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On Quasi-Sasakian 3-Manifolds Admitting η-Ricci Solitons

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Abstract. The object of the present paper is to prove that in a quasi-Sasakian 3-manifold admitting η-Ricci soliton, the structure function β is a constant. As a consequence we obtain several important results.

1. Introduction

An almost contact metric manifold M and its almost contact metric structure (ϕ, ξ, η, g) are said to be quasi-Sasakian if the structure is normal and the fundamental 2-form Φ is closed. The notion of quasi-Sasakian structure was introduced by Blair [6] to unify Sasakian and cosympletic structures. Tanno [33] also added some remarks on quasi-Sasakian structures. The properties of quasi-Sasakian manifolds have been studied by several authors, viz., De et al. [17], Gonzalez and Chinea [22], Kanemaki ([26], [27]) and Oubina [31]. Kim [25] studied quasi-Sasakian manifolds and proved that fibred Riemannian spaces with invariant fibres normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosympletic structure. Recently, quasi-Sasakian manifolds have been the subject of growing interest in view of finding the significant applications to Physics, in particular to super gravity and magnetic theory ([1], [2]). Quasi-Sasakian structures have wide applications in the mathematical analysis of string theory ([3], [21]). On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Olszak [30] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [29]. Next he has proved that if the manifold is additionally conformally flat with β = constant, then (a) the manifold is locally a product of R and two-dimensional Kaehlerian space of constant Gauss curvature (the cosympletic case), or (b) the manifold is of constant positive curvature (the non-cosympletic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure).
In 1982, Hamilton [23] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold as follows:

$$\frac{\partial}{\partial_t} g_{ij} = -2S_{ij}, \tag{1}$$

where g_{ij} are components of a time dependent family of Riemannian metrics and S_{ij} are the components of Ricci tensor of a manifold.

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Ricci solitons are special solutions of the Ricci flow equation (1) of the form $g_{ij} = \sigma(t)\psi_t^* g_{ij}$ with the initial condition $g_{ij}(0) = g_{ij}$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function.
A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [11]. On the manifold M, a Ricci soliton is a triplet (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where £ is the Lie derivative. Metrics satisfying (1) are interesting and useful in physics and are often referred as quasi-Einstein ([12],[13]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scaling, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [20] who discusses some aspects of it.
The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively. Ricci solitons have been studied by several authors such as ([15], [16], [23], [24], [35], [36]) and many others.

As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [10]. This notion has also been studied in [11] for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a 4-tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants, and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3}$$

where S is the Ricci tensor associated to g. In this connection we may mention the works of Blaga ([7], [8], [9]), Prakasha et al. [32] and Majhi et al. [28]. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ). If $\mu \neq 0$, then the η-Ricci soliton is named as the proper η-Ricci soliton.
Motivated by the above studies we characterize quasi-Sasakian 3-manifolds admitting η-Ricci Solitons.

Definition 1.1. *A conformally flat Riemannian manifold M is said to be of quasi-constant curvature [14] if its curvature tensor $\bar{R}$ is of type (0,4) satisfies the condition*

$$\begin{aligned} \bar{R}(X, Y, U, W) &= p[g(Y, U)g(X, W) - g(X, U)g(Y, W)] + q[g(X, W)H(Y)H(U) + g(Y, U)H(X)H(W) \\ &\quad - g(X, U)H(Y)H(W) - g(Y, W)H(X)H(U)] \end{aligned} \tag{4}$$

for all vector fields X, Y, U, W on M, where p and q are scalars, H is a non-zero 1-form and $\bar{R}(X, Y, U, W) = g(R(X, Y)U, W)$, R is the curvature tensor of type (1,3). Throughout the paper, we consider X, Y, Z, U, W as arbitrary vector fields on M.

The paper is organized as follows: After introduction, in Section 2 we discuss some preliminaries of the quasi-Sasakian 3-manifolds. Section 3 is devoted to study our main theorem. Our main Theorem can be presented as follows:

Theorem 1.2. *The structure function of a non-cosympletic quasi-Sasakian 3-manifold admitting η-Ricci soliton is constant.*

As a consequence of the main Theorem 1.2, we obtain some important corollaries:

Corollary 1.3. *A non-cosympletic quasi-Sasakian 3-manifold admitting η-Ricci soliton is η-Einstein.*

Corollary 1.4. *A non-cosympletic quasi-Sasakian 3-manifold admitting η-Ricci soliton is a manifold of quasi-constant curvature.*

Corollary 1.5. *The scalar curvature of a non-cosympletic quasi-Sasakian 3-manifold admitting η-Ricci soliton is constant.*

Corollary 1.6. *A non-cosympletic quasi-Sasakian 3-manifold admitting η-Ricci soliotons is locally ϕ-symmetric.*

Corollary 1.7. *A non-cosympletic quasi-Sasakian 3-manifold admitting η-Ricci soliton can be obtained by a homothetic deformation of a Sasakian structure.*

2. Quasi-Sasakian 3-manifolds

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that ([4], [5])

$$\phi^2 X = -X + \eta(X)\xi, \tag{5}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad X, Y \in \chi(M), \tag{6}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{7}$$

where $\chi(M)$ denotes the collection of all smooth vector fields of M. Let Φ be the fundamental 2-form of M defined by

$$\Phi(X,Y) = g(X, \phi Y) = -g(\phi X, Y). \tag{8}$$

A three dimensional almost contact metric manifold M is quasi-Sasakian if and only if [30]

$$\nabla_X \xi = -\beta\phi X, \tag{9}$$

where ∇ is the Levi-Civita connection and β is a smooth function on the manifold.
For a quasi-Sasakian 3-manifold it is known that [17]

$$\xi\beta = 0. \tag{10}$$

For a 3-dimensional quasi-Sasakian manifold, we know

$$(\nabla_X \phi)Y = \beta\{g(X,Y)\xi - \eta(Y)X\}, \tag{11}$$

$$(\nabla_X \eta)Y = -\beta g(\phi X, Y). \tag{12}$$

In consequence of (12) we have

$$(\pounds_\xi \eta)Y = 0. \tag{13}$$

The Riemannian curvature tensor R of a 3-dimensional quasi-Sasakian manifold is given by [29]

$$\begin{aligned} R(X,Y)Z \;=\; & g(Y,Z)[(\frac{r}{2} - \beta^2)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi + \eta(X)(\phi \mathrm{grad}\beta) - \mathrm{d}\beta(\phi X)\xi] \\ & -g(X,Z)[(\frac{r}{2} - \beta^2)Y + (3\beta^2 - \frac{r}{2})\eta(Y)\xi + \eta(Y)(\phi grad\beta) - d\beta(\phi Y)\xi] \\ & +[(\frac{r}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y)]X \\ & -[(\frac{r}{2} - \beta^2)g(X,Z) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Z) - \eta(X)d\beta(\phi Z) - \eta(Z)d\beta(\phi X)]Y \\ & -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{14}$$

where Q is the Ricci operator, that is, $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the manifold.
In a 3-dimensional quasi-Sasakian manifold, the Ricci tensor S is given by [29]

$$\begin{aligned} S(X,Y) \;=\; & (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y) \\ & -\eta(X)d\beta(\phi Y) - \eta(Y)d\beta(\phi X). \end{aligned} \tag{15}$$

3. Proof of the main theorem

Let us consider a non-cosympletic quasi-Sasakian 3-manifold M admitting η-Ricci soliton. Then we have

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{16}$$

Using (15) in (16), we get

$$\begin{aligned}(\pounds_V g)(X,Y) &= -(r-2\beta^2+2\lambda)g(X,Y) - (6\beta^2 - r + 2\mu)\eta(X)\eta(Y)\\ &\quad +2\eta(X)d\beta(\phi Y) + 2\eta(Y)d\beta(\phi X).\end{aligned} \tag{17}$$

Differentiating the equation (17) covariantly with respect to W and using (12), we obtain

$$\begin{aligned}(\nabla_W \pounds_V g)(X,Y) &= -\{dr(W) - 4\beta d\beta(W)\}g(X,Y) - \{12\beta d\beta(W) - dr(W)\}\eta(X)\eta(Y)\\ &\quad +\beta(6\beta^2 - r + 2\mu)\{g(\phi W,X)\eta(Y) + g(\phi W,Y)\eta(X)\}\\ &\quad -2\beta g(\phi W,X)d\beta(\phi Y) - 2\beta g(\phi W,Y)d\beta(\phi X)\\ &\quad +2\eta(X)g(\nabla_W \mathrm{grad}\beta, \phi Y) + 2\eta(Y)g(\nabla_W\mathrm{grad}\beta,\phi X).\end{aligned} \tag{18}$$

According to Yano ([34], p. 23) we have the following well known formula

$$(\pounds_V\nabla_X g - \pounds_X\nabla_V g - \nabla_{[V,X]}g)(Y,Z) = -g((\pounds_V\nabla)(X,Y),Z) - g((\pounds_V\nabla)(X,Z),Y).$$

Making use of the parallelism of the metric g in the above formula we have

$$(\nabla_X\pounds_V g)(Y,Z) = g((\pounds_V\nabla)(X,Y),Z) + g((\pounds_V\nabla)(Z,X),Y), \tag{19}$$

for any vector fields X, Y, Z.
Since $\pounds_V\nabla$ is symmetric tensor of type (1,2), that is, $(\pounds_V\nabla)(X,Y) = (\pounds_V\nabla)(Y,X)$, the above equation yields

$$g((\pounds_V\nabla)(X,Y),Z) = \frac{1}{2}(\nabla_X\pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y\pounds_V g)(Z,X) - \frac{1}{2}(\nabla_Z\pounds_V g)(X,Y), \tag{20}$$

for any vector fields X, Y, Z.
From (20) it follows that

$$2g((\pounds_V\nabla)(X,Y),Z) = (\nabla_X\pounds_V g)(Y,Z) + (\nabla_Y\pounds_V g)(Z,X) - (\nabla_Z\pounds_V g)(X,Y). \tag{21}$$

With the help of (18), from (21) we deduce

$$\begin{aligned}2g((\pounds_V\nabla)(X,Y),Z) &= -\{dr(X) - 4\beta d\beta(X)\}g(Y,Z) - \{dr(Y) - 4\beta d\beta(Y)\}g(X,Z)\\ &\quad +\{dr(Z) - 4\beta d\beta(Z)\}g(X,Y) - \{12\beta d\beta(X) - dr(X)\}\eta(Y)\eta(Z)\\ &\quad -\{12\beta d\beta(Y) - dr(Y)\}\eta(Z)\eta(X) + \{12\beta d\beta(Z) - dr(Z)\}\eta(X)\eta(Y)\\ &\quad +2\beta(6\beta^2 - r + 2\mu)\{g(\phi X,Z)\eta(Y) + g(\phi Y,Z)\eta(X)\}\\ &\quad -4\beta g(\phi X,Z)d\beta(\phi Y) - 4\beta g(\phi Y,Z)d\beta(\phi X)\\ &\quad -2\eta(Y)g(\phi\nabla_X\mathrm{grad}\beta,Z) + 2\eta(Z)g(\nabla_X\mathrm{grad}\beta,\phi Y)\\ &\quad +2\eta(Z)g(\nabla_Y\mathrm{grad}\beta,\phi X) - 2\eta(X)g(\phi\nabla_Y\mathrm{grad}\beta,Z)\\ &\quad -2\eta(X)g(\nabla_{\phi Y}\mathrm{grad}\beta,Z) - 2\eta(Y)g(\nabla_{\phi X}\mathrm{grad}\beta,Z),\end{aligned} \tag{22}$$

and hence we have

$$\begin{aligned}2(\pounds_V\nabla)(X,Y) &= -\{dr(X) - 4\beta d\beta(X)\}Y - \{dr(Y) - 4\beta d\beta(Y)\}X\\ &\quad +\{\mathrm{grad}r - 4\beta\mathrm{grad}\beta\}g(X,Y) - \{12\beta d\beta(X) - dr(X)\}\eta(Y)\xi\\ &\quad -\{12\beta d\beta(Y) - dr(Y)\}\eta(X)\xi + \{12\beta\mathrm{grad}\beta - \mathrm{grad}r\}\eta(X)\eta(Y)\\ &\quad +2\beta(6\beta^2 - r + 2\mu)\{\eta(Y)\phi X + \eta(X)\phi Y\} - 4\beta d\beta(\phi Y)\phi X - 4\beta d\beta(\phi X)\phi Y\\ &\quad -2\eta(Y)\phi(\nabla_X\mathrm{grad}\beta) + 2g(\nabla_X\mathrm{grad}\beta,\phi Y)\xi\\ &\quad +2g(\nabla_Y\mathrm{grad}\beta,\phi X)\xi - 2\eta(X)\phi(\nabla_Y\mathrm{grad}\beta)\\ &\quad -2\eta(X)\nabla_{\phi Y}\mathrm{grad}\beta - 2\eta(Y)\nabla_{\phi X}\mathrm{grad}\beta.\end{aligned} \tag{23}$$

Replacing Y by ξ in (23) yields

$$\begin{aligned}(\pounds_V\nabla)(X,\xi) \quad = \quad & -4\beta d\beta(X)\xi + 4\beta\eta(X)\text{grad}\beta + \beta(6\beta^2 - r + 2\mu)\phi X - \phi(\nabla_X\text{grad}\beta) \\ & +g(\nabla_\xi\text{grad}\beta,\phi X)\xi - \eta(X)\phi(\nabla_\xi\text{grad}\beta) - \nabla_{\phi X}\text{grad}\beta \end{aligned} \tag{24}$$

Differentiating the equation (24) covariantly with respect to Y and using (11) and (12) we get

$$\begin{aligned}(\nabla_Y\pounds_V\nabla)(X,\xi) \quad = \quad & -4\{d\beta(X)d\beta(Y)\xi + \beta g(\nabla_Y\text{grad}\beta, X)\xi - \beta^2 d\beta(X)\phi Y\} + 4\{d\beta(Y)\eta(X)\text{grad}\beta \\ & -\beta^2 g(\phi Y, X)\text{grad}\beta + \beta\eta(X)\nabla_Y\text{grad}\beta\} + (6\beta^2 - r + 2\mu)d\beta(Y)\phi X \\ & +\beta\{12\beta d\beta(Y) - dr(Y)\}\phi X + \beta^2(6\beta^2 - r + 2\mu)g(X,Y)\xi \\ & +\beta^2(6\beta^2 - r + 2\mu)\eta(X)Y - \beta g(\nabla_X\text{grad}\beta, Y)\xi + \beta\eta(\nabla_X\text{grad}\beta)Y \\ & +\nabla_Y g(\nabla_\xi\text{grad}\beta,\phi X)\xi - g(\nabla_Y\nabla_\xi\text{grad}\beta,\phi X)\xi \\ & -g(\nabla_\xi\text{grad}\beta,\nabla_Y\phi X)\xi - \beta g(\nabla_\xi\text{grad}\beta,\phi X)\phi Y \\ & +\beta g(\phi Y, X)\phi(\nabla_\xi\text{grad}\beta) - \beta\eta(X)g(\nabla_\xi\text{grad}\beta, Y)\xi \\ & +\beta\eta(X)\eta(\nabla_\xi\text{grad}\beta)Y - \nabla_Y\nabla_{\phi X}\text{grad}\beta. \end{aligned} \tag{25}$$

Now we state a well known Lemma:

Lemma 3.1. *(Poincare Lemma): In Riemannian manifold $d^2 = 0$, where d is the exterior differential operator, that is,*

$$g(\nabla_X\text{grad}\zeta, Y) = g(\nabla_Y\text{grad}\zeta, X), \tag{26}$$

for any two vector fields X, Y and for any smooth function ζ.

It is well known that

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z). \tag{27}$$

Putting $V = Z = \xi$ in (27) and then using (25), (10) and (26) yields

$$\begin{aligned}(\pounds_\xi R)(X,Y)\xi \quad = \quad & 4\beta^2 d\beta(Y)\phi X - 4\beta^2 d\beta(X)\phi Y + 4d\beta(X)\eta(Y)\text{grad}\beta - 4d\beta(Y)\eta(X)\text{grad}\beta \\ & -2\beta^2 g(\phi X, Y)grad\beta + \beta\eta(Y)\nabla_X\text{grad}\beta - \beta\eta(X)\nabla_Y\text{grad}\beta \\ & +(6\beta^2 - r + 2\mu)d\beta(X)\phi Y - (6\beta^2 - r + 2\mu)d\beta(Y)\phi X \\ & +12\beta^2 d\beta(X)\phi Y - 12\beta^2 d\beta(Y)\phi X - \beta dr(X)\phi Y + \beta dr(Y)\phi X \\ & -\beta^2(6\beta^2 - r + 2\mu)\eta(Y)X + \beta^2(6\beta^2 - r + 2\mu)\eta(X)Y \\ & +\beta\eta(\nabla_Y\text{grad}\beta)X - \beta\eta(\nabla_X\text{grad}\beta)Y \\ & +\nabla_X g(\nabla_\xi\text{grad}\beta,\phi Y)\xi - g(\nabla_X\nabla_\xi\text{grad}\beta,\phi Y)\xi \\ & -g(\nabla_\xi\text{grad}\beta,\nabla_X\phi Y)\xi - \nabla_Y g(\nabla_\xi\text{grad}\beta,\phi X)\xi \\ & +g(\nabla_Y\nabla_\xi\text{grad}\beta,\phi X)\xi + g(\nabla_\xi\text{grad}\beta,\nabla_Y\phi X)\xi \\ & -\beta g(\nabla_\xi\text{grad}\beta,\phi Y)\phi X + \beta g(\nabla_\xi\text{grad}\beta,\phi X)\phi Y \\ & +2\beta g(\phi X, Y)\phi(\nabla_\xi\text{grad}\beta) \\ & -\beta\eta(Y)g(\nabla_\xi\text{grad}\beta, X)\xi + \beta\eta(X)g(\nabla_\xi\text{grad}\beta, Y)\xi \\ & +\beta\eta(Y)\eta(\nabla_\xi\text{grad}\beta)X - \beta\eta(X)\eta(\nabla_\xi\text{grad}\beta)Y \\ & +\nabla_Y\nabla_{\phi X}\text{grad}\beta - \nabla_X\nabla_{\phi Y}\text{grad}\beta. \end{aligned} \tag{28}$$

From (28) we can easily obtain that

$$\begin{aligned}(\pounds_\xi R)(X,\xi)\xi \quad = \quad & 4d\beta(X)\text{grad}\beta - \beta\eta(X)\nabla_\xi\text{grad}\beta \\ & +\beta\nabla_X\text{grad}\beta + \beta^2(6\beta^2 - r + 2\mu)\eta(X)\xi \\ & -\beta^2(6\beta^2 - r + 2\mu)X - 2\beta\eta(\nabla_X\text{grad}\beta)\xi \\ & -\nabla_\xi g(\nabla_\xi\text{grad}\beta,\phi X)\xi + g(\nabla_\xi\nabla_\xi\text{grad}\beta,\phi X)\xi \\ & +g(\nabla_\xi\text{grad}\beta,\nabla_\xi\phi X)\xi + \nabla_\xi\nabla_{\phi X}\text{grad}\beta. \end{aligned} \tag{29}$$

In view of (14) we infer

$$R(X,\xi)\xi = -\beta^2 X + \beta^2\eta(X)\xi. \tag{30}$$

Taking Lie differentiation of (30) along ξ and using (10) and (13) yields

$$(\pounds_\xi R)(X,\xi)\xi = -\beta^2[\xi, X]. \tag{31}$$

Equating (29) and (31) and then taking inner product with ξ we get

$$\begin{aligned}-\beta g(\nabla_X \mathrm{grad}\beta, \xi) - \nabla_\xi g(\nabla_\xi \mathrm{grad}\beta, \phi X) + g(\nabla_\xi\nabla_\xi \mathrm{grad}\beta, \phi X)\\ +g(\nabla_\xi \mathrm{grad}\beta, \nabla_\xi \phi X) + g(\nabla_\xi\nabla_{\phi X}\mathrm{grad}\beta, \xi) = -\beta^2\eta(\nabla_\xi X).\end{aligned} \tag{32}$$

Replacing X by ϕX in (32) and after simplification we infer

$$2\beta^2(X\beta) = \beta g(\mathrm{grad}\beta, \nabla_\xi X). \tag{33}$$

If X be a unit vector, then $g(X,X) = 1$, from which it follows that $g(\nabla_\xi X, X) = 0$, and hence we have

$$\nabla_\xi X = 0. \tag{34}$$

Using (34) in (33) we get

$$X\beta = g(\mathrm{grad}\beta, X) = 0. \tag{35}$$

In view of (35) and (10) we can say that the structure function β is constant along any unit vector field X. Let Y be any arbitrary non-zero vector field. Then $\frac{Y}{\|Y\|}$ is a unit vector. Hence from (35) it follows that

$$g(\mathrm{grad}\beta, \frac{Y}{\| Y \|}) = 0, \tag{36}$$

which yields

$$g(\mathrm{grad}\beta, Y) = Y\beta = 0, \tag{37}$$

for any vector field Y.

This completes the proof of the main Theorem. □

Moreover, using the fact that β = constant, in (15) we have

$$S(X,Y) = (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y). \tag{38}$$

Therefore, the manifold becomes η-Einstein and hence complete the proof of the Corollary 1.3.

Since β is constant, then the curvature tensor R is of the form

$$\begin{aligned}R(X,Y)Z \;=\; & g(Y,Z)\{(\frac{r}{2} - \beta^2)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi\} - g(X,Z)\{(\frac{r}{2} - \beta^2)Y + (3\beta^2 - \frac{r}{2})\eta(Y)\xi\}\\ &+\{(\frac{r}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z)\}X - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}\\ &+\{(\frac{r}{2} - \beta^2)g(X,Z) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Z)\}Y,\end{aligned} \tag{39}$$

from which it follows that

$$R(X,Y)Z = p[g(Y,Z)X - g(X,Z)Y] + q[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \tag{40}$$

where $p = \frac{r}{2} - \beta^2$ and $q = 3\beta^2 - \frac{r}{2}$. That is the manifold reduces to a manifold of quasi-constant curvature. Thus the Corollary 1.4 is proved.

In [29], Olszak proved the following:

Theorem 3.2. *[29] A 3-dimensional quasi-Sasakian manifold M is conformally flat if and only if its structure function β fulfils the following two conditions*

$$r - 10\beta^2 = constant., \tag{41}$$

$$(\nabla_X d\beta)(Y) = \beta(3\beta^2 - \frac{r}{2})\{g(X,Y) - \eta(X)\eta(Y)\} + \beta\eta(X)d\beta(\phi Y) + \beta\eta(Y)d\beta(\phi X). \tag{42}$$

Since we consider a 3-dimensional quasi-Sasakian manifold, then it is conformally flat. Also we have β =constant. Then from the equation (42) we infer

$$\beta(3\beta^2 - \frac{r}{2})\{g(X,Y) - \eta(X)\eta(Y)\} = 0. \tag{43}$$

Contracting X and Y in (43), we obtain

$$2\beta(3\beta^2 - \frac{r}{2}) = 0, \tag{44}$$

and hence

$$r = 6\beta^2. \tag{45}$$

Thus from the Theorem 1.2 and the equation (45), the proof of the Corollary 1.5 directly follows.

In [18] De and Sarkar proved the following:

Theorem 3.3. *[18] A three-dimensional non-cosympletic quasi-Sasakian manifold with constant structure function β is locally ϕ-symmetric if and only if the scalar curvature is constant.*

In view of the Theorem 1.2, Corollary 1.5 and Theorem 3.3, the Corollary 1.6 is proved.

With the help of the Theorem 1.2, equations (45) and (14) we deduce

$$R(X,Y)Z = \beta^2[g(Y,Z)X - g(X,Z)Y]. \tag{46}$$

In [29], Olszak prove the following:

Theorem 3.4. *[29] Let M is a quasi-Sasakian manifold of positive constant curvature K. Then $K \geq 0$ and (a) if $K = 0$, the manifold is cosympletic, (b) if $K > 0$, the quasi-Sasakian structure is obtained by a homothetic deformation of a Sasakian structure.*

By the hypothesis the manifold is non-cosympletic, so we have $\beta \neq 0$ and hence $\beta^2 > 0$, that is, the manifold is of positive constant curvature. Therefore, the proof of the Corollary 1.7 follows from the Theorem 3.4.

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