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Generalized quantum exponential function and its applications

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Abstract. This article aims to present (q,h)-analogue of exponential function which unifies, extends h- and q-exponential functions in a convenient and efficient form. For this purpose, we introduce generalized quantum binomial which serves as an analogue of an ordinary polynomial. We state (q,h)-analogue of Taylor series and introduce generalized quantum exponential function which is determined by Taylor series in generalized quantum binomial. Furthermore, we prove existence and uniqueness theorem for a first order, linear, homogeneous IVP whose solution produces an infinite product form for generalized quantum exponential function. We conclude that both representations of generalized quantum exponential function are equivalent. We illustrate our results by ordinary and partial difference equations. Finally, we present a generic dynamic wave equation which admits generalized trigonometric, hyperbolic type of solutions and produces various kinds of partial differential/difference equations.

1. Introduction

The h-analysis and the concepts of difference equations date back to seventeenth century, to the celebrated fundamentals of calculus of Isaac Newton [3]. In the nineteenth century, Boole [10] published the calculus of finite differences. On the other hand, the origins of the q-analysis started in the eighteenth century by Euler [13] and developed by Gauss, Ramanujan and Jacobi [23]. After Jacsons's pioneering articles [17–22], that systematically presented q-calculus, the number of studies in q- and h-analysis have been grown rapidly in many areas of mathematics such as in number theory-combinatorics [4], in orthogonal polynomials [5, 6], in ordinary and partial differential equations [1, 28], in mathematical physics [2, 26, 27], in approximation theory [12] and also in physical disciplines such as quantum mechanics and relativity [30]. For the very detailed information about q-calculus, we refer to [24].

In the literature, the study of discrete dynamical equations is widely investigated in two main separate discrete sets: h-lattice

$$h\mathbb{Z} := \{hx : x \in \mathbb{Z},\ h > 0\}, \tag{1}$$

and q-numbers

$$\mathbb{K}_q := \{q^n : \quad n \in \mathbb{Z}, \quad q \in \mathbb{R}, \quad q \neq 1\} \cup \{0\}. \tag{2}$$

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The parameters h and q were not chosen accidentally. The parameter q is the first letter of "quantum" as well as it denotes the number of elements in finite fields while the parameter h is used to remind the Planck's constant in quantum mechanics. The set (1) and (2) recover $\mathbb{R}$ as $h \to 0$ and $q \to 1$, respectively. This fact can be also governed by limiting h-derivative and q-derivative [17]

$$D_h f(x) := \frac{f(x+h) - f(x)}{h}, \qquad D_q f(x) := \frac{f(qx) - f(x)}{(q-1)x} \tag{3}$$

respectively as

$$\lim_{h \to 0} D_h f(x) = \frac{df}{dx} = \lim_{q \to 1} D_q f(x).$$

For almost a century, the scientists have been presenting the discretization of continous equations in two separate approaches: h-discretization and q-discretization. Referring to different behaviours of such quantum derivatives, each discretization has dissimilar features such as dissimilarity in elementary functions and their properties (polynomials, exponential functions etc.) Therefore, it is not straightforward to intensify on a unified discrete set, instead of studying in $h\mathbb{Z}$ and $\mathbb{K}_q$, separately. The main objective of the concept of time scales, introduced by Hilger [16], is unification and extension of not only such discrete settings but also any type of continuous and discrete sets. In [11], a special discrete time scale $\mathbb{T}_{(q,h)}$ is introduced which unifies and extends h- and q-analysis in order to present (q,h)-fractional calculus. Some contributions on the so-called (q,h)-analysis have being created such as representation of Newton's binomial formula with (q,h)-binomial coefficients [7], (q,h)-analogue of Laplace transform [25] and (q,h)-analogue of quantum splines [14].

The definition of exponential function on time scales is implicit and inapplicable. The fundamental purpose of the current article is to introduce (q,h)-analogue of exponential function in a precise, concrete and efficient form, which unifies and extends h-, q-exponential functions. For this purpose, we first present (q,h)-analogue of Taylor's formula, which leads us to discover the form of (q,h)-analogue of polynomials. We introduce the generalized quantum binomial $(x - x_0)^n_{q,h}$ whose role in $\mathbb{T}_{(q,h)}$ is as significant as the role of $(x - x_0)^n$ in ordinary calculus, for any point $x_0 \in \mathbb{R}$. We state and prove (q,h)-analogue of Taylor series via generalized quantum binomials from which we introduce generalized quantum exponential function. Another way to construct the generalized quantum exponential function is to present it as a solution of a first order, linear, homogeneous (q,h)-initial value problem. We prove existence and uniqueness of its solution, where the solution appears to be in an infinite product form. In the light of uniqueness, both representations of generalized quantum exponential functions (infinite series and infinite product) are concluded to be equivalent. As a consequence of generalized quantum exponential function, we introduce the generalized quantum trigonometric and hyperbolic functions.

This paper is organized as follows. In Section 2, we present prelimanary properties of delta and nabla (q,h)-derivatives. Section 3 is devoted to present generalized quantum Taylor's formula and generalized quantum binomial. As an application, Cauchy-Euler type of (q,h)-difference equation whose solution is in the form of the generalized quantum binomial is presented. Furthermore, we introduce (q,h)-analogue of partial derivatives and present a (q,h)-analogue of a partial difference equation admitting a traveling wave-like solution. In Section 4, we introduce the generalized quantum exponential function and as direct consequences generalized quantum trigonometric and hyperbolic functions. As an application, we present a dynamic wave equation on $\mathbb{T}_{(q,h)} \times \mathbb{T}_{(\tilde{q},\tilde{h})}$. This generic equation produces various kinds of partial differential/difference equations, which can be regarded as a generalized wave equation.

2. Basic notions

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers. For any time scale $\mathbb{T}$, the *forward jump operator* σ and the *backward jump operator* ρ are defined by

$$\sigma(x) := \inf\{s \in \mathbb{T} : s > x\}, \qquad \rho(x) := \sup\{s \in \mathbb{T} : s < x\}. \tag{4}$$

In [11], a two-parameter time scale $\mathbb{T}_{(q,h)}$ is introduced as follows

$$\mathbb{T}_{(q,h)} := \{q^n x + [n]h : x \in \mathbb{R}, \quad n \in \mathbb{Z}, \quad h, q \in \mathbb{R}^+, \quad q \neq 1\} \cup \left\{ \frac{h}{1-q} \right\}, \tag{5}$$

where $[n] := \dfrac{q^n - 1}{q - 1}$. The forward and backward jump operators act on every $x \in \mathbb{T}_{(q,h)}$ as

$$\sigma^n(x) = q^n x + [n]h, \qquad \rho^n(x) = q^{-n}(x - [n]h), \quad n \in \mathbb{N}.$$

Clearly, $\mathbb{T}_{(q,h)}$ is a set of *regular discrete points* [15], namely

$$(\sigma \circ \rho)(x) = (\rho \circ \sigma)(x) = x, \quad x \in \mathbb{T}_{(q,h)}. \tag{6}$$

The classicial derivative can be approximated in various derivatives such as h-derivative, q-derivative [17], symmetric h-derivative, symmetric q-derivative [24] or conformable derivative [8]. In addition, it is possible to introduce more general quantum derivatives which comprise and extend h-derivative and q-derivative.

Definition 2.1. *Let $f : \mathbb{T}_{(q,h)} \to \mathbb{R}$ be any function. The delta (q,h)-derivative of f, denoted by $D_{(q,h)}(f)$, is defined by*

$$D_{(q,h)} f(x) := \frac{f(\sigma(x)) - f(x)}{\sigma(x) - x} = \frac{f(qx + h) - f(x)}{(q-1)x + h}. \tag{7}$$

while the nabla (q,h)-derivative of f, denoted by $\tilde{D}_{(q,h)}$, is given as

$$\tilde{D}_{(q,h)} f(x) := \frac{f(x) - f(\rho(x))}{x - \rho(x)} = \frac{f(x) - f(\frac{x-h}{q})}{x - (\frac{x-h}{q})}. \tag{8}$$

Note that, delta (q,h)-derivative $D_{(q,h)}$, recovers h-derivative, q-derivative and standard derivative in the appropriate limits of q and h. Similarly, this fact holds for nabla (q,h)-derivative. In addition, we can define *symmetric (q,h)-derivative* by

$$\bar{D}_{(q,h)} f(x) := \frac{f(\sigma(x)) - f(\rho(x))}{\sigma(x) - \rho(x)} = \frac{f(qx + h) - f(\frac{x-h}{q})}{qx + h - (\frac{x-h}{q})}. \tag{9}$$

In this case, (9) recovers the *symmetric h-derivative* and the *symmetric q-derivative* respectively

$$\lim_{q \to 1} \bar{D}_{(q,h)} f(x) = \frac{f(x+h) - f(x-h)}{2h}, \qquad \lim_{h \to 0} \bar{D}_{(q,h)} f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}.$$

The generalized symmetric quantum derivative (9) will play an important role in algebra such as in quantum groups.

Proposition 2.2. *Let $f(x)$ be a function.*

(i) $D_{(q,h)} f(x) = 0$ *if and only if* $f(x)$ *is constant.*

(ii) $D_{(q,h)} f(x) = D_{(q,h)} g(x)$ *for all* $x \in \mathbb{R}$ *if and only if* $f(x) = g(x) + c$ *with some constant* c.

(iii) $D_{(q,h)} f(x) = c_1$ *if and only if* $f(x) = c_1 x + c_2$ *where* c_1, c_2 *are constants.*

Proof. It is sufficient to prove if parts of the statements, since only if parts follow directly from the definition of delta (q,h)-derivative. Consider

$$D_{(q,h)}f(x) = \frac{f(qx+h)-f(x)}{(q-1)x+h} = 0,$$

which implies that $f(qx+h) = f(x)$ for all $x \in \mathbb{R}$ i.e., f is a constant function. Using (i) for $h(x) := f(x) - g(x)$, it follows that h is constant. For (iii), let $D_{(q,h)}f(x) = c_1$, then we have

$$(E-1)f = c_1[(q-1)x+h],$$

where $E(f(x)) := f(\sigma(x)) = f(qx+h)$. Thus

$$\begin{aligned} f(x) &= (E-1)^{-1}c_1[(q-1)x+h] = -c_1\sum_{i=0}^{\infty}E^i[(q-1)x+h] \\ &= -c_1\left\{[(q-1)x+h]\sum_{i=0}^{\infty}q^i + h\sum_{i=0}^{\infty}[i]\right\} = c_1x + h\left\{\frac{c_1}{q-1} - \sum_{i=0}^{\infty}[i]\right\} = c_1x + c_2, \end{aligned}$$

provided that $q < 1$. □

Similarly Proposition (2.2) holds for nabla (q,h)-derivative.

Proposition 2.3. *Let f, g be arbitrary functions. Then product and quotient rules of $D_{(q,h)}$ and $\tilde{D}_{(q,h)}$ are as follows:*

$$\begin{aligned} D_{(q,h)}[f(x)g(x)] &= f(x)D_{(q,h)}g(x) + g(qx+h)D_{(q,h)}f(x) \\ &= g(x)D_{(q,h)}f(x) + f(qx+h)D_{(q,h)}g(x), \end{aligned}$$

$$\begin{aligned} \tilde{D}_{(q,h)}[f(x)g(x)] &= f(x)\tilde{D}_{(q,h)}g(x) + g(\frac{x-h}{q})\tilde{D}_{(q,h)}f(x) \\ &= g(x)\tilde{D}_{(q,h)}f(x) + f(\frac{x-h}{q})\tilde{D}_{(q,h)}g(x), \end{aligned}$$

$$\begin{aligned} D_{(q,h)}\left[\frac{f(x)}{g(x)}\right] &= \frac{g(x)D_{(q,h)}f(x) - f(x)D_{(q,h)}g(x)}{g(x)g(qx+h)} \\ &= \frac{g(qx+h)D_{(q,h)}f(x) - f(qx+h)D_{(q,h)}g(x)}{g(x)g(qx+h)}, \\ \tilde{D}_{(q,h)}\left[\frac{f(x)}{g(x)}\right] &= \frac{g(x)\tilde{D}_{(q,h)}f(x) - f(x)\tilde{D}_{(q,h)}g(x)}{g(x)g(\frac{x-h}{q})} \\ &= \frac{g(\frac{x-h}{q})\tilde{D}_{(q,h)}f(x) - f(\frac{x-h}{q})\tilde{D}_{(q,h)}g(x)}{g(x)g(\frac{x-h}{q})}. \end{aligned}$$

The proofs follow immediately from the definitions of delta and nabla (q,h)-derivatives.

3. Generalization of quantum Taylor's formula and quantum binomial

We begin this section by presenting a motivation example which inspire us to introduce (q,h)-analogues of Taylor's formula and polynomials.

Example 3.1. *Let us consider the delta (q,h)-derivative of the monomial x^n, $n \in \mathbb{Z}^+$. By the use of the standard binomial expansion, we have*

$$D_{(q,h)}(x^n) = \frac{(qx+h)^n - x^n}{(q-1)x+h} = \frac{\sum\limits_{i=0}^{n} \binom{n}{i}(qx)^i h^{n-i} - x^n}{(q-1)x+h}. \tag{10}$$

If $h \to 0$, we have

$$\lim_{h\to 0} D_{(q,h)}(x^n) = \frac{(qx)^n - x^n}{(q-1)x} = [n]x^{n-1},$$

We stress that $[n]x^{n-1}$ tends to nx^{n-1} as $q \to 1$. To be more precise, the monomial x^n has the same nice structure in q-calculus similar to its behaviour in ordinary calculus. However, if $q \to 1$ (10) reduces to

$$\lim_{q\to 1} D_{(q,h)}(x^n) = \frac{(x+h)^n - x^n}{h} = \sum_{i=1}^{n} \binom{n}{i-1} x^{i-1}h^{n-i},$$

which implies that in h-calculus x^n does not behave as properly as it behaves in ordinary calculus or q-calculus.

On account of such differences between h- and q-calculus, it is worthwhile to discover (q,h)-analogue of a polynomial, acting similar to the polynomials in ordinary calculus.

Theorem 3.2. *Let $x_0 \in \mathbb{R}$ and $\{P_0(x), P_1(x), ...\}$ be a sequence of polynomials satisfying the conditions simultaneously*

(i) $P_0(x_0) = 1$ *and* $P_i(x_0) = 0$*, for* $i = 1, 2, 3, \ldots$

(ii) $degP_i = i$ *for* $i = 0, 1, 2\ldots$

(iii) $D_{(q,h)}P_i = P_{i-1}$ *for* $i \geq 1$.

Then any polynomial $f(x)$ of degree n, admits the generalized quantum Taylor's Formula

$$f(x) = \sum_{i=0}^{n} (D^i_{(q,h)}f)(x_0)P_i(x), \tag{11}$$

where $D^i_{(q,h)}$ stands for the delta (q,h)-derivative of order i and

$$P_i(x) = \frac{(x-x_0)(x-(qx_0+h))(x-(q^2x_0+[2]h)...(x-(q^{i-1}x_0+[i-1]h))}{[i]!}. \tag{12}$$

Here $[i]! := [i].[i-1]...[2].[1]$ and for convention $[0]! = 1$.

Proof. Let W be the vector space of polynomials of dimension $n+1$. Since $degP_i = i$, for each i, then $B := \{P_0(x), P_1(x), ...P_n(x)\}$ is a linearly independent set of polynomials with $|B| = n+1$. Therefore B spans W and B becomes a basis for W. In other words, any polynomial $f(x) \in W$ can be written as a linear combination of polynomials in B

$$f(x) = \sum_{i=0}^{n} a_i P_i(x). \tag{13}$$

Now we need to construct the polynomials P_i. In general, such polynomials have the following form

$$P_i(x) := c_i(x-x_0)(x-x_1)....(x-x_{i-1}), \tag{14}$$

where x_i' s are functions of x_0, q and h. Clearly by the setting (14), $deg(P_i) = i$, and $P_i(x_0) = 0$, for any $i \geq 1$. Assume $P_0(x) = c_0 \equiv 1$. In order to fulfill the condition (iii), we need to compare the leading coefficients of x^i in $D_{(q,h)}P_i$ and P_{i-1}, from which we obtain

$$c_i = \frac{1}{[i]!}. \tag{15}$$

Comparing the coefficient of x^{i-1} in $D_{(q,h)}P_i$ and P_{i-1}, we derive

$$x_1 = qx_0 + h, \qquad x_2 = q^2x_0 + qh + h. \tag{16}$$

Inductively, we obtain

$$x_{i-1} = q^{i-1}x_0 + [i-1]h. \tag{17}$$

Thus, (14), (15) and (17) imply that P_i is of the form (12). To find the constants a_i explicitly, we first use the condition (i) on (13) and we obtain $f(x_0) = a_0$. Since $D_{(q,h)}$ is a linear operator, we have

$$D_{(q,h)}f(x) = \sum_{i=0}^{n} a_i D_{(q,h)}P_i(x) = \sum_{i=1}^{n} a_i P_{i-1}(x),$$

where we also used the condition (iii). At $x = x_0$, we obtain $D_{(q,h)}f(x_0) = a_1$. Similarly, applying $D_{(q,h)}$, i times to $f(x)$, we get

$$D^i_{(q,h)}f(x_0) = a_i, \quad i \geq 0,$$

which finishes the proof. □

Definition 3.3. *We define the generalized quantum binomial, (q,h)-analogue of $(x-x_0)^n$, as the polynomial*

$$(x-x_0)^n_{q,h} := \begin{cases} 1 & if \quad n = 0, \\ \prod_{i=1}^{n} \left(x - (q^{i-1}x_0 + [i-1]h)\right) & if \quad n > 0, \end{cases} \tag{18}$$

where $x_0 \in \mathbb{R}$.

Remark 3.4. *The generalized quantum binomial* (18) *reduces to q-binomial [24],*

$$(x-x_0)^n_q = (x-x_0)(x-qx_0)...(x-q^{n-1}x_0),$$

as $h \to 0$ and to h-binomial

$$(x-x_0)^n_h = (x-x_0)(x-(x_0+h))(x-(x_0+2h))....(x-(x_0+(n-1)h)),$$

as $q \to 1$. Furthermore, it approximates the ordinary binomial

$$\lim_{(q,h)\to(1,0)} (x-x_0)^n_{(q,h)} = (x-x_0)^n.$$

When $x_0 = 0$, the generalized quantum binomial (18) *becomes*

$$(x-0)^n_{q,h} = x(x-h)(x-[2]h)...(x-[n-1]h)$$

and its reduction to $\mathbb{K}_q$ reduces to the monomial $(x-0)^n_q \equiv x^n$, unlike the reduction on $h\mathbb{Z}$, which appears as

$$x(x-h)(x-2h)....(x-(n-1)h)) \neq x^n.$$

Nevertheless, the generalized quantum binomial $(x-x_0)^n_{q,h}$ plays the same role in $\mathbb{T}_{(q,h)}$ as $(x-x_0)^n$ plays in ordinary calculus, which is proved in the following Proposition.

Proposition 3.5. *The generalized quantum binomial* (18) *admits the following identities:*

(i) $D_{(q,h)}(x-x_0)^n_{q,h} = [n](x-x_0)^{n-1}_{q,h}, \quad n = 1,2,....$

(ii) $D^k_{(q,h)}(x-x_0)^n_{q,h} = P(n,k)(x-x_0)^{n-k}_{q,h}, \quad 0 \le k \le n,$

where P stands for the permutation coefficient

$$P(n,k) := \frac{[n]!}{[n-k]!}. \tag{19}$$

Proof. (i) Using the definition of delta (q,h)-derivative (7) on generalized quantum binomial (18), we obtain

$$\begin{aligned} D_{(q,h)}(x-x_0)^n_{q,h} &= \frac{(qx+h-x_0)^n_{q,h} - (x-x_0)^n_{q,h}}{(q-1)x+h} \\ &= \frac{q^{n-1}(qx+h-x_0) - (x-q^{n-1}x_0 - (1+q+...+q^{n-2})h)}{(q-1)x+h} \cdot (x-x_0)^{n-1}_{q,h} \\ &= \frac{(q^n-1)x + (1+q+...q^{n-1})h}{(q-1)x+h} \cdot (x-x_0)^{n-1}_{q,h} \\ &= [n](x-x_0)^{n-1}_{q,h}. \end{aligned}$$

The Leibnitz rule (ii) is a direct consequence of part (i). □

Example 3.6. *We present a (q,h)-analogue of Cauchy-Euler type of equation as*

$$\alpha_1(x-qa-h)D_{(q,h)}y + \alpha_2 y = 0, \quad a \in \mathbb{R}, \quad \alpha_1, \alpha_2 \in \mathbb{R}/\{0\}.$$

By Proposition (3.5)*, one can show that $y = (x-a)^r_{q,h}$ is a solution provided that $r = 2$ and*

$$\alpha_1[2] + \alpha_2 = 0.$$

It is possible to consider an n-th order Cauchy-Euler type of equation

$$\alpha_n B_0 D^n_{(q,h)}y + \alpha_{n-1}B_1 D^{n-1}_{(q,h)}y + ... + \alpha_1 B_{n-1}D_{(q,h)}y + \alpha_0 y = 0,$$

where $\alpha_i \in \mathbb{R}/\{0\}$, for all $i = 0,1,...,n$ and the functions B_i depend on x,q,h and the parameter a

$$B_i := (x-q^i a - [i]h)(x-q^{i+1}a - [i+1]h)...(x-q^{n-1}a - [n-1]h).$$

Here, it is possible to show that $y = (x-a)^r_{q,h}$ is a solution provided that $r = n$ and

$$\sum_{i=0}^{n} \alpha_i P(n,i) = 0, \tag{20}$$

where $P(n,i)$ is the permutation coefficient defined in (19)*. The equation* (20) *can be considered as a characteristic equation which provides a relation between the coefficients α_i, $i = 0,1,..,n$.*

As another illustration, we present (q,h)-analogue of a partial difference equation. For that purpose, we first introduce partial delta and nabla (q,h)-derivatives.

Definition 3.7. *We define the partial delta (q,h)-derivative of $f(x,t) : \mathbb{T}_{(q,h)} \times \mathbb{T}_{(q,h)} \to \mathbb{R}$ as*

$$\partial^x_{(q,h)} f(x,t) := \frac{f(qx+h,t) - f(x,t)}{(q-1)x+h}, \tag{21}$$

and the partial nabla (q,h)-derivative of $f(x,t)$ as

$$\tilde{\partial}^x_{(q,h)} f(x,t) := \frac{f(x,t) - f(\frac{x-h}{q},t)}{x - (\frac{x-h}{q})}, \tag{22}$$

Example 3.8. *Consider the first order partial (q,h)-difference equation*

$$\partial^x_{(q,h)} u(x,t) + \tilde{\partial}^t_{(q,h)} u(x,t) = 0. \tag{23}$$

By the use of Proposition 3.5, one can show that the generalized quantum binomial $(x-t)^n_{q,h}$ satisfies

$$\partial^x_{(q,h)} (x-t)^n_{q,h} = -\tilde{\partial}^t_{(q,h)} (x-t)^n_{q,h} = [n](x-t)^{n-1}_{q,h}\ ,$$

which implies that

$$(\partial^x_{(q,h)} + \tilde{\partial}^t_{(q,h)})(x-t)^n_{q,h} = 0.$$

In ordinary calculus, a formal power series can be given as

$$f(x) = \sum_{i=0}^{\infty} c_i x^i, \tag{24}$$

which can be regarded as a polynomial of infinite degree in a way that we can differentiate or integrate on the series formally without questioning the convergence of the series. Inspired by (24), it is possible to introduce a formal power series in terms of generalized quantum binomial $(x-x_0)^i_{q,h}$ as follows

$$f(x) := \sum_{i=0}^{\infty} c_i (x-x_0)^i_{q,h}\ , \tag{25}$$

which can be considered as a polynomial of infinite degree. Therefore, we can operate on this series formally,

$$D_{(q,h)} f(x) = \sum_{i=0}^{\infty} c_i D_{(q,h)} \left((x-x_0)^i_{q,h} \right) = \sum_{i=1}^{\infty} c_i [i] (x-x_0)^{i-1}_{q,h}\ , \tag{26}$$

By a similar fashion, it is possible to represent the formal power series as

$$g(x-t) := \sum_{i=0}^{\infty} c_i (x-t)^i_{q,h}\ . \tag{27}$$

We conclude that $g(x-t)$ is a solution of the partial (q,h)-difference equation (23) which is in the form of a travelling wave with velocity 1. Note that by Remark 3.4, as $h \to 0$ the equation (23) reduces to $(\partial^x_q + \tilde{\partial}^t_q)u(x,t) = 0$, whose solution can be given in terms q−binomial as $g(x-t) = \sum_{i=0}^{\infty} c_i (x-t)^i_q$, while as $q \to 1$ the equation (23) reduces to $(\partial^x_h + \tilde{\partial}^t_h)u(x,t) = 0$ which admits a solution in terms h−binomial as $g(x-t) = \sum_{i=0}^{\infty} c_i (x-t)^i_h$. Furthermore as $(q,h) \to (1,0)$, the equation (23) reduces to $u_x + u_t = 0$. The power series solution derived from (27), $g(x-t) = \sum_{i=0}^{\infty} c_i (x-t)^i$ is in the integral surface as expected.

4. Generalized Quantum Exponential Function

It is possible to extend the Theorem 3.2, on the vector space of formal power series equipped with the same conditions. In this section, we first aim to present (q,h)-analogue of Taylor Series.

Theorem 4.1. *Any formal power series $f(x)$ can be written in terms of generalized quantum Taylor series about $x=0$*

$$f(x) = \sum_{i=0}^{\infty} (D^i_{(q,h)} f)(0) \frac{(x-0)^i_{q,h}}{[i]!}. \tag{28}$$

Proof. We seek for polynomials of order i with arbitrary constants c_i, in the form

$$P_i(x) = c_i (x-0)^i_{q,h},$$

such that P_i satisfies the conditions of Theorem 3.2. To fulfill $P_0(0) = 1$ we assume $P_0(x) = c_0 = 1$. Clearly, $P_1(x) = c_1(x-0)^1_{q,h}$ satisfies $P_1(0) = 0$. To have $D_{(q,h)} P_1(x) = P_0(x) = 1$, we get $c_1 = 1$. Processing in the same way, we obtain

$$P_2(x) := c_2 (x-0)^2_{q,h} = \frac{(x-0)^2_{q,h}}{[2]}.$$

Inductively, P_i yields as

$$P_i(x) := \frac{(x-0)^i_{q,h}}{[i]!},$$

which verifies the conditions $P_i(0) = 0$ for $i \geq 1$ and $D_{(q,h)} P_i(x) = P_{i-1}(x)$ by Proposition 3.5. Now for any formal power series $f(x)$, we have

$$f(x) := \sum_{i=0}^{\infty} a_i P_i(x),$$

where a_i are some constants. Since $P_0(0) = 1$ and $P_i(0) = 0$ for $i \geq 1$, we have $f(0) = a_0$. Acting $D_{(q,h)}$ on f, we obtain

$$D_{(q,h)} f(x) = \sum_{i=0}^{\infty} a_i D_{(q,h)} P_i(x) = \sum_{i=1}^{\infty} a_i P_{i-1}(x),$$

which implies that $D_{(q,h)} f(0) = a_1$ by using the conditions $P_0(0) = 1$ and $P_i(0) = 0$ for $i \geq 1$. Similarly acting $D^k_{(q,h)}$ on f, we get

$$D^k_{(q,h)} f(x) = \sum_{i=k}^{\infty} a_i P_{i-k}(x),$$

which implies $D^k_{(q,h)} f(0) = a_k$, $k \geq 0$. Therefore $f(x)$ yields as in (28). $\square$

In order to introduce (q,h)-analogue of an exponential function $f(x)$, we expect it to admit (q,h)-analogue of Taylor series expansion (28) and to satisfy

$$f(0) = 1, \quad D_{(q,h)} f(x) = \alpha f(x), \tag{29}$$

for any nonzero constant α. The conditions (29) imply that $D^2_{(q,h)}f(x) = \alpha^2 f(x)$ and thus $D^2_{(q,h)}f(0) = \alpha^2$. Thus, such $f(x)$ has to satisfy

$$D^i_{(q,h)}f(0) = \alpha^i, \quad \forall i \geq 0. \tag{30}$$

Hence, Theorem 4.1 and the constraint (30) enable us to determine f uniquely as

$$f(x) = \sum_{i=0}^{\infty} \frac{\alpha^i (x-0)^i_{q,h}}{[i]!}. \tag{31}$$

Moreover the series (31) is absolutely convergent with radius of converge $R = \infty$. Indeed, by utilizing Ratio test on (31) and triangle inequality, we conclude that

$$\begin{aligned}
\lim_{i\to\infty}\left|\frac{\alpha^{i+1}(x-0)^{i+1}_{q,h}}{[i+1]_q!}\cdot\frac{[i]!}{\alpha^i(x-0)^i_{q,h}}\right| &= \alpha \lim_{i\to\infty}\left|\frac{(x-(1+q+..+q^{i-1})h}{1+q+..+q^i}\right| \\
&\leq \alpha \lim_{i\to\infty}\left[\left|\frac{x}{1+q+..+q^i}\right| + \left|\frac{(1+q+..+q^{i-1})h}{1+q+..+q^i}\right|\right] \\
&\leq \alpha \lim_{i\to\infty}\left[\left|\frac{x}{1+q+..+q^i}\right| + h + \left|\frac{q^i h}{1+q+..+q^i}\right|\right] = 2\alpha h < 1,
\end{aligned}$$

provided that $q > 1$ and h is sufficiently small. One can show that the remainder of the series (31)

$$R_n(x) = \frac{D^{n+1}_{(q,h)}f(c)}{[n+1]!}\cdot(x-0)^{n+1}_{q,h} = \frac{\alpha^{n+1}f(c)}{[n+1]!}\cdot(x-0)^{n+1}_{q,h}, \qquad 0 < c < x,$$

converges to 0.

Definition 4.2. *We define generalized quantum exponential function, denoted by* $\exp_{(q,h)}(\alpha x)$ *as*

$$\exp_{(q,h)}(\alpha x) := \sum_{i=0}^{\infty} \frac{\alpha^i (x-0)^i_{q,h}}{[i]!}, \tag{32}$$

where α is arbitrary nonzero constant.

Clearly $\exp_{(q,h)}(0) = 1$. As $h \to 0$, the generalized quantum exponential function (32) becomes

$$\exp_{(q,0)}(\alpha x) = \sum_{i=0}^{\infty} \frac{\alpha^i (x-0)^i_q}{[i]!} = \sum_{i=0}^{\infty} \frac{(\alpha x)^i}{[i]!}. \tag{33}$$

In this case, if also $\alpha = 1$, (33) reduces to the so-called q-exponential function [24]

$$\exp_{(q,0)}(x) = \sum_{i=0}^{\infty} \frac{x^i}{[i]!}. \tag{34}$$

Note that, additive property for q-exponential identity does not hold

$$\exp_{(q,0)}(x)\cdot\exp_{(q,0)}(y) \neq \exp_{(q,0)}(x+y) \quad \text{if} \quad yx \neq qxy.$$

As $q \to 1$, the generalized quantum exponential function (32) implies

$$\exp_{(1,h)}(\alpha x) = \sum_{i=0}^{\infty} \frac{\alpha^i (x-0)^i_h}{i!} = \sum_{i=0}^{\infty}[\frac{x}{h}(\frac{x}{h}-1)\cdots(\frac{x}{h}-(i-1))]\frac{(\alpha h)^i}{i!} = \sum_{i=0}^{\infty}\binom{\frac{x}{h}}{i}(\alpha h)^i = (1+\alpha h)^{\frac{x}{h}}. \tag{35}$$

Unlike $\mathbb{K}_q$ case, additive property for h-exponential function

$$\exp_{(1,h)}(\alpha x) \cdot \exp_{(1,h)}(\alpha y) = (1+\alpha h)^{\frac{x}{h}}(1+\alpha h)^{\frac{y}{h}} = (1+\alpha h)^{\frac{x+y}{h}} = \exp_{(1,h)}(\alpha(x+y))$$

holds. Additionally, if $\alpha = 1$, (35) reduces to the so-called h-exponential function

$$\exp_{(1,h)}(x) = (1+h)^{\frac{x}{h}}. \tag{36}$$

Proposition 4.3. *The generalized quantum exponential function* (32) *obeys the chain rule*

$$D_{(q,h)} \exp_{(q,h)}(\alpha x) = \alpha \exp_{(q,h)}(\alpha x),$$

where α is a nonzero constant.

Proof. Consider the delta (q,h)-derivative of (32)

$$D_{(q,h)} \exp_{(q,h)}(\alpha x) = \sum_{i=0}^{\infty} \frac{\alpha^i}{[i]!} D_{(q,h)}(x-0)^i_{q,h} = \alpha \sum_{i=1}^{\infty} \frac{\alpha^{i-1}}{[i-1]!}(x-0)^{i-1}_{q,h} = \alpha \exp_{(q,h)}(\alpha x),$$

where we used the Proposition 3.5. □

Notice that, as $q \to 1$, we get

$$D_{(1,h)} \exp_{(1,h)}(\alpha x) = D_h(1+\alpha h)^{\frac{x}{h}} = \frac{(1+\alpha h)^{\frac{x+h}{h}} - (1+\alpha h)^{\frac{x}{h}}}{h} = \alpha(1+\alpha h)^{\frac{x}{h}}. \tag{37}$$

As $h \to 0$, we have

$$D_{(q,0)} \exp_{(q,0)}(\alpha x) = D_q \sum_{i=0}^{\infty} \frac{\alpha^i (x-0)^i_q}{[i]!} = \sum_{i=0}^{\infty} \frac{\alpha^i D_q(x^i)}{[i]!} = \sum_{i=1}^{\infty} \frac{\alpha^i x^{i-1}}{[i-1]!} = \alpha \cdot \exp_{(q,0)}(\alpha x).$$

Even though in [24], it is claimed that the chain rule fails for exponential function defined by (36), here we proved the Proposition 4.3 and therefore its reduction (37) is correct. Hence, rather than the h-exponential function defined by (36) in the literature, it is reasonable to introduce it in a more proper form

$$\exp_{(1,h)}(\alpha x) := (1+\alpha h)^{\frac{x}{h}}, \quad \alpha \in \mathbb{R},$$

so that the chain rule holds.
As an alternative point of view, one can introduce the generalized quantum exponential function on $\mathbb{T}_{(q,h)}$ as the solution of linear, homogenous (q,h)-initial value problem

$$D_{(q,h)} y(x) = \alpha y(x), \qquad \alpha \in \mathbb{R}/\{0\}, \tag{38}$$

$$y(x_0) = 1. \tag{39}$$

This iterative construction is inspired by the exponential function on time scales. For $\mathbb{T} = \mathbb{T}_{(q,h)}$, we present such construction in a detailed process.
Using the definition of delta (q,h)-derivative (7), we obtain the recurrence relation

$$y(qx+h) = \{1 + \alpha\,[(q-1)\,x + h]\} \cdot y(x). \tag{40}$$

If $x \in \mathbb{T}_{(q,h)}$ then $x = q^n x_0 + [n]h$ for some $n \in \mathbb{Z}$. Now we use the reduction formula (40) to obtain $y(x)$ as a function of x.
For $x = x_0$, using the initial condition (39) we have

$$\begin{aligned} y(qx_0 + h) &= \{1 + \alpha\,[(q-1)\,x_0 + h]\} \cdot y(x_0) \\ &= 1 + \alpha[(q-1)x_0 + h]. \end{aligned}$$

For $x = qx_0 + h$, we have

$$\begin{aligned} y(q^2x_0 + [2]h) &= \{1 + \alpha[(q-1)(qx_0 + h) + h]\} \cdot y(qx_0 + h) \\ &= (1 + \alpha q[(q-1)x_0 + h]) \cdot (1 + \alpha[(q-1)x_0 + h]). \end{aligned}$$

Similarly for $x = q^2x_0 + [2]h$, one can obtain

$$y(q^3x_0 + [3]h) = (1 + \alpha q^2[(q-1)x_0 + h]) \cdot (1 + \alpha q[(q-1)x_0 + h]) \cdot (1 + \alpha[(q-1)x_0 + h]).$$

If we continue this process, the solution of (38)-(39) can be found as

$$y(q^nx_0 + [n]h) = \prod_{i=0}^{n-1} \left(1 + \alpha q^i[(q-1)x_0 + h]\right)$$

As a result, we can construct a solution of the IVP (38)-(39) by

$$y(x) = \prod_{s \in (x_0, x)_{\mathbb{T}_{(q,h)}}} (1 + \alpha[(q-1)s + h]). \tag{41}$$

Being the solution of linear, first order, homogeneous (q, h)-IVP (38)-(39); the function (41) can be regarded as generalized quantum exponential function

$$e_\alpha(x, x_0) := \prod_{s \in (x_0, x)_{\mathbb{T}_{(q,h)}}} (1 + \alpha[(q-1)s + h]).$$

More generally, let us consider variable coefficient linear homogenous (q, h)-IVP

$$D_{(q,h)}y(x) = \alpha(x)y(x), \tag{42}$$
$$y(x_0) = 1. \tag{43}$$

Theorem 4.4. *Let $\alpha(x)$ be a function such that $\alpha(x) \neq \frac{-1}{q^ih}$ for all x and for all integer i. Let $x_0 \in \mathbb{T}_{(q,h)}$. Then*

$$e_{\alpha(x)}(x, x_0) := \prod_{s \in (x_0, x)_{\mathbb{T}_{(q,h)}}} (1 + \alpha(x)[(q-1)s + h]), \tag{44}$$

is the unique solution of the initial value problem (42)-(43) *on* $\mathbb{T}_{(q,h)}$.

Proof. Intuitively we may set

$$e_\alpha(x)(x_0, x_0) = \prod_{s \in (x_0, x_0)_{\mathbb{T}_{(q,h)}}} (1 + \alpha[(q-1)s + h]) = 1$$

to satisfy the initial condition (43). In order to verify (42), we apply delta (q, h)-derivative to (44),

$$\begin{aligned} D_{(q,h)}e_\alpha(x, x_0) &= \frac{e_\alpha(qx + h, x_0) - e_\alpha(x, x_0)}{(q-1)x + h} \\ &= \frac{\prod_{s \in (x_0, qx+h)_{\mathbb{T}_{(q,h)}}} (1 + \alpha[(q-1)s + h]) - \prod_{s \in (x_0, x)_{\mathbb{T}_{(q,h)}}} (1 + \alpha[(q-1)s + h])}{(q-1)x + h} \\ &= \frac{\prod_{s \in (x_0, x)_{\mathbb{T}_{(q,h)}}} (1 + \alpha[(q-1)s + h]) [1 + \alpha((q-1)x + h) - 1]}{(q-1)x + h} = \alpha e_\alpha(x, x_0). \end{aligned}$$

Now we prove the uniqueness of the solution. Assume that $\phi(x)$ is also a solution of (42)-(43). Then using the quotient rule, we have

$$\begin{aligned} D_{(q,h)}\left(\frac{\phi(x)}{e_{\alpha(x)}(x,x_0)}\right) &= \frac{e_{\alpha(x)}(x,x_0)\cdot D_{(q,h)}(\phi(x)) - \phi(x)\cdot D_{(q,h)}(e_{\alpha(x)}(x,x_0))}{e_{\alpha(x)}(qx+h,x_0)\cdot e_{\alpha(x)}(x,x_0)} \\ &= \frac{e_{\alpha(x)}(x,x_0)\cdot\alpha(x)\cdot\phi(x) - \phi(x)\cdot\alpha(x)\cdot e_{\alpha(x)}(x,x_0)}{e_{\alpha(x)}(qx+h,x_0)\cdot e_{\alpha(x)}(x,x_0)} = 0. \end{aligned}$$

Then by the Proposition 2.2, $\dfrac{\phi(x)}{e_\alpha(x,x_0)}$ is constant. On the other hand, by the virtue of (43), we have $\dfrac{\phi(x_0)}{e_\alpha(x_0,x_0)} = 1$ which implies that $\phi(x) = e_\alpha(x,x_0)$. To be more precise, solution of the IVP (42)-(43) is unique. □

Theorem 4.4 assures that the generalized quantum exponential functions (32) and (44) are equivalent for any constant α and $x_0 = 0$. Furthermore, the reductions obtained from both forms (32) and (44) of generalized quantum exponential functions are also equivalent. Let $q \to 1$, then $\mathbb{T}_{(q,h)}$ reduces to $h\mathbb{Z}$. In this case

$$e_\alpha(x,x_0) = \prod_{s\in(x_0,x)_{h\mathbb{Z}}}(1+\alpha h) = (1+\alpha h)^{\frac{x-x_0}{h}} \tag{45}$$

which is the exponential function on $h\mathbb{Z}$ [9]. Note that the reduction (45) is equivalent to (35) whenever $x_0 = 0$. Let $h \to 0$, then $\mathbb{T}_{(q,h)}$ reduces to $\mathbb{K}_q$. In this case

$$e_\alpha(x,x_0) = \prod_{s\in(x_0,x)_{\mathbb{K}_q}}(1+\alpha(q-1)s) \tag{46}$$

which is the exponential function on $\mathbb{K}_q$ [9]. Note that the reduction (46) is equivalent to (33) whenever $x_0 = 0$.

One can introduce the generalized trigonometric functions, as the solutions of linear, homogenous (q,h) initial value problems. For instance, the (q,h)-IVP

$$\begin{aligned} &D^2_{(q,h)}y(x) + y(x) = 0, \\ &y(0) = 0, \;\; D_{(q,h)}y(0) = 1, \end{aligned}$$

admits the unique solution $y(x) = \frac{1}{2i}(\exp_{(q,h)}(ix) - \exp_{(q,h)}(-ix))$ which can be introduced as generalized quantum sine function. By a similar fashion other generalized quantum trigonometric functions and hyperbolic functions can be defined.

Definition 4.5. *We introduce generalized quantum sine function, denoted by* $\sin_{(q,h)}(x)$, *as*

$$\sin_{(q,h)}(x) := \frac{\exp_{(q,h)}(ix) - \exp_{(q,h)}(-ix)}{2i}, \tag{47}$$

and generalized quantum cosine function, denoted by $\cos_{(q,h)}(x)$, *as*

$$\cos_{(q,h)}(x) := \frac{\exp_{(q,h)}(ix) + \exp_{(q,h)}(-ix)}{2}. \tag{48}$$

All other generalized quantum trigonometric functions can be defined analogously as

$$\tan_{(q,h)}(x) := \frac{\sin_{(q,h)}(x)}{\cos_{(q,h)}(x)}, \qquad \cot_{(q,h)}(x) := \frac{\cos_{(q,h)}(x)}{\sin_{(q,h)}(x)}, \tag{49}$$

$$\sec_{(q,h)}(x) := \frac{1}{\cos_{(q,h)}(x)}, \qquad \csc_{(q,h)}(x) := \frac{1}{\sin_{(q,h)}(x)}. \tag{50}$$

Definition 4.6. *We introduce generalized quantum hyperbolic functions,*

$$\sinh_{(q,h)}(x) := \frac{\exp_{(q,h)}(x) - \exp_{(q,h)}(-x)}{2}, \tag{51}$$

and

$$\cosh_{(q,h)}(x) := \frac{\exp_{(q,h)}(x) + \exp_{(q,h)}(-x)}{2}. \tag{52}$$

Proposition 4.7. *The generalized quantum trigonometric functions have the following properties*

$$D_{(q,h)} \sin_{(q,h)}(x) = \cos_{(q,h)}(x), \quad D_{(q,h)} \cos_{(q,h)}(x) = -\sin_{(q,h)}(x) \tag{53}$$

whereas the generalized quantum hyperbolic functions admit the properties

$$D_{(q,h)} \sinh_{(q,h)}(x) = \cosh_{(q,h)}(x), \quad D_{(q,h)} \cosh_{(q,h)}(x) = \sinh_{(q,h)}(x). \tag{54}$$

Proof is a direct consequence of the Proposition 4.3. Delta (q,h)-derivatives of other trigonometric functions follow immediately.

We emphasize that the generalized quantum trigonometric functions, hyperbolic functions and their properties constructed in the Proposition 4.7, recover standart versions, q-versions of such functions given in [24] and h-versions presented in [29] in the appropriate limits of q and h.

Example 4.8. *Consider the second order partial (q,h)-difference equation on $\mathbb{T}_{(q,h)} \times \mathbb{T}_{(\bar{q},\bar{h})}$*

$$(\partial^t_{(q,h)})^2 u(x,t) - c^2 (\partial^x_{(\bar{q},\bar{h})})^2 u(x,t) = 0, \quad c \in \mathbb{R}/\{0\}, \tag{55}$$

which can be regarded as generalized wave equation. Note that, the quantum parameters $q, \bar{q}$ and $h, \bar{h}$ need not to be equal. Let us apply the method of seperation of variables as $u(x,t) = f(x)g(t)$. We have

$$(\partial^t_{(q,h)})^2 u(x,t) = f(x) \cdot ((D^t_{(q,h)})^2 g(t)), \qquad (\partial^x_{(\bar{q},\bar{h})})^2 u(x,t) = g(t) \cdot ((D^x_{(\bar{q},\bar{h})})^2 f(x)),$$

which implies that

$$\frac{(D^x_{(\bar{q},\bar{h})})^2 f}{f} = \frac{(D^t_{(q,h)})^2 g}{c^2 g} = K, \tag{56}$$

where K is an arbitrary constant, since both sides of (56) *are seperated with respect to only one variable.* (56) *produces two ordinary (q,h)-difference equations*

$$(D^x_{(\bar{q},\bar{h})})^2 f - Kf = 0, \tag{57}$$

$$(D^t_{(q,h)})^2 g - Kc^2 g = 0. \tag{58}$$

Case (i) $K = 0$: In the light of Proposition 2.2, the equations (57) *and* (58) *imply that $f(x) = c_1 x + c_2$ and $g(t) = c_3 t + c_4$ respectively where $c_i, \ i = 1,2,3,4$ are arbitrary constants. Therefore solution of* (55) *can be presented as*

$$u(x,t) = \tilde{c_1} xt + \tilde{c_2} x + \tilde{c_3} t + \tilde{c_4}.$$

Case (ii) $K > 0$: In this case, starting with $f(x) = \exp_{(\bar{q},\bar{h})}(rx)$ for (57)*, the Proposition 4.3 enables to obtain the characteristic equation $r^2 - K = 0$, which has roots $r_{1,2} = \mp \sqrt{K}$. Then $f(x) = c_1 \exp_{(\bar{q},\bar{h})}(\sqrt{K}x) + c_2 \exp_{(\bar{q},\bar{h})}(-\sqrt{K}x)$. By a similar fashion,* (58) *implies that $g(t) = c_3 \exp_{(q,h)}(c\sqrt{K}t) + c_4 \exp_{(q,h)}(-c\sqrt{K}t)$. Hence, a solution of* (55) *is of the form*

$$u(x,t) = \left(\tilde{c_1} \cosh_{(\bar{q},\bar{h})}(\sqrt{K}x) + \tilde{c_2} \sinh_{(\bar{q},\bar{h})}(\sqrt{K}x)\right) \cdot \left(\tilde{c_3} \cosh_{(q,h)}(c\sqrt{K}t) + \tilde{c_4} \sinh_{(q,h)}(c\sqrt{K}t)\right).$$

Case (iii) $K<0$: The characteristic equation of (57) *is $r^2-K=0$, which has roots $r_{1,2}=\mp i\sqrt{-K}$. Therefore the solution of* (57) *can be found as*

$$f(x)=c_1\cos_{(\bar{q},\bar{h})}(\sqrt{-K}x)+c_2\sin_{(\bar{q},\bar{h})}(\sqrt{-K}x).$$

In a similar way, (58) *enables us to derive*

$$g(t)=c_3\cos_{(q,h)}(c\sqrt{-K}t)+c_4\sin_{(q,h)}(c\sqrt{-K}t).$$

Hence, we end up with

$$u(x,t)=\left(\tilde{c_1}\cos_{(\bar{q},\bar{h})}(\sqrt{-K}x)+\tilde{c_2}\sin_{(\bar{q},\bar{h})}(\sqrt{-K}x)\right)\cdot\left(\tilde{c_3}\cos_{(q,h)}(c\sqrt{-K}t)+\tilde{c_4}\sin_{(q,h)}(c\sqrt{-K}t)\right).$$

To sum up, we conclude that a general solution of (55) *arises as*

$$u(x,t)=\begin{cases}\tilde{c_1}xt+\tilde{c_2}x+\tilde{c_3}t+\tilde{c_4} & ,K=0;\\ \left(\tilde{c_1}\cosh_{(\bar{q},\bar{h})}(\sqrt{K}x)+\tilde{c_2}\sinh_{(\bar{q},\bar{h})}(\sqrt{K}x)\right)\cdot\left(\tilde{c_3}\cosh_{(q,h)}(c\sqrt{K}t)+\tilde{c_4}\sinh_{(q,h)}(c\sqrt{K}t)\right) & ,K>0;\\ \left(\tilde{c_1}\cos_{(\bar{q},\bar{h})}(\sqrt{-K}x)+\tilde{c_2}\sin_{(\bar{q},\bar{h})}(\sqrt{-K}x)\right)\cdot\left(\tilde{c_3}\cos_{(q,h)}(c\sqrt{-K}t)+\tilde{c_4}\sin_{(q,h)}(c\sqrt{-K}t)\right) & ,K<0.\end{cases}\tag{59}$$

Here the constants $\tilde{c_i}$'s in u_i depend on the arbitrary constants c_i, $i=1,2,3,4$.

Remark 4.9. *We notice that the equation* (55) *is a generic dynamic equation which produces various kinds of partial differential/ difference type of equations. Using various choices of limits $q\to1$, $h\to0$, $\bar{q}\to1$, $\bar{h}\to0$ in* (55)*, it is possible to obtain fifteen more different kinds of partial differential/ difference equations. For instance, as $h\to0$ and $\bar{q}\to1$, we obtain a q-difference-$\bar{h}$-difference equation*

$$(\partial_q^t)^2u(x,t)-c^2(\partial_{\bar{h}}^x)^2u(x,t)=0,$$

on $\mathbb{K}_q\times\bar{h}\mathbb{Z}$. Here, the solution arises as

$$u(x,t)=\begin{cases}\tilde{c_1}xt+\tilde{c_2}x+\tilde{c_3}t+\tilde{c_4} & ,K=0;\\ (\tilde{c_1}(1+\sqrt{K}\bar{h})^{x/\bar{h}}+\tilde{c_2}(1-\sqrt{K}\bar{h})^{x/\bar{h}})\cdot(\tilde{c_3}\cosh_q(c\sqrt{K}t)+\tilde{c_4}\sinh_q(c\sqrt{K}t)) & ,K>0;\\ (\tilde{c_1}(1+i\sqrt{-K}\bar{h})^{x/\bar{h}}+\tilde{c_2}(1-i\sqrt{-K}\bar{h})^{x/\bar{h}})\cdot(\tilde{c_3}\cos_q(c\sqrt{-K}t)+\tilde{c_4}\sin_q(c\sqrt{-K}t)) & ,K<0.\end{cases}$$

In this case if also $q\to1$, we derive a differential-$\bar{h}$-difference equation on $\mathbb{R}\times\bar{h}\mathbb{Z}$

$$(\partial^t)^2u(x,t)-c^2(\partial_{\bar{h}}^x)^2u(x,t)=0,$$

or if $\bar{h}\to0$, we have a differential-q-difference equation on $\mathbb{K}_q\times\mathbb{R}$

$$(\partial_q^t)^2u(x,t)-c^2(\partial^x)^2u(x,t)=0.$$

When $\bar{h}=h\to0$, the equation (55) *reduces to*

$$(\partial_q^t)^2u(x,t)-c^2(\partial_{\bar{q}}^x)^2u(x,t)=0,$$

on $\mathbb{K}_q\times\mathbb{K}_{\bar{q}}$ and its solution becomes

$$u(x,t)=\begin{cases}\tilde{c_1}xt+\tilde{c_2}x+\tilde{c_3}t+\tilde{c_4} & ,K=0;\\ \left(\tilde{c_1}\cosh_{\bar{q}}(\sqrt{K}x)+\tilde{c_2}\sinh_{\bar{q}}(\sqrt{K}x)\right)\cdot\left(\tilde{c_3}\cosh_q(c\sqrt{K}t)+\tilde{c_4}\sinh_q(c\sqrt{K}t)\right) & ,K>0;\\ \left(\tilde{c_1}\cos_{\bar{q}}(\sqrt{-K}x)+\tilde{c_2}\sin_{\bar{q}}(\sqrt{-K}x)\right)\cdot\left(\tilde{c_3}\cos_q(c\sqrt{-K}t)+\tilde{c_4}\sin_q(c\sqrt{-K}t)\right) & ,K<0.\end{cases}$$

As $\bar{q}=q\to1$, the equation (55) *produces*

$$(\partial_h^t)^2u(x,t)-c^2(\partial_{\bar{h}}^x)^2u(x,t)=0,$$

on $h\mathbb{Z}\times\bar{h}\mathbb{Z}$ and its solution is

$$u(x,t)=\begin{cases}\tilde{c_1}xt+\tilde{c_2}x+\tilde{c_3}t+\tilde{c_4} & ,K=0\\ (\tilde{c_1}(1+\sqrt{K}\bar{h})^{x/\bar{h}}+\tilde{c_2}(1-\sqrt{K}\bar{h})^{x/\bar{h}})(\tilde{c_3}(1+c\sqrt{K}h)^{t/h}+\tilde{c_4}(1-c\sqrt{K}h)^{t/h}) & ,K>0\\ (\tilde{c_1}(1+i\sqrt{-K}\bar{h})^{x/\bar{h}}+\tilde{c_2}(1-i\sqrt{-K}\bar{h})^{x/\bar{h}})(\tilde{c_3}(1+ic\sqrt{-K}h)^{t/h}+\tilde{c_4}(1-ic\sqrt{-K}h)^{t/h}) & ,K<0\end{cases}$$

All reductions under the appropriate limits generate classical wave equation. We emphasize that for the generalized quantum binomial, we face with

$$(x-0)^n_{q,h} \cdot (t-0)^n_{q,h} \neq (x+t)^n_{q,h}.$$

As a direct consequence, generalized quantum exponential function fails to admit the additivity property, i.e.,

$$\exp_{(q,h)}(x) \cdot \exp_{(q,h)}(t) \neq \exp_{(q,h)}(x+t).$$

This is the reason why we cannot express a solution of (55), in the form

$$f(x-ct)+g(x+ct),$$

where $f(x-ct) = \exp_{(q,h)}(x-ct)$ and $g(x+ct) = \exp_{(q,h)}(x+ct)$.

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