we have

$$\tau/\kappa = \frac{s+a}{c} \quad \text{and} \quad (\tau/\kappa)' = \frac{1}{c}.$$

Consequently, using equations (7)–(8) and (33), we have

$$\boldsymbol{\alpha}(s) = c\,(\tau/\kappa)\,\mathbf{t} + c\,\mathbf{b} = \frac{\boldsymbol{\omega}_d(s)}{(\tau/\kappa)'}.$$

Conversely, if $\boldsymbol{\alpha}(s)$ is of them form (32), we have $\langle \boldsymbol{\alpha}(s), \mathbf{n}(s)\rangle = 0$ for $s \in I$, since $\boldsymbol{\omega}_d = (\tau/\kappa)\,\mathbf{t} + \mathbf{b}$, and thus $\boldsymbol{\alpha}(s)$ is a rectifying curve. ■

Proposition 4.5. *A unit–speed Frenet curve $\boldsymbol{\alpha}(s) : I \to \mathbb{E}^3$ of class C^k, $k \geq 4$ with Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a rectifying curve if and only if the circular helix associated with it is given by*

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\frac{c}{\sqrt{c^2+s^2}}, \frac{s}{\sqrt{c^2+s^2}}, \tan^{-1}(s/c)\right), \tag{34}$$

where c is a non–zero constant.

Proof : Suppose that $\boldsymbol{\alpha}(s)$ is a unit–speed rectifying curve. Then by equation (4), we have

$$\frac{\tau(s)}{\kappa(s)} = as + b,$$

where $a \neq 0$, b are constants. The re–parametrization $s \to s - b/a$ yields

$$\frac{\tau(s)}{\kappa(s)} = as$$

and since $\boldsymbol{\alpha}(s)$ is not a general helix, its associated circular helix is given by Theorem 4.1 as

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+(as)^2}}, \frac{as}{\sqrt{1+(as)^2}}, \tan^{-1}(as)\right),$$

which is the required form (34) with $c = a^{-1}$.

Conversely, suppose that the unit speed curve has the associated circular helix (34). Then from equation (27) we have

$$\frac{\tau}{\kappa} = \frac{s}{c},$$

i.e, the torsion/curvature ratio of $\boldsymbol{\alpha}(s)$ is a non–trivial linear function of arc length, and hence it is a rectifying curve. ■

Remark 4.1. Recall that there are essentially two ways to generate rectifying curves: through the dilated centrodes of a Frenet curve, and by the dilation of certain spherical curves. Note that for each rectifying curve $\boldsymbol{\alpha}(s)$, there is a unique unit–speed curve $\boldsymbol{\gamma}(s)$ (excluding great circles) on the unit sphere S^2 with center at the origin [7] such that

$$\boldsymbol{\alpha}(s) = a\sec(s+s_0)\,\boldsymbol{\gamma}(s),$$

where $a \neq 0$ and s_0 are constants. However, this expression does not define a unit–speed curve — if s_α is arc length along $\boldsymbol{\alpha}(s)$, its parametric speed (assuming that $a > 0$) is

$$v_\alpha = \frac{\mathrm{d}s_\alpha}{\mathrm{d}s} = |\boldsymbol{\alpha}'(s)| = a\sec^2(s+s_0), \tag{35}$$

since $|\boldsymbol{\gamma}(s)| = |\boldsymbol{\gamma}'(s)| = 1$, $\langle\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s)\rangle = 0$. The curvature κ_α and torsion τ_α of $\boldsymbol{\alpha}(s)$ are given by equation (6). Integrating (35), the arc length of $\boldsymbol{\alpha}(s)$ is $s_\alpha = a\tan(s+s_0) + b$ for some constant b, and using the re–parameterization $s_\alpha \to s_\alpha - b$ and equation (6), we obtain

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{s}{a}.$$

Since $\boldsymbol{\alpha}(s)$ is a rectifying curve, it is not a general helix, and its associated circular helix is thus obtained from (27) as

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+(s/a)^2}}, \frac{s/a}{\sqrt{1+(s/a)^2}}, \tan^{-1}(s/a)\right),$$

which is in agreement with the expression as given in Proposition 4.5.

5. Pythagorean-hodograph curves

Although the unit–speed parameterization offers an intrinsic approach to the differential geometry of space curves, it is incompatible with simple (rational) curves when $\kappa \not\equiv 0$ [12]. The *Pythagorean–hodograph curves* [8] offer a useful compromise between the conflicting requirements of relating the parameter to the curve intrinsic geometry, while maintaining a rational form. We consider now the results of the preceding sections in the context of the Pythagorean–hodograph curves, with non–unit–speed parameterizations.

Definition 5.1. A polynomial/rational curve $\boldsymbol{\alpha}(\xi) = (x(\xi), y(\xi), z(\xi))$, with a general parameter ξ, is called a *Pythagorean–hodograph* (PH) *curve* if the components of its hodograph (derivative) $\boldsymbol{\alpha}'(\xi) = (x'(\xi), y'(\xi), z'(\xi))$ satisfy

$$x'^2(\xi) + y'^2(\xi) + z'^2(\xi) = \sigma^2(\xi) \tag{36}$$

for some polynomial/rational function $\sigma(\xi)$.

Here $\sigma(\xi)$ represents the *parametric speed* of $\boldsymbol{\alpha}(\xi)$, i.e., the derivative

$$\sigma(\xi) = |\boldsymbol{\alpha}'(\xi)| = \frac{\mathrm{d}s}{\mathrm{d}\xi}$$

of its arc length s with respect to the parameter ξ. Polynomial/rational PH curves have rational tangents $\mathbf{t}(\xi) = \boldsymbol{\alpha}'(\xi)/|\boldsymbol{\alpha}'(\xi)|$. However, they differ with regard to the arc length function,

$$s(\xi) = \int \sigma(\xi)\,\mathrm{d}\xi\,. \tag{37}$$

For a polynomial PH curve, $\sigma(\xi)$ is a polynomial, so $s(\xi)$ is evidently also a polynomial. But for a rational PH curve, $\sigma(\xi)$ is a rational function, and its integral does not (in general) yield a rational arc length function $s(\xi)$.

A polynomial PH curve is generated [8] from a quaternion polynomial

$$\mathcal{A}(\xi) = u(\xi) + v(\xi)\,\mathbf{i} + p(\xi)\,\mathbf{j} + q(\xi)\,\mathbf{k} \tag{38}$$

and its conjugate $\mathcal{A}^*(\xi) = u(\xi) - v(\xi)\,\mathbf{i} - p(\xi)\,\mathbf{j} - q(\xi)\,\mathbf{k}$ by integrating the product

$$\begin{aligned} \boldsymbol{\alpha}'(\xi) \;&=\; \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi) \;=\; [\,u^2(\xi) + v^2(\xi) - p^2(\xi) - q^2(\xi)\,]\,\mathbf{i} \\ &\quad + 2\,[\,u(\xi)q(\xi) + v(\xi)p(\xi)\,]\,\mathbf{j} \;+\; 2\,[\,v(\xi)q(\xi) - u(\xi)p(\xi)\,]\,\mathbf{k}\,, \end{aligned} \tag{39}$$

and the resulting PH curve $\boldsymbol{\alpha}(\xi)$ has the parametric speed

$$\sigma(\xi) = |\mathcal{A}(\xi)|^2 = u^2(\xi) + v^2(\xi) + p^2(\xi) + q^2(\xi)\,. \tag{40}$$

Definition 5.2. A polynomial/rational curve $\boldsymbol{\alpha}(\xi) = (x(\xi), y(\xi), z(\xi))$, with a general parameter ξ, is called a *double Pythagorean–hodograph* (DPH) *curve* if $|\boldsymbol{\alpha}'(\xi)|$ and $|\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|$ are *both* polynomial/rational functions.

It may be shown [9] that the polynomial PH curve defined by (39) satisfies

$$|\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|^2 = \sigma^2(\xi)\,\rho(\xi), \tag{41}$$

where $\rho(\xi)$ is the polynomial defined in terms of the components of (38) as

$$\rho = 4\,[\,(up' - u'p + vq' - v'q)^2 \;+\; (uq' - u'q - vp' + v'p)^2\,]. \tag{42}$$

Thus, if $\boldsymbol{\alpha}(\xi)$ is a polynomial DPH curve, $\rho(\xi)$ must be a perfect square, i.e., for some polynomial $\omega(\xi)$ we have

$$\rho(\xi) \;=\; \omega^2(\xi)\,. \tag{43}$$

Lemma 5.1. *The set of all polynomial/rational curves with a rational Frenet–Serret apparatus is identical to the set of all polynomial/rational DPH curves.*

Proof : Recall [19] that, for a curve $\boldsymbol{\alpha}(\xi)$ with a general parameterization, the Frenet–Serret apparatus $(\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b})$ is given by

$$\left(\frac{|\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''|}{|\boldsymbol{\alpha}'|^3}, \frac{\langle\boldsymbol{\alpha}'\times\boldsymbol{\alpha}'', \boldsymbol{\alpha}'''\rangle}{|\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''|^2}, \frac{\boldsymbol{\alpha}'}{|\boldsymbol{\alpha}'|}, \frac{\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''}{|\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''|}\times\frac{\boldsymbol{\alpha}'}{|\boldsymbol{\alpha}'|}, \frac{\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''}{|\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''|}\right). \tag{44}$$

Thus $|\boldsymbol{\alpha}'(\xi)|$ and $|\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|$ being polynomial/rational functions is clearly sufficient and necessary for a rational Frenet–Serret apparatus. ∎

Note that the centrodes $\boldsymbol{\omega}(\xi) = \tau(\xi)\,\mathbf{t}(\xi) + \kappa(\xi)\,\mathbf{b}(\xi)$ and dilated centrodes $\boldsymbol{\omega}_d(\xi) = \boldsymbol{\omega}(\xi)/\kappa(\xi)$ of polynomial/rational DPH curves are rational curves.

Lemma 5.2. *If a polynomial/rational curve $\boldsymbol{\alpha}(\xi)$ is a general helix, it must be a polynomial/rational PH curve.*

Proof : This result is a consequence of the fact that, since $\mathbf{t}(\xi) = \boldsymbol{\alpha}'(\xi)/|\boldsymbol{\alpha}'(\xi)|$, the helix condition $\langle \mathbf{t}(\xi), \mathbf{u} \rangle = \cos\psi$ is equivalent [11] to

$$\langle \boldsymbol{\alpha}'(\xi), \mathbf{u} \rangle = \cos\psi\, |\boldsymbol{\alpha}'(\xi)|. \tag{45}$$

For any polynomial/rational curve $\boldsymbol{\alpha}(\xi)$, the left–hand side of equation (45) is clearly a polynomial/rational function, but $\boldsymbol{\alpha}(\xi)$ must be a PH curve for the right–hand side to also be a polynomial/rational function. ∎

Remark 5.1. It is known [9, 10] that every helical polynomial PH curve must also be a DPH curve, although there exist polynomial DPH curves of degree 7 and higher that are not helical.

Lemma 5.3. *If a polynomial PH curve $\boldsymbol{\alpha}(\xi)$ is a general helix satisfying* (2)*, the triple product $\langle\, \boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi), \boldsymbol{\alpha}'''(\xi)\,\rangle$ must be proportional to the cube of a polynomial $\omega(\xi)$.*

Proof : Since every helical polynomial PH curve $\boldsymbol{\alpha}(\xi)$ is a polynomial DPH curve, the polynomial (42) that appears in equation (41) must be of the form (43) for some polynomial $\omega(\xi)$. Thus, $\boldsymbol{\alpha}(\xi)$ has a torsion/curvature ratio of the form

$$\frac{\tau(\xi)}{\kappa(\xi)} = \frac{\langle\, \boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi), \boldsymbol{\alpha}'''(\xi)\,\rangle}{\omega^3(\xi)}.$$

This is constant only if the numerator and denominator are proportional. ∎

Lemma 5.4. *If a polynomial/rational curve $\boldsymbol{\alpha}(\xi)$ is a slant helix, it must be a polynomial/rational DPH curve.*

Proof : Since the principal normal to $\boldsymbol{\alpha}(\xi)$ is defined by

$$\mathbf{n}(\xi) = \frac{\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)}{|\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|} \times \frac{\boldsymbol{\alpha}'(\xi)}{|\boldsymbol{\alpha}'(\xi)|},$$

the slant helix condition $\langle \mathbf{n}(\xi), \mathbf{u} \rangle = \cos\phi$ reduces to

$$\langle\, [\, \boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)\,] \times \boldsymbol{\alpha}'(\xi), \mathbf{u}\,\rangle = \cos\phi\, |\boldsymbol{\alpha}'(\xi)|\, |\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|.$$

Again, the left–hand side of this equation is a polynomial/rational function if $\boldsymbol{\alpha}(\xi)$ is a polynomial/rational curve, so it can only be satisfied when $|\boldsymbol{\alpha}'(\xi)|$ and $|\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|$ are *both* polynomial/rational functions — i.e., when $\boldsymbol{\alpha}(\xi)$ is a polynomial/rational DPH curve. ∎

Lemma 5.4 has been noted by Monterde [20]. Since the Salkowski curves — with constant curvature and non–constant torsion — discussed in [20] are rational slant helices, they are also rational DPH curves. Furthermore, as a corollary to Theorem 3.3 and Lemma 5.4, we deduce the following.

Corollary 5.1. *The centrode $\boldsymbol{\omega}(\xi) = \tau(\xi)\mathbf{t}(\xi) + \kappa(\xi)\mathbf{b}(\xi)$ of a Salkowski curve $\boldsymbol{\alpha}(\xi) : I \to E^3$ is a rational DPH curve.*

Finally, we consider how rectifying curves fit in the context of PH curves.

Lemma 5.5. *If a polynomial curve $\boldsymbol{\alpha}(\xi)$ is a rectifying curve, it must be a DPH curve.*

Proof : It is shown in [4] that a rectifying curve must be expressible in terms of its arc length $s(\xi)$, tangent $\mathbf{t}(\xi)$ and binormal $\mathbf{b}(\xi)$, and constants p, q as

$$\boldsymbol{\alpha}(\xi) = (s(\xi) + p)\, \mathbf{t}(\xi) + q\, \mathbf{b}(\xi)\,.$$

Substituting for $\mathbf{t}(\xi)$, $\mathbf{b}(\xi)$ and clearing denominators, this is equivalent to

$$|\boldsymbol{\alpha}'(\xi)|\,|\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|\,\boldsymbol{\alpha}(\xi) \;=\; |\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|\,(s(\xi)+p)\,\boldsymbol{\alpha}'(\xi) + \; q\,|\boldsymbol{\alpha}'(\xi)|\,\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)\,.$$

For a polynomial curve $\boldsymbol{\alpha}(\xi)$ to satisfy this condition, $|\boldsymbol{\alpha}'(\xi)|$, $|\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)|$, and $s(\xi)$ must be polynomials. These are precisely the defining properties of a polynomial DPH curve. Specifically, substituting from (37) and (40)–(43) we obtain the polynomial condition

$$\sigma(\xi)\,\omega(\xi)\,\boldsymbol{\alpha}(\xi) = \omega(\xi)\,(s(\xi)+p)\,\boldsymbol{\alpha}'(\xi) + q\,\boldsymbol{\alpha}'(\xi)\times\boldsymbol{\alpha}''(\xi)\,.$$

If the quaternion polynomial (38) is of degree m, the expression on the left and first term on the right of this equation are of equal degree $6m-1$, while the second term on the right is of of degree $4m-1$. ∎

The degree considerations in the preceding proof show that the existence of polynomial DPH rectifying curves is not *prima facie* impossible, although actually constructng them and identifying their simplest instances is a non–trivial task, which we do not attempt at present. Similar considerations apply to the study rational DPH rectifying curves, with the additional complication that such curves do not, in general, have rational arc length functions.

Acknowledgement

This work is supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

References

[1] J. V. Beltran and J. Monterde, A characterization of quintic helices, *J. Comput. Appl. Math.* 206 (1) (2007) 116–121.
[2] S. Breuer and D. Gottlieb, Explicit characterizations of spherical curves, *Proc. Amer. Math. Soc.* 27 (1971) 126–127.
[3] C. Cao, S. T. Xu, and L. C. Wang, An algorithm for protein helix assignment using helix geometry, *PLoS One* 10 (7), article e0129674 (2015).
[4] B–Y. Chen, When does the position vector of a space curve always lie in its rectifying plane?, *Amer. Math. Monthly* 110 (2003) 147–152.
[5] B–Y. Chen and F. Dillen, Rectifying curves as centrodes and extremal curves, *Bull. Inst. Math. Academia Sinica* 33 (2) (2005)77–90 .
[6] F. F. Chen, *Introduction to Plasma Physics*, Springer, New York 1974.
[7] S. Deshmukh, B–Y. Chen, and S. H. Alshammari, On rectifying curves in Euclidean 3–space, *Turk. J. Math.*, 42 (2018) 609–620.
[8] R. T. Farouki, *Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable*, Springer, Berlin 2008.
[9] R. T. Farouki, C. Giannelli, and A. Sestini, Helical polynomial curves and double Pythagorean hodographs I. Quaternion and Hopf map representations, *J. Symb. Comput.* 44 (2) (2009) 161–179.
[10] R. T. Farouki, C. Giannelli, and A. Sestini, Helical polynomial curves and double Pythagorean hodographs II. Enumeration of low-degree curves, *J. Symb. Comput.* 44 (2009) 307–332.
[11] R. T. Farouki, C. Y. Han, C. Manni, and A. Sestini, Characterization and construction of helical polynomial space curves, *J. Comput. Appl. Math.* 162 (2004) 365–392.
[12] R. T. Farouki and T. Sakkalis, Rational space curves are not "unit speed," *Comput. Aided Geom. Design* 24 (2007) 238–240.
[13] S. Gertzbein, J. Seligman, R. Holtby, K. Chan, N. Ogston, A. Kapasouri, M. Tile, and B. Cruickshank, Centrode patterns and segmental instability in degenerative disk disease, *Spine* 10 (1985) 257–261.
[14] S. Izumiya and N. Takeuchi, New special curves and developable surfaces, *Turk. J. Math.* 28 (2004) 153–163.
[15] D–S. Kim, H–S. Chung, and K–H. Cho, Space curves satisfying $\tau/\kappa = a\,s+b$, *Honam Math. J.* 1 (1993) 5–9.
[16] D. Laugwitz, *Differential and Riemannian Geometry*, Academic Press, New York 1965.
[17] G. H. Martin, *Kinematics and Dynamics of Machines*, 2nd edition, McGraw–Hill, New York 1982.
[18] J. McCoy, Helices for mathematical modelling of proteins, nucleic acids and polymers, *J. Math. Anal. Appl.* 347 (2008) 255–265.
[19] R. S. Millman and G. D. Parker, *Elements of Differential Geometry*, Prentice-Hall, New Jersey 1977.
[20] J. Monterde, Salkowski curves revisited: a family of curves with constant curvature and non-constant torsion, *Comput. Aided Geom. Design* 26 (2009) 271–278.
[21] N. R. Pereira, Scaling invariance of helical curve motion and soliton equations, *J. Math. Phys.* 19 (4) (1978) 898–900.
[22] E. Salkowski, Zur Transformationen von Raumkurven, *Math. Annalen* 66 (4) (1909) 517–557.
[23] J. E. Shigley and J. J. Uicker, Jr., *Theory of Machines and Mechanisms*, McGraw–Hill, New York 1980.
[24] P. H. Strubecker, *Differentialgeometrie I. Kurventheorie der Ebene und des Raumes*, Walter de Gruyter, Berlin 1964.
[25] D. J. Struik, *Lectures on Classical Differential Geometry*, Dover (reprint), New York 1950.
[26] J. D. Watson and F. H. Crick, Molecular structures of nucleic acids, *Nature* 171 (1953) 737–738.
[27] P. J. Weiler and E. R. Bogoch, Kinematics of the distal radioulnar joint in rheumatoid arthritis: an *in vivo* study using centrode analysis, *J. Hand Surgery* 20A (6) (1995) 937–943.
[28] Y. Wong, A global formulation of the condition for a curve to lie in a sphere, *Monatshefte Math.* 67 (1963) 363–365.
[29] Y. Wong, On an explicit characterization of spherical curves, *Proc. Amer. Math. Soc.* 34 (1972) 239–242.
[30] H. Yeh and J. I. Abrams, *Principles of Mechanics of Solids and Fluids*, Vol. 1, McGraw-Hall, New York 1960.