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A Measure of Inaccuracy in Concomitants of Ordered Random Variables under Farlie-Gumbel-Morgenstern Family

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Abstract. In communication theory, for possible outcomes of an experiment, we have two basic problems for the statement of the experimenter: we may not have enough information (vague statement) or some of the information may be incorrect, which make inaccurate in either or both of these situations. In this article, a measure of inaccuracy and its residual between distributions of concomitants of generalized order statistics (*gos*) and parent random variable are extended. Results of inaccuracy for family distributions and stochastic comparisons are obtained. Furthermore, some properties of the proposed measure are discussed. The unique characterization of the distribution function of parent random variable by the inaccuracy is shown.

1. Introduction

The Farlie-Gumbel-Morgenstern (*FGM*) family is a highly flexible class of bivariate distributions, it was originally introduced by Morgenstern [21] for Cauchy marginal distribution. This structure was investigated by Gumbel [9] for exponential marginal distribution and further was generalized by Farlie [7]. Accordingly, in the present study we deal with the distribution theory and applications of concomitants of the *FGM* family of bivariate distributions, which is specified by the Cumulative distribution function (cdf) and Probability density function (pdf), respectively, as follows:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))], \tag{1}$$

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \\ &= f_X(x)f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)], \end{aligned} \tag{2}$$

where $-1 \le \alpha \le 1$, $f_X(x)$, $f_Y(y)$, and $F_X(x)$, $F_Y(y)$ are the marginal pdf's and cdf's of X and Y respectively. The association parameter α is known as the dependence parameter of the random variables X and Y. If α is zero, then X and Y are independent. This system provides a general expression of bivariate distribution, as this model depends on the marginal distributions. Since both the bivariate cdf and pdf are given in terms of

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marginal distributions, it is easy to generate a random sample from the *FGM* distribution. Thus, members of this family can be used in simulation studies, especially when weak dependence between variates is of interest. It follows that the conditional density of Y given $X = x$ is given by:

$$\begin{aligned} f_{Y|X}(y \mid x) &= \frac{f_{X,Y}(x, y)}{f_X(x)} \\ &= f_Y(y)[1 + \alpha(2F_X(x) - 1)(2F_Y(y) - 1)], -1 \le \alpha \le 1. \end{aligned} \tag{3}$$

The concept of *gos* is a general model which contains all types of ordered observations such as order statistics, sequential order statistics and *kth* record values as a special cases of *gos*. It was introduced by Kamps [13] as follow: let $n \in \mathbb{N}, k \ge 1, m_1, ..., m_{n-1} \in \mathbb{R}, M_r = \sum_{j=r}^{n-1} m_j, 1 \le r \le n-1$, be parameters such that $\gamma_r = k+n-r+M_r \ge 1$ for all $r \in \{1, 2, ..., n-1\}$, and let $\tilde{m} = (m_1, ..., m_{n-1}) \in \mathbb{R}^{n-1}$. If $m_1 = m_2 = \ldots = m_{n-1} = m$, the pdf of $X_{(r,n,m,k)}$ can be written as:

$$f_{(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1 - F(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{4}$$

where $c_{r-1} = \prod_{j=1}^{r} \gamma_j$, $g_m(z) = h_m(z) - h_m(0)$, $0 < z < 1$,

$$h_m(z) = \begin{cases} \frac{-(1-z)^{m+1}}{m+1}, & m \ne -1, \\ \\ -\ln(1-z), & m = -1. \end{cases}$$

Originally, David et al. [3] studied the concomitants of order statistics. For some bivariate population with cdf $F(x, y)$, let (X_i, Y_i), $i = 1, 2, ..., n$, be n pairs of independent random variables. Let $X_{(r;n)}$ be the *rth* order statistics, then Y associated with $X_{(r;n)}$ is called the concomitant of *rth* order statistics and is denoted by $Y_{[r;n]}$. The pdf and cdf of $Y_{[r;n]}$ are given by:

$$g_{[r;n]}(y) = g_{Y_{[r;n]}}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y \mid x) f_{(r;n)}(x) dx, \tag{5}$$

$$G_{[r;n]}(y) = \int_{-\infty}^{\infty} F_{Y|X}(y \mid x) f_{(r;n)}(x) dx, \tag{6}$$

where $f_{(r;n)}(x)$ is the pdf of $X_{(r;n)}$. The double sampling can give an application of concomitants, as we can investigate for example a group of patients, such that their weights are ordered before they response to a treatment ($X_{(i;n)}$'s), then record their weights after obtaining the treatment ($Y_{[i;n]}$'s). We can see that $Y_{[i;n]}$'s need not have a similar order, therefore, $Y_{[i;n]}$'s are concomitants of the order statistics $X_{(i;n)}$'s.

The concept of entropy was introduced by Shannon [28] in the information theory literature. The Shannon entropy (uncertainty) of a continuous random variable X_1 measures the average reduction of uncertainty of X_1. The Shannon entropy for a non negative random variable X_1 with pdf $f_X(x) = f_1(x)$ is defined as:

$$H(X_1) = H(f_1) = -\int_0^{\infty} f_1(x) \ln f_1(x) dx. \tag{7}$$

Divergence measures are used to quantify the dissimilarity of two probability distributions. They are equal to zero if and only if the distributions are the same. An important and well-known divergence measure was introduced by Kullback and Leibler [18]. The Kullback-Leibler divergence (information divergence) for two non negative continuous random variables X_1 and X_2 with pdf's f_1 and f_2, respectively, is given by:

$$K(X_1, X_2) = K(f_1, f_2) = \int_0^{\infty} f_1(x) \ln\left(\frac{f_1(x)}{f_2(x)}\right) dx, \tag{8}$$

$K(X_1, X_2)$ is non negative, invariant under one-to-one transformation of (X_1, X_2) and it is not symmetric. Adding (7) and (8), we get:

$$\begin{aligned} H(f_1) + K(f_1, f_2) &= -\int_0^{\infty} f_1(x) \ln f_2(x) dx \\ &= I(f_1, f_2), \end{aligned} \tag{9}$$

which is Kerridge measure of inaccuracy associated with random variables X_1 and X_2 as an expansion (generalization) of uncertainty, see Kerridge [17]. If we consider F_1 as the actual distribution function then F_2 can be interpreted as reference distribution function. In survival analysis and life testing, the current age of the system under consideration is also taken into account. Thus, for calculating the remaining uncertainty of a system which has survived up to time t, the measures defined in (7), (8) and (9) are not suitable. Ebrahimi [5] considered a random variable $X_t = (X - t)|X > t, t \geq 0$ and defined uncertainty of such a system, given by:

$$H(f_1; t) = -\int_t^{\infty} \left(\frac{f_1(x)}{\overline{F}_1(t)}\right) \ln\left(\frac{f_1(x)}{\overline{F}_1(t)}\right) dx, \tag{10}$$

where $\overline{F}_1(t) = 1 - F_1(t)$ is survival function. Clearly when $t = 0$, (10) reduce to (7). Taneja et al. [24] defined the dynamic measure of inaccuracy associated with two residual lifetime distributions F_1 and F_2 corresponding to the Kerridge measure of inaccuracy given by:

$$I(f_1, f_2; t) = -\int_t^{\infty} \left(\frac{f_1(x)}{\overline{F}_1(t)}\right) \ln\left(\frac{f_2(x)}{\overline{F}_2(t)}\right) dx, \tag{11}$$

where $\overline{F}_2(t) = 1 - F_2(t)$ is survival function. Clearly for $t = 0$, (11) reduces to (9).

In this paper, we propose the measure of inaccuracy and residual inaccuracy of concomitants of *gos*. There are many articles and several books published on concomitants of order statistics, but not much attention has been given to the study of inaccuracy properties for concomitants of *gos*. Several authors have worked on information theoretic aspects of order statistics, for details refer to Ebrahimi et al. [6] and Zarezadeh and Asadi [29]. Recently, Thapliyal and Taneja [25] have introduced the concept of inaccuracy using order statistics. They have proposed the measure of inaccuracy between the *rth* order statistics and the parent random variable and proved a characterization result for it. Thapliyal and Taneja [26] have proposed the measure of residual inaccuracy of order statistics and prove a characterization result for it. For a further view of the literature survey on inaccuracy measure see Kundu and Nanda [19], Kayal et al. [14], Psarrakos and Di Crescenzo [22], Kayal et al. [16], Kayal and Sunoj [15] and Di Crescenzo et al. [4]. The rest of this dissertation is organized as follows: Section 2 derives the inaccuracy and its dynamic residual of the distribution of the rth concomitants of *gos* and $f(y)$ and vice versa. Moreover, studies some properties of a special cases of *gos*. Section 3 presents some results of inaccuracy for some specific distributions. Also, achieves the upper bound of the residual inaccuracy. Besides, considering some characterization results.

2. A measure of inaccuracy and residual inaccuracy of concomitants of *gos*

In this section, we use *gos* to obtain the inaccuracy of concomitants of *FGM* distributions. Under the *FGM* family, the cdf and pdf of the concomitant of *gos* $Y_{[r,n,m,k]}$, $1 \leq r \leq n$, is given by Beg and Ahsanullah [1], respectively, as follows:

$$\begin{aligned} G_{[r,n,m,k]}(y) &= \int_0^\infty F_{Y|X}(y \mid x) f_{(r,n,m,k)}(x)dx \\ &= F_Y(y)\left[1 - \alpha D^*(r,n,m,k)(1 - F_Y(y))\right], \end{aligned} \tag{12}$$

$$g_{[r,n,m,k]}(y) = f_Y(y)\left[1 + \alpha D^*(r,n,m,k)(2F_Y(y) - 1)\right], \tag{13}$$

where $f_{(r,n,m,k)}(x)$ is the pdf of *gos* $X_{[r,n,m,k]}$ defined in (4), $D^*(r,n,m,k) = 1 - \frac{2\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r(\gamma_i+1)}$, with parameters $n \in \mathbb{N}$, $k \geq 1$, $m \in \mathbb{R}$, such that $\gamma_r = k + (n-r)(m+1) \geq 1$, for all $1 \leq r \leq n$. Tahmasebi and Behboodian [23] introduced the Shannon entropy for concomitants of *gos* of *FGM* distributions by the following theorem:

Theorem 2.1. *If $Y_{[r,n,m,k]}$ is the concomitant of rth gos, then, from (7) and (13), the Shannon entropy of $Y_{[r,n,m,k]}$ for $1 \leq r \leq n$, $\alpha \neq 0$, $-1 \leq \alpha \leq 1$ is given by:*

$$\begin{aligned} H(Y_{[r,n,m,k]}) &= W(r,\alpha,n,m,k) + H(Y)(1 - \alpha D^*(r,n,m,k)) \\ &\quad - 2\alpha D^*(r,n,m,k)\phi_f(y), \end{aligned} \tag{14}$$

where

$$\begin{aligned} W(r,\alpha,n,m,k) = \frac{1}{4\alpha D^*(r,n,m,k)} &\Big[(1 - \alpha D^*(r,n,m,k))^2 \\ &\times \ln(1 - \alpha D^*(r,n,m,k)) - (1 + \alpha D^*(r,n,m,k))^2 \\ &\times \ln(1 + \alpha D^*(r,n,m,k))\Big] + \frac{1}{2}, \end{aligned} \tag{15}$$

$$\phi_f(y) = \int_0^\infty F_Y(y) f_Y(y) \ln f_Y(y) \mathrm{d}y. \tag{16}$$

A measure of inaccuracy associated with distribution of *rth* concomitant of *gos* and parent distribution function $f_Y(y)$, analogous to the Kerridge measure of inaccuracy between two density functions f_1 and f_2 given by (9), is as follows:

$$\begin{aligned} I_n(g_{[r,n,m,k]}(y), f_Y(y)) &= -\int_0^\infty g_{[r,n,m,k]}(y) \ln f_Y(y)dy \\ &= -\int_0^\infty f_Y(y)\left[1 + \alpha D^*(r,n,m,k)(2F_Y(y) - 1)\right] \ln f_Y(y)dy \\ &= (1 - \alpha D^*(r,n,m,k))H(Y) - 2\alpha D^*(r,n,m,k)\phi_f(y). \end{aligned} \tag{17}$$

Clearly, from (14) and (17), the Kullback-Leibler divergence is given by

$$K(g_{[r,n,m,k]}(y), f_Y(y)) = W(r,\alpha,n,m,k). \tag{18}$$

A measure of inaccuracy associated with parent distribution function $f_Y(y)$ and distribution of *rth* concomitant of *gos* is as follows:

$$\begin{aligned} I_n(f_Y(y), g_{[r,n,m,k]}(y)) &= -\int_0^\infty f_Y(y)\ln g_{[r,n,m,k]}(y)dy \\ &= -\int_0^\infty f_Y(y)\ln f_Y(y)\left[1+\alpha D^*(r,n,m,k)(2F_Y(y)-1)\right]dy \\ &= 1+H(Y)-\frac{(1+\alpha D^*(r,n,m,k))}{2\alpha D^*(r,n,m,k)}\ln(1+\alpha D^*(r,n,m,k)) \\ &\quad +\frac{(1-\alpha D^*(r,n,m,k))}{2\alpha D^*(r,n,m,k)}\ln(1-\alpha D^*(r,n,m,k)). \end{aligned} \tag{19}$$

Remark 2.1. *Under order statistics (with $m = 0$ and $k = 1$) and record value (with $m = -1$ and $k = 1$) as a special cases of gos, denote I_n^{os} and I_n^{rv} the inaccuracy of order statistics and record value, respectively, we get:*

1. *If n is odd and $r = \frac{n+1}{2}$, or $\alpha = 0$, then we have $D^*(r,n,m,k) = 0$, from (17) and (19), we have $I_n^{os}(g_{[r,n,m,k]}(y), f_Y(y)) = H(Y)$ and $I_n^{os}(f_Y(y), g_{[r,n,m,k]}(y)) = H(Y)$, respectively.*
2. *If n is even, $r = \frac{n}{2}+1$ and n replaced with $n+1$, or $\alpha = 0$, then we have $D^*(r,n+1,m,k) = 0$, from (17) and (19), we have $I_n^{os}(g_{[r,n+1,m,k]}(y), f_Y(y)) = H(Y)$ and $I_n^{os}(f_Y(y), g_{[r,n+1,m,k]}(y)) = H(Y)$, respectively..*
3. *If $\lambda \geq 1$ is an integer number and we change r to $r\lambda$ and n to $(n+1)\lambda-1$, then from (17) and (19), we have $I_n^{os}(g_{[r,n,m,k]}(y), f_Y(y)) = I_n^{os}(g_{[r\lambda,(n+1)\lambda-1,m,k]}(y), f_Y(y))$ and $I_n^{os}(f_Y(y), g_{[r,n,m,k]}(y)) = I_n^{os}(f_Y(y), g_{[r\lambda,(n+1)\lambda-1,m,k]}(y))$, respectively.*
4. *If $r = n = 2^b-1$, then $I_n^{os}(g_{[2^b-1,2^b-1,m,k]}(y), f_Y(y)) = I_n^{rv}(g_{[b,b,m,k]}(y), f_Y(y))$.*

Analogous to (11), the dynamic residual measure of inaccuracy associated with two residual lifetime distributions $G_{[r,n,m,k]}(y)$ and $F_Y(y)$, respectively, is given by:

$$\begin{aligned} I_n(g_{[r,n,m,k]}(y), f_Y(y);t) &= -\int_t^\infty \left(\frac{g_{[r,n,m,k]}(y)}{\overline{G}_{[r,n,m,k]}(t)}\right)\ln\left(\frac{f_Y(y)}{\overline{F}_Y(t)}\right)dy \\ &= \ln\overline{F}_Y(t)-\frac{1}{\overline{G}_{[r,n,m,k]}(t)}\int_t^\infty g_{[r,n,m,k]}(y)\ln f_Y(y)dy \\ &= \ln\overline{F}_Y(t)-\frac{1}{\overline{G}_{[r,n,m,k]}(t)}\int_t^\infty f_Y(y)[1+\alpha D^*(r,n,m,k) \\ &\quad\times(2F_Y(y)-1)]\ln f_Y(y)dy \\ &= \ln\overline{F}_Y(t)-\frac{1}{\overline{G}_{[r,n,m,k]}(t)}[(1-\alpha D^*(r,n,m,k))A_f(Y;t) \\ &\quad+2\alpha D^*(r,n,m,k)\phi_f(y;t)], \end{aligned} \tag{20}$$

where $\phi_f(y;t) = \int_t^\infty F_Y(y)f_Y(y)\ln f_Y(y)\mathrm{d}y$, $A_f(Y;t) = \int_t^\infty f_Y(y)\ln f_Y(y)\mathrm{d}y$.

Proposition 2.1. *Let $Q = f_Y(q) < \infty$, where $q = \sup\{y : f_Y(y) \leq M\}$, M is the mode of the distribution then*

$$I_n(g_{[r,n,m,k]}(y), f_Y(y);t) \geq \ln\overline{F}_Y(t)-\ln M. \tag{21}$$

Proof. As q is the mode of the distribution, hence $\ln f_Y(y) \leq \ln Q$. Using this fact in (20) we get our result.

The dynamic residual measure of inaccuracy associated with two residual lifetime distributions $F_Y(y)$

and $G_{[r,n,m,k]}(y)$, respectively, is given by:

$$\begin{aligned}
I_n(f_Y(y), g_{[r,n,m,k]}(y); t) &= -\int_t^{\infty} \left(\frac{f_Y(y)}{\overline{F}_Y(t)}\right) \ln\left(\frac{g_{[r,n,m,k]}(y)}{\overline{G}_{[r,n,m,k]}(t)}\right) dy \\
&= \ln \overline{G}_{[r,n,m,k]}(t) - \frac{1}{\overline{F}_Y(t)} \int_t^{\infty} f_Y(y) \ln g_{[r,n,m,k]}(y) dy \\
&= \ln \overline{G}_{[r,n,m,k]}(t) - \frac{1}{\overline{F}_Y(t)} [\frac{(1-\alpha D^*(r,n,m,k))}{2\alpha D^*(r,n,m,k)} \ln(1+\alpha D^*(r,n,m,k)) \\
&\quad + \ln(1+\alpha D^*(r,n,m,k)) - A_f(Y;t) - \overline{F}_Y(t) \\
&\quad - \frac{(1-\alpha D^*(r,n,m,k))}{2\alpha D^*(r,n,m,k)} \ln((1-\alpha D^*(r,n,m,k)) + 2\alpha D^*(r,n,m,k) F_Y(t)) \\
&\quad + (1-\alpha D^*(r,n,m,k)) - F_Y(t) \ln(2\alpha D^*(r,n,m,k) F_Y(t))].
\end{aligned} \tag{22}$$

3. Results of inaccuracy for a family distribution and applications

We consider the scale family F of distributions with distribution function F of the form

$$F_Y(y) = 1 - \exp(-\eta g(y)), y \geq g^{-1}(0), \eta > 0, \tag{23}$$

where g is assumed to be differentiable on $(g^{-1}(0), \infty)$ and strictly increasing, with $\lim_{y\to\infty} g(y) = \infty$. The family F includes the Pareto and exponential distributions and Weibull distributions; for further details, see Cramer and Kamps [2].

Next we want to prove an important property of inaccuracy measure using some properties of stochastic ordering. For that we present the following definitions:

1. A random variable X is said to be less than Y in likelihood ratio ordering (denoted by $X \leq^{lr} Y$) if $\frac{f_X(x)}{g_Y(x)}$ is non increasing in x.
2. A random variable X is said to be less than Y in the stochastic ordering (denoted by $X \leq^{st} Y$) if $\overline{F}_X(x) \leq \overline{G}_Y(x)$ for all x, where $\overline{F}_X(x)$ and $\overline{G}_Y(x)$ are the survival functions of X and Y respectively.

In the following remark we will discuss the constant $D^*(r,n,m,k)$:

Remark 3.1. *Since $D^*(r,n,m,k) = 1 - \frac{2\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r (\gamma_i+1)}$, with parameters $n \in \mathbb{N}$, $k \geq 1$, $m \in \mathbb{R}$, such that $\gamma_r = k + (n-r)(m+1) \geq 1$, for all $1 \leq r \leq n$, then we can note the following:*

1. $-1 \leq D^*(r,n,m,k) < 1$ *as* $0 < \frac{\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r (\gamma_i+1)} < 1$, $0 < \frac{2\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r (\gamma_i+1)} \leq 2$.
2. $D^*(r,n,m,k)$ *is positive (negative) if* $0 < \frac{2\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r (\gamma_i+1)} < 1$ $(1 < \frac{2\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r (\gamma_i+1)} \leq 2)$.

$D^*(r,n,m,k)$ *could be zero if* $\frac{2\prod_{j=1}^r \gamma_j}{\prod_{i=1}^r (\gamma_i+1)} = 1$, *which can be hold in many cases, for example:*

1. $m \in \mathbb{R}$, $n = r = k = 1$ *then* $\gamma_r = 1$.
2. $m = -1$, $n \in \mathbb{N}$, $r = k = 1$.
3. n *is even*, $r = \frac{n}{2} + 1$, $m = 0$, $k = 2$.
4. n *is odd*, $r = \frac{n+1}{2}$, $m = 0$, $k = 1$.

Theorem 3.1. *Let $Y_1, Y_2, \ldots, Y_n$ be n concomitant of gos. Let F_Y denote their distribution function defined in (23) with $\eta \geq 1$ is an integer number. If the following assumptions are satisfied:*

1. *f_Y is decreasing in its support.*
2. *from Remark (3.1), $0 < \alpha \leq 1, -1 \leq D^*(r,n,m,k) < 0$ (or $-1 \leq \alpha < 0, 0 < D^*(r,n,m,k) < 1$).*
3. *$0 < 2F_Y(y) - 1 < 1$,*

then the corresponding inaccuracy defined in (20) is decreasing function of n.

Proof. As f_Y is decreases in its support, moreover, the product $\alpha D^*(r,n,m,k)$ is negative and $0 < 2F_Y(y) - 1 < 1$, hence

$$\frac{g_{[r,n+1,m,k]}(y)}{g_{[r,n,m,k]}(y)} = \frac{[1 + \alpha D^*(r,n+1,m,k)(2F_Y(y) - 1)]}{[1 + \alpha D^*(r,n,m,k)(2F_Y(y) - 1)]}, t \leq y < \infty, \tag{24}$$

is a decreasing function. This implies that $Y_{[r,n+1,m,k]} \leq^{lr} Y_{[r,n,m,k]}$ which implies $Y_{[r,n+1,m,k]} \leq^{st} Y_{[r,n,m,k]}$, refer to Shaked and Shanthikumar [27]. Also, from the family defined in (23) with $\eta \geq 1$ is an integer number, we note that $\phi_f(y;t) = \int_t^\infty F_Y(y) f_Y(y) \ln f_Y(y) dy$, $A_f(Y;t) = \int_t^\infty f_Y(y) \ln f_Y(y) dy$ are decreasing functions of η. Also from (17) and (20), the inaccuracy and the residual inaccuracy of the concomitant of *gos* are given by, respectively

$$I_n(g_{[r,n,m,k]}(y), f_Y(y)) = (1 - \alpha D^*(r,n,m,k))H(Y) - 2\alpha D^*(r,n,m,k)\phi_f(y),$$

and

$$\begin{aligned} I_n(g_{[r,n,m,k]}(y), f_Y(y); t) = \ln \overline{F}_Y(t) - \frac{1}{\overline{G}_{[r,n,m,k]}(t)}&[(1 - \alpha D^*(r,n,m,k))A_f(Y;t) \\ &+ 2\alpha D^*(r,n,m,k)\phi_f(y;t)], \end{aligned}$$

then, we have

$$I_n(g_{[r,n,m,k]}(y), f_Y(y)) - I_n(g_{[r,n+1,m,k]}(y), f_Y(y)) \geq 0,$$

and

$$I_n(g_{[r,n,m,k]}(y), f_Y(y); t) - I_n(g_{[r,n+1,m,k]}(y), f_Y(y); t) \geq 0.$$

This completes the proof.

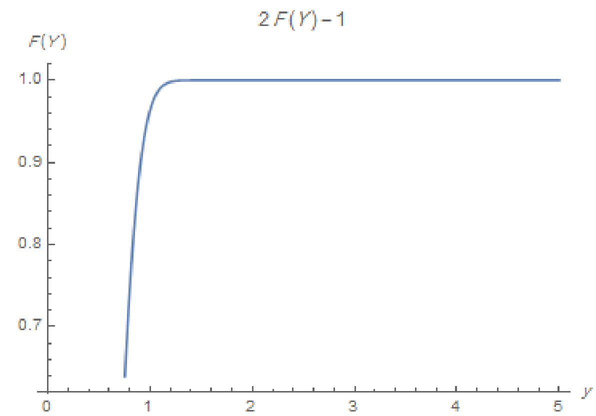

Figure 1: $2F_Y(y) - 1$.

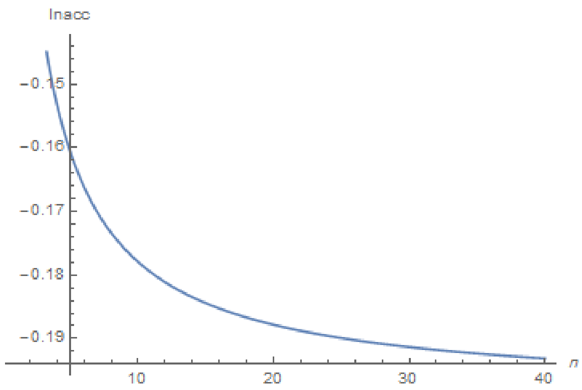

Figure 2: Inaccuracy of Weibull distribution.

In Figure 1, we plot $2F_Y(y) - 1$ for Weibull distribution with $\lambda = 4$, $c = 3$ which satisfy the assumption $0 < 2F_Y(y) - 1 \leq 1$. In Figure 2, we plot the inaccuracy of the distribution in Figure 1 of order statistics for $n = 1, 2, ..., 40$.

In the following subsections, we will apply the previous results to some subfamilies of the family defined in (23) when the they are Weibull and Pareto, and obtain its inaccuracy.

Remark 3.2. *In the computation of the following subsections, we use some important formulas as follows, see Golomb [8] and Jain and Srivastava [12]:*

1. *Let* $T(t) = \int_{-\infty}^{\infty} [f(x)]^t dx$, *then* $[-\frac{\partial T(t)}{\partial t}]_{t=1} = H(X)$, $t \geq 1$.
2. *Let* $A(X) = \int_{-\infty}^{\infty} u(x) f(x) \ln f(x) dx$, $U(t) = \int_{-\infty}^{\infty} u(x)[f(x)]^t dx$, *then* $[\frac{\partial U(t)}{\partial t}]_{t=1} = A(X)$, $t \geq 1$.

3.1. *Weibull distribution*

With the cdf of Weibull distribution:

$$F_Y(y) = 1 - e^{-\lambda\, y^c}, \; y \geq 0, \; c, \lambda > 0, \tag{25}$$

Theorem 3.2. *If* $Y_{[r,n,m,k]}$ *is the concomitant of rth gos for Weibull distribution from (25) then, from (17), the inaccuracy of* $Y_{[r,n,m,k]}$ *for* $1 \leq r \leq n$, $\alpha \neq 0$, $-1 \leq \alpha \leq 1$ *is given by:*

$$\begin{aligned} I_n(g_{[r,n,m,k]}(y), f_Y(y)) = \frac{-1}{2c} & [2\nu - c(2 + 2\nu + \alpha D^*(r,n,m,k)) + (c-1)\alpha D^*(r,n,m,k) \ln 4 \\ & +2c \ln c + 2 \ln \lambda] \, , \end{aligned} \tag{26}$$

where $\nu = -\Gamma'(1)$ *is the Euler's constant.*

Proof. From (17) and (25), we have:

$$\begin{aligned} I_n(g_{[r,n,m,k]}(y), f_Y(y)) &= (1 - \alpha D^*(r,n,m,k))H(Y) - 2\alpha D^*(r,n,m,k)\phi_f(y) \\ &= -(1 - \alpha D^*(r,n,m,k)) \int_0^{\infty} f_Y(y) \ln f_Y(y) \mathrm{d}y \\ &\quad - 2\alpha D^*(r,n,m,k) \int_0^{\infty} F_Y(y) f_Y(y) \ln f_Y(y) \mathrm{d}y \\ &= -(1 - \alpha D^*(r,n,m,k))\, A_1 - 2\alpha D^*(r,n,m,k)\, A_2. \end{aligned} \tag{27}$$

To find

$$A_1 = -H(Y) = \int_0^\infty c\lambda y^{c-1}\exp(-\lambda y^c)(\ln c\lambda)y^{c-1}\exp(-\lambda y^c)\mathrm{d}y,$$

first, we want to obtain

$$T(t) = \int_0^\infty [f(y)]^t \mathrm{d}y = \int_0^\infty (c\lambda)^t y^{t(c-1)}\exp(-t\lambda y^c)\mathrm{d}y.$$

Let $y^c = v \Rightarrow \mathrm{d}y = c^{-1}v^{\frac{1}{c}-1}\mathrm{d}v$, then

$$\begin{aligned} T(t) &= c^{t-1}\int_0^\infty v^{\frac{t(c-1)+1}{c}-1}\exp(-tv)\mathrm{d}v \\ &= c^{t-1}\lambda^t(t\lambda)^{-(\frac{t(c-1)+1}{c})}\Gamma\left(\frac{t(c-1)+1}{c}\right). \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{\partial T(t)}{\partial t} = T'(t) = {} & Qc^{t-1}\lambda^t(\ln c\lambda)\Gamma\left(\frac{t(c-1)+1}{c}\right) + Q'c^{t-1}\lambda^t \\ & \times\Gamma\left(\frac{t(c-1)+1}{c}\right) + Qc^{t-1}\lambda^t(\frac{c-1}{c})\Gamma'\left(\frac{t(c-1)+1}{c}\right), \end{aligned}$$

where $Q = (t\lambda)^{-(\frac{t(c-1)+1}{c})}$, $Q' = Q(-(\frac{t(c-1)+1}{tc}) - (\frac{c-1}{c})\ln t\lambda)$.

$$\Longrightarrow T'(1) = A_1 = \ln c\lambda - \frac{c-1}{c}\nu + \frac{c-1}{c}\ln\lambda - 1. \tag{28}$$

To find

$$A_2 = A_1 - \int_0^\infty c\lambda y^{c-1}\exp(-\lambda y^c)(\ln c\lambda)y^{c-1}\exp(-\lambda y^c)\mathrm{d}y,$$

first, we want to obtain

$$\begin{aligned} U(t) &= \int_0^\infty \exp(-\lambda y^c)[f(y)]^t\mathrm{d}y = \int_0^\infty (c\lambda)^t y^{t(c-1)}\exp(-\lambda(t+1)y^c)\mathrm{d}y \\ &= c^{t-1}\lambda^t(t+1)^{-(\frac{t(c-1)+1}{c})}\Gamma\left(\frac{t(c-1)+1}{c}\right). \end{aligned}$$

$$\Longrightarrow U'(1) = I - \frac{c-1}{2c}\nu - \frac{1}{2}(-\frac{1}{2} + \frac{1-c}{c}\ln(2\lambda)) + \frac{1}{2}\ln(c\lambda), \tag{29}$$

where $\nu = -\Gamma'(1) = 0.57722$ is the Euler's constant. By substituting (28) and (29) in (27) the result follows.

3.2. *Pareto distribution*

With the cdf of Pareto distribution:

$$F_Y(y) = \quad 1 - y^{-c},\ y \geq 1,\ c > 0, \tag{30}$$

Theorem 3.3. *If $Y_{[r,n,m,k]}$ is the concomitant of rth gos for Pareto distribution from (30) then, from (17), the inaccuracy of $Y_{[r,n,m,k]}$ for $1 \leq r \leq n$, $\alpha \neq 0$, $-1 \leq \alpha \leq 1$ is given by:*

$$I_n(g_{[r,n,m,k]}(y), f_Y(y)) = \frac{1+c}{2c}(2 + \alpha D^*(r,n,m,k)) - \ln c. \tag{31}$$

Proof. The proof is similar to the previous theorem.

In the following, we will describe the upper bound limit between the residual inaccuracy and the entropy.

3.3. Upper bound for residual inaccuracy

The lower and upper bounds are very important to determine the lowest and highest limits, which are used in many applications in information theory. Moreover, to know where the residual inaccuracy lies between, thereby making us capable to prevent or avoid it. We derive the upper bound for the residual inaccuracy under the condition that the pdf for $Y_{[r,n,m,k]}$ is less than 1. Note that

$$\begin{aligned} I_n(g_{[r,n,m,k]}(y), f_Y(y); t) &= -\int_t^\infty \left(\frac{g_{[r,n,m,k]}(y)}{\overline{G}_{[r,n,m,k]}(t)}\right) \ln\left(\frac{f_Y(y)}{\overline{F}_Y(t)}\right) dy \\ &= \ln \overline{F}_Y(t) - \frac{1}{\overline{G}_{[r,n,m,k]}(t)} \int_t^\infty g_{[r,n,m,k]}(y) \ln f_Y(y) dy. \end{aligned}$$

We know that, for $t > 0$, $\ln \overline{F}_Y(t) \le 0$. Using this we get:

$$\begin{aligned} I_n(g_{[r,n,m,k]}(y), f_Y(y); t) &\le -\frac{1}{\overline{G}_{[r,n,m,k]}(t)} \int_t^\infty g_{[r,n,m,k]}(y) \ln f_Y(y) dy \\ &\le -\frac{1}{\overline{G}_{[r,n,m,k]}(t)} \int_0^\infty g_{[r,n,m,k]}(y) \ln f_Y(y) dy. \end{aligned}$$

Hence

$$I_n(g_{[r,n,m,k]}(y), f_Y(y); t) \le \frac{H(Y_{[r,n,m,k]})}{\overline{G}_{[r,n,m,k]}(t)}, \tag{32}$$

when $t \longrightarrow 0$ we get the equality, $H(Y_{[r,n,m,k]})$ is the Shannon entropy based on the concomitant of *gos*. This upper bound provides the best possible residual inaccuracy (as we try to avoid this inaccurate), which is widely used for accurately predicting the inaccuracy and for other information measures and properties. Noting that, writing the upper bound of the residual inaccuracy in terms of the shannon entropy make it quite informative, as the entropy is widely studied in many works of literature.

3.4. Some characterization results

Gupta et al. [10] studied characterizations of entropy of order statistics. Here, in this section, we present some characterization results based on inaccuracy of the concomitant of *gos*. Consider a problem of finding sufficient condition for the uniqueness of the solution of the initial value problem (IVP)

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0,$$

where f is a given function of two variables whose domain is a region $D \subset \mathbb{R}^2$, (x_0, y_0) is a specified point in D, y is an unknown function. By the solution of the IVP on an interval $I \subset \mathbb{R}$, we mean a function $\phi(x)$ such that (i) ϕ is differentiable on I, (ii) the graph of ϕ lies in D, (iii) $\phi(x_0) = y_0$ and (iv) $\phi'(x) = f(x, \phi(x))$, for all $x \in I$. The following theorem together with other results will help in proving our characterization result.

Theorem 3.4. *Let the function f be defined and continuous in a domain $D \subset \mathbb{R}^2$, and let f satisfy a Lipschitz condition (with respect to y) in D, namely*

$$|f(x, y_1) - f(x, y_2)| \le k|y_1 - y_2|, k > 0,$$

for every points (x, y_1) and (x, y_2) in D. Then the function $y = \phi(x)$ satisfying the IVP $y' = f(x, y)$ and $\phi(x_0) = y_0$, $x \in I$, is unique.

Proof. See Gupta and Kirmani [11].

For any function $f(x, y)$ of two variables defined in $D \subset \mathbb{R}^2$, we now present a sufficient condition which guarantees that the Lipschitz condition is satisfied in D.

Lemma 3.1. *Suppose that the function f is continuous in a convex region $D \subset \mathbb{R}^2$. Suppose further that $\frac{\partial f}{\partial y}$ exists and is continuous in D. Then, the function f satisfies Lipschitz condition in D.*

Proof. See Gupta and Kirmani [11].
We now present our characterization results.

Remark 3.3. *The pdf $g_{[r,n,m,k]}$ is differentiable if its derivative exists at each point in its domain. This can be seen if we apply it in well-known distributions, therefor, its domain will be identified.*

Theorem 3.5. *Let Y be a non-negative continuous random variable with distribution function F_Y. Let the dynamic inaccuracy of the concomitant of rth gos, based on a random sample of size n be denoted by $I_n(g_{[r,n,m,k]}(y), f_Y(y); t) < \infty$, $t \geq 0$. Then $I_n(g_{[r,n,m,k]}(y), f_Y(y); t)$ characterizes the distribution. With considering that $g_{[r,n,m,k]}$ is differentiable.*

Proof. Suppose there exists two functions F_{1Y} and F_{2Y} such that:

$$I_n(g_{1[r,n,m,k]}(y), f_{1Y}(y); t) = I_n(g_{2[r,n,m,k]}(y), f_{2Y}(y); t),$$

for all $t > 0$. Then

$$I'_n(g_{i[r,n,m,k]}(y), f_{iY}(y); t) = \xi_{i[r,n,m,k]}(t)[I_n(g_{i[r,n,m,k]}(y), f_{iY}(y); t) - 1 + \xi_{i[r,n,m,k]}(t)], i = 1, 2, \tag{33}$$

where $\xi_{i[r,n,m,k]}(t) = \frac{g_{i[r,n,m,k]}(t)}{\bar{G}_{i[r,n,m,k]}(t)}$ is the hazard rate of the concomitant of rth gos, $i = 1, 2$. Differentiating the above equation with respect to t and simplifying, we get:

$$\xi'_{i[r,n,m,k]}(t) = \frac{\xi_{i[r,n,m,k]}(t)}{\xi_{i[r,n,m,k]}(t) + I'_n(g_{i[r,n,m,k]}(y), f_{iY}(y); t)}[I''_n(g_{i[r,n,m,k]}(y), f_{iY}(y); t) \\ - \xi_{i[r,n,m,k]}(t)I'_n(g_{i[r,n,m,k]}(y), f_{iY}(y); t)], i = 1, 2.$$

Suppose now

$$I_n(g_{1[r,n,m,k]}(y), f_{iY}(y); t) = I_n(g_{2[r,n,m,k]}(y), f_{iY}(y); t) = g(t).$$

Then for all $t \geq 0$,

$$\xi'_{i[r,n,m,k]}(t) = \psi(t, \xi_{i[r,n,m,k]}(t)),$$

for $i = 1, 2$, where

$$\psi(t, y) = \frac{y}{y + g'(t)}[g''(t) - yg'(t)].$$

It follows from Theorem (3.1) and Lemma (3.4) that $\xi'_{1[r,n,m,k]}(t) = \xi'_{2[r,n,m,k]}(t)$, which prove the characterization result.

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