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Numerical Solutions of a System of Singularly Perturbed Reaction-Diffusion Problems

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Abstract. This paper addresses the numerical approximation of solutions to a coupled system of singularly perturbed reaction-diffusion equations. The components of the solution exhibit overlapping boundary and interior layers. Sinc procedure can control the oscillations in computed solutions at boundary layer regions naturally because the distribution of Sinc points is denser at near the boundaries. Also the obtained results show that the proposed method is applicable even for small perturbation parameter as $\epsilon = 2^{-30}$. The convergence analysis of proposed technique is discussed, it is shown that the approximate solutions converge to the exact solutions at an exponential rate. Numerical experiments are carried out to demonstrate the accuracy and efficiency of the method.

1. Introduction

Singularly perturbed problems arise in several branches of engineering and applied mathematics, including heat and mass transfer in chemical and nuclear engineering, linearized Navier-Stokes equation at high Reynolds number, control theory, etc.
In this article, we consider the following system of m coupled singularly perturbed reaction-diffusion equations:

$$\begin{cases} \mathbf{L}\mathbf{u} := -\epsilon\mathbf{u}'' + \mathbf{A}(x)\mathbf{u} = \mathbf{f}(x), \\ \mathbf{u}(0) = 0, \mathbf{u}(1) = 0, \qquad x \in (0,1), \end{cases} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, ..., u_m)^T$ and $\mathbf{f} = (f_1, f_2, ..., f_m)^T$ are column vectors, $\mathbf{A} = (a_{ij}(x))_{i,j=1}^{m}$ is an $m \times m$ matrix that entries of f_i and a_{ij} are assumed to lie in $C^2[0,1]$. In addition, ϵ is a small diffusion parameter whose presence makes a singularly perturbed system. We are interested in the singularly perturbed case where ϵ is much smaller than 1, in which case the solutions of these problems have boundary layers, which are rapid changes of the solution close to the boundary, near $x = 0$ and $x = 1$.
Coupled systems appear in many applications, notably turbulent interaction of waves and currents [35],

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diffusion processes in electroanalytic chemistry [33], optimal control and certain resistance-capacitor electrical circuits [15] and biological processes implications [2], [10], [11].
To satisfy the standard maximum principle chapter 1 [27] and [36], we assume that the coupling matrix $\mathbf{A}$ is a strictly diagonally dominant L_0-matrix(i.e., diagonal entries are positive and off-diagonal entries are non-positive) with

$$\min_{x\in[0,1],1\leq i\leq m}\left(\sum_{j=1}^{m} a_{ij}(x)\right)\geq\beta>0.$$

It is well known that the problems of the type (1) are difficult to solve efficiently, using standard numerical techniques when the diffusion parameters are very small. To obtain a reliable numerical solution for these problems, it is advantageous to use a mesh that concentrates the nodes inside the boundary layers. In this context, the Sinc method can control the oscillations in computed solutions at boundary layer regions naturally because the concentration of Sinc nods is denser at near the boundaries.
In recent years, various numerical methods have been developed for coupled system of singularly perturbed reaction-diffusion problems. Matthews et al. [22] provided a method for the numerical solution of system of equations of (1) on Shishkin mesh using classical finite difference scheme. Madden and Stynes [23] presented a uniformly convergent numerical method for system of reaction-diffusion BVPs. Lin and Stynes [19] gave a balanced finite element method based on piece-wise quadratic splines for a system of singularly perturbed reaction-diffusion two-point boundary value problems. Also, Chen et al. [8] derived collocation method for a coupled system of singularly perturbed linear equations, their method was based on rational spectral collocation method in barycentric form with sinh transform. Clavero et al. [9] developed an almost third order finite difference scheme on a piecewise uniform Shishkin mesh for singularly perturbed reaction-diffusion systems. In [20] for reaction-diffusion systems with an arbitrary number of equations, second order of uniform convergence of central differences scheme was proved by Linß and Madden. Das and Natesan [12] considered a uniformly convergent hybrid scheme for singularly perturbed system of reaction-diffusion based on cubic spline approximation. We can point out to many other efficient methods for solving system of singularly perturbed equations as [4], [6], [7], [13], [14], [16], [17], [18], [24], [34], [37] and [38].

In this paper, we apply the Sinc-Galerkin method to solve the coupled system of singularly perturbed reaction-diffusion problems. Sinc method has been studied extensively and found to be a very effective technique, particularly for problems with singular solutions and those on unbounded domain. Sinc method originally introduced by Stenger [31] which is based on the Whittaker-Shannon-Kotel' nikov sampling theorem for entire functions. The books [21] and [32] provide excellent overviews of the existing Sinc methods for solving ODEs and PDEs. The efficiency of the Sinc method has been formally proved by many researchers Bialecki [5], Bao et al [3] Babolian et al [1], Nurmuhammada et al. [25], Okayama et al. [26] and Rashidinia and Nabati [30].
Recently, Rashidinia et al. in [28] and [29] considered efficiency of Sinc method on singularly perturbed one-dimensional parabolic convection-diffusion problems that their solutions have oscillatory behavior at near the boundaries. In this work, we will present that the Sinc scheme be useful for the system of singularly perturbed reaction-diffusion equations too.
The paper is organized as follows: In section 2, we review some basic facts about the Sinc approximation. In section 3, the Sinc-Galerkin method is developed for solving of a coupled system of singularly perturbed reaction-diffusion equations, In section 4, the convergence analysis of proposed method is given. Some numerical examples will be presented in section 5, and at the end we conclude implementation, application and efficiency of proposed scheme.

2. Preliminaries

The goal of this section is to recall notations and definitions of the Sinc function and state some known theorems that are important for this paper.

The Sinc function is defined on $-\infty < x < \infty$ by

$$Sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For $h > 0$ we will denote the Sinc basis functions by

$$S(j,h)(x) = sinc(\frac{x - jh}{h}), \qquad j = 0, \pm 1, \pm 2, ...$$

let f be a function defined on $(-\infty, \infty)$ then for $h > 0$ the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh) S(j,h)(x),$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of Whittaker cardinal expansions have been studied and are thoroughly surveyed in Stenger [32]. These properties are derived in the infinite strip D_d of the complex plane where $d > 0$

$$D_d = \{\zeta = \xi + i\eta : |\eta| < d \leq \frac{\pi}{2}\}.$$

Approximations can be constructed for infinite, semi-finite, and finite intervals. But in this paper we construct approximation on the interval $(0, 1)$,we consider the conformal map

$$\phi(z) = \ln(\frac{z}{1-z}), \tag{2}$$

which maps the eye-shaped region

$$D_E = \{z = x + iy; |\arg(\frac{z}{1-z})| < d \leq \frac{\pi}{2}\},$$

onto the infinite strip D_d.
For the Sinc method, the basis functions on the interval $(0, 1)$ for $z \in D_E$ are derived from the composite translated Sinc function:

$$S_j(z) = S(j,h) \circ \phi(z) = sinc(\frac{\phi(z) - jh}{h}). \tag{3}$$

The function

$$z = \phi^{-1}(\omega) = \frac{e^{\omega}}{1 + e^{\omega}},$$

is an inverse mapping of $\omega = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\psi(u) = \phi^{-1}(u) \in D_E : -\infty < u < \infty\} = (0, 1).$$

The Sinc grid points $z_k \in (0, 1)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, ... \tag{4}$$

Definition 2.1 (Lund and Bowers [21]). *Let $B(D_E)$ is the class of functions f which are analytic in D_E such that*

$$\int_{\psi(u+\Sigma)} |f(z)|dz \to 0, \qquad as \quad u \to \pm\infty \tag{5}$$

where $\Sigma = \{i\eta : |\eta| < d \leq \frac{\pi}{2}\}$ and satisfy

$$\aleph(f) \equiv \int_{\partial D_E} |f(z)|dz < \infty, \tag{6}$$

where ∂D_E represents the boundary of D_E.

Definition 2.2 (Lund and Bowers [21]). *Let $L_\alpha(D_E)$ be the set of all analytic function u in D_E, for which there exists a constant C such that*

$$|u(z)| \leq C\frac{|\rho(z)|^\alpha}{[1+|\rho(z)|]^{2\alpha}}, \qquad z \in D_E, \quad 0 < \alpha \leq 1. \tag{7}$$

where $\rho(z) = e^{\phi(z)}$.

Theorem 2.3 (Stenger [32]). *If $\phi' \in B(D_E)$, then for all $x \in \Gamma$*

$$\Big|u(x) - \sum_{j=-N}^{N} u(x_j)S_j(x)\Big| \leq \frac{2\aleph(u\phi')}{\pi d}e^{-\pi d/h},$$

moreover, if $|u(x)| \leq c_1 e^{-\alpha|\phi(x)|}, x \in \Gamma$ for some positive constant C_1 and α, and also $h = \sqrt{\pi d/\alpha N}$ then

$$\sup_{x\in\Gamma}\Big|u(x) - \sum_{j=-N}^{N} u(x_j)S_j(x)\Big| \leq C_1 N^{1/2}\exp\Big(-(\pi d\alpha N)^{1/2}\Big),$$

where C_1 depends only on u, d and α.

Theorem 2.4 (Lund and Bowers [21]). *Let $F \in B(D_E)$ and ϕ be a conformal map with constants α and C_2 so that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \leq C_2\exp\Big(-\alpha|\phi(x)|\Big), \qquad x \in \Gamma,$$

by selecting $h = \sqrt{\pi d/\alpha N}$, then the Sinc trapezoidal quadrature rule is

$$\int_0^1 F(x)dx = h\sum_{j=-N}^{N}\frac{F(x_j)}{\phi'(x_j)} + o\Big(\exp\Big(-(\pi d\alpha N)^{1/2}\Big)\Big).$$

The Sinc-Galerkin method requires that the derivatives of composite Sinc function be evaluated at the nodes. We need to recall the following lemma.

Lemma 2.5 (Lund and Bowers [21]). *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (3). Then*

$$\delta_{jk}^{(0)} = [S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{8}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 0, & j=k, \\ \frac{(-1)^{(k-j)}}{k-j}, & j\neq k, \end{cases} \tag{9}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j=k, \\ \frac{-2(-1)^{(k-j)}}{(k-j)^2}, & j\neq k, \end{cases} \tag{10}$$

in relations (8-10) h is step size and x_k is Sinc grid given by (4).

It is convenient to define the following matrices:

$$\mathbf{I}^{(l)} = [\delta_{jk}^{(l)}], l = 0,1,2, \tag{11}$$

where $\delta_{jk}^{(l)}$ denotes the (j,k)th element of the matrix $\mathbf{I}^{(l)}$. Note that the matrix $\mathbf{I}^{(2)}$ and $\mathbf{I}^{(1)}$ are symmetric and skew-symmetric matrices respectively, also $\mathbf{I}^{(0)}$ is identity matrix.

3. The Sinc-Galerkin method

In this section, we apply the Sinc-Galerkin method for solution of coupled system of singularly perturbed linear equations (1) with the unknown vector function $\mathbf{u} = (u_1, u_2, ..., u_m)^T$. The m coupled system of singularly perturbed BVP (1) can be written in the following form as

$$\mathbf{L}u_i := -\epsilon u_i^{''} + \sum_{r=1}^{m} a_{ir}(x)u_r(x) = f_i(x), \tag{12}$$

$$u_i(0) = u_i(1) = 0, \qquad i = 1,2,...,m. \tag{13}$$

The approximate solution for $u_i(x)(i = 1,2,...,m)$ is represented by formula

$$u_i(x) \approx \bar{u}_i(x) = \sum_{j=-N}^{N} c_j^i S_j(x), \quad i = 1,2,...,m, \tag{14}$$

where $S_j(x)$ is function $S(j,h)\circ\phi(x)$ for some fixed step size h. The unknown coefficients c_j^i in relation (14) are determined by orthogonalizing the residual $\mathbf{L}\bar{u}_i - f_i(x)$ with respect to the basis function $\{S_k\}_{k=-N}^{N}$, i.e,

$$0 = \Big\langle L\bar{u}_i - f_i(x), S_k\Big\rangle = \Big\langle -\epsilon\bar{u}_i^{''}(x), S_k\Big\rangle + \Big\langle \sum_{r=1}^{m} a_{ir}(x)\bar{u}_r(x), S_k\Big\rangle - \Big\langle f_i(x), S_k\Big\rangle, i = 1,2,...,m \tag{15}$$

where $\Big\langle .,. \Big\rangle$ represents the inner product defined by

$$\langle f, \eta\rangle = \int_0^1 f(x).\eta(x)\omega(x)dx. \tag{16}$$

Using integrating by parts for the first integral term in the right hand side of (15) we have

$$\Big\langle -\epsilon\bar{u}_i^{''}(x), S_k\Big\rangle = B_T + \int_0^1 \bar{u}_i(x)\Big(-\epsilon S_k(x)\omega(x)\Big)^{''}dx, \tag{17}$$

$$B_T = \Big\{\bar{u}_i^{'}S_k\omega - \bar{u}_i(S_k\omega)^{'}\Big\}(x)\Big|_0^1.$$

Suppose that $B_T = 0$, then we apply the Sinc quadrature rule in Theorem 2 to the last two integrals in the right hand side of (15) and the integral in the right hand side of (17), we can obtain the following approximations:

$$\Big\langle -\epsilon\bar{u}_i''(x), S_k\Big\rangle \approx h\sum_{j=-N}^{N}\sum_{l=0}^{2}\frac{\bar{u}_i(x_j)}{\phi'(x_j)h^l}\delta_{kj}^{(l)}g_{2,l}(x_j), \tag{18}$$

$$\Big\langle \sum_{r=1}^{m} a_{ir}(x)\bar{u}_r(x), S_k\Big\rangle \approx h\sum_{r=1}^{m}\frac{a_{ir}(x_k)\bar{u}_r(x_k)\omega(x_k)}{\phi'(x_k)} \tag{19}$$

$$\Big\langle f_i, S_k\Big\rangle \approx h\frac{f_i(x_k)\omega(x_k)}{\phi'(x_k)}, \tag{20}$$

where

$$g_{2,2} = -\epsilon\omega(x)(\phi')^2(x), \quad g_{2,1} = -\epsilon\omega(x)\phi''(x) - 2\epsilon\omega'(x)\phi'(x), \quad g_{2,0} = -\epsilon\omega''(x)$$

The weight function $\omega(x)$ in the Sinc-Galerkin inner product (16) may be chosen for a variety of reasons. Although other reasons exist, a choice we make here is due to the requirement that the boundary terms B_T vanish. For the case of second-order problem in the Sinc-Galerkin method, a convenient choice for the weight function is given by Stenger [32] as

$$\omega(x) = \frac{1}{\phi'(x)}.$$

Replacing each terms of (15) with the approximations in (18-20), and replacing $u_i(x_j)$ by c_j^i and dividing by h, finally we obtain the discrete Sinc-Galerkin system for determination of the unknown coefficients $\left\{c_j^i\right\}_{j=-N}^{N}$ as

$$\sum_{j=-N}^{N}\left\{\sum_{l=0}^{2}\frac{1}{h^l}\delta_{kj}^{(l)}\frac{g_{2,l}(x_j)}{\phi'(x_j)}c_j^i\right\} + \sum_{r=1}^{m}\frac{a_{ir}(x_k)\omega(x_k)}{\phi'(x_k)}c_k^r = \frac{f_i(x_k)\omega(x_k)}{\phi'(x_k)}, \tag{21}$$

$$k = -N, -N+1, ..., N, \quad i = 1, 2, ..., m, \;\; M = 2N+1.$$

To obtain a matrix representation of the equations (21), we define the $M \times M$ diagonal matrix as follow:

$$\mathbf{D}\big(s(x)\big) = \begin{pmatrix} s(x_{-N}) & 0 & 0 & \dots & 0 \\ 0 & s(x_{-N+1}) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & & s(x_N) \end{pmatrix}$$

By using the above definitions and notations in (11), the system (21) can be represented by the following matrix form:

$$\mathbf{A}_i\mathbf{C}^i + \sum_{r=1}^{m}\mathbf{B}_{ir}\mathbf{C}^r = \mathbf{E}_i \qquad i = 1, ..., m, \tag{22}$$

where $\mathbf{C}^i$ and $\mathbf{E}_i$ are M-vector and $\mathbf{A}_i$ and $\mathbf{B}_{ir}$ are $M \times M$ matrices as:

$$\mathbf{A}_i = \epsilon \left\{ \frac{-1}{h^2}\mathbf{I}^{(2)} + \frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}\left(\frac{\phi''}{(\phi')^2}\right) + \mathbf{D}\left(\frac{-1}{\phi'}\left(\frac{1}{\phi'}\right)''\right)\right\},$$

$$\mathbf{B}_{ir} = \mathbf{D}\Big(\frac{a_{ir}}{(\phi')^2}\Big), \qquad \mathbf{E}_i = \mathbf{D}\Big(\frac{1}{(\phi')^2}\Big).\mathbf{F}_i,$$

$$\mathbf{F}_i = \Big(f_i(x_{-N}), f_i(x_{-N+1}), ..., f_i(x_N)\Big)^T,$$

$$\mathbf{C}^i = \Big(c^i_{-N}, c^i_{-N+1}, ..., c^i_N\Big)^T.$$

So that the system of equations in (22) is a system of linear equations with $m \times M$ equations in $m \times M$ unknowns, the coefficient matrix of the system (22) can be denoted by the following block matrix:

$$\mathcal{A} = \begin{pmatrix} \mathbf{A}_1 + \mathbf{B}_{11} & \mathbf{B}_{12} & ... & \mathbf{B}_{1m} \\ \mathbf{B}_{21} & \mathbf{A}_2 + \mathbf{B}_{22} & ... & \mathbf{B}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_{m1} & \mathbf{B}_{m2} & ... & \mathbf{A}_m + \mathbf{B}_{mm} \end{pmatrix}, \tag{23}$$

the arising block linear system can be denoted as

$$\mathcal{A}\mathbf{C} = \mathbf{P}, \tag{24}$$

where:

$$\mathbf{P} = \Big(\mathbf{F}_1, \mathbf{F}_2, ..., \mathbf{F}_m\Big)^T,$$
$$\mathbf{C} = \Big(\mathbf{C}^0, \mathbf{C}^1, ..., \mathbf{C}^m\Big)^T.$$

Remark 3.1. *Notice that the approach in the Sinc method is to select the basis functions so that each S_j in (14) satisfies the homogeneous boundary conditions, if instead of homogeneous boundary conditions given in (1), the following nonhomogeneous boundary conditions are specified*

$$\boldsymbol{u}(0) = \boldsymbol{p}, \qquad \boldsymbol{u}(1) = \boldsymbol{q},$$

then we reformulate the problem (1) by applying the following transformation

$$\boldsymbol{v}(x) = \boldsymbol{u}(x) + (x-1)\boldsymbol{p} - x\boldsymbol{q},$$

to convert the boundary conditions to homogeneous.

4. Convergence analysis

In this section, we show that the approximate solution $\overline{u}_i(x)$ given in (14) converges to the exact solution $u_i(x)$ of (12). In order to establish a bound of $|u_i(x) - \overline{u}_i(x)|$ for $i = 1, 2, ..., m$, we first need to get a bound of $\|(\mathcal{A}\widetilde{\mathbf{C}} - \mathbf{P})^i\|_2$ which $(\mathcal{A}\widetilde{\mathbf{C}} - \mathbf{P})^i$ is yield from i–th block of system $\mathcal{A}\widetilde{\mathbf{C}} - \mathbf{P}$ which $\widetilde{\mathbf{C}}$ is a vector defined by

$$\widetilde{\mathbf{C}} = \Big(\widetilde{\mathbf{C}}^1, \widetilde{\mathbf{C}}^2, ..., \widetilde{\mathbf{C}}^m\Big)^T, \tag{25}$$

where

$$\widetilde{\mathbf{C}}^i = \Big(u_i(x_{-N}), u_i(x_{-N+1}), ..., u_i(x_N)\Big)^T, \tag{26}$$

where $u_i(x_j)$ are exact value of solution (12) at the Sinc points. To this aim, we show the following lemma.

Lemma 4.1. *Suppose $\mathcal{A}$ and $\boldsymbol{P}$ are the same as those obtained in (24) and $\widetilde{\mathbf{C}}$ is defined in (25) , let $h = \sqrt{\pi d/\alpha N}$, and $\phi' \in B(D_E)$ for all $x \in \Gamma$. Then there exists a constant k_i independent of N such that*

$$\left\|(\mathcal{A}\widetilde{\mathbf{C}} - \boldsymbol{P})^i\right\|_2 \leq k_i N^{3/2} \exp\Big(-(\pi d\alpha N)^{1/2}\Big), \qquad i = 0, 1, ..., m \tag{27}$$

Proof. For simplicity, we denote $\lambda_k^i = \Big(\mathcal{A}\widetilde{\mathbf{C}} - \mathbf{P}\Big)_k^i$ for $k = -N, -N+1, ..., N$, we know that λ_k^i is k-th component the system (24) for i−th block. By using orthogonalizing the residual $Lu_i - f_i(x)$ with respect to the basis function $\{S_k\}_{k=-N}^{N}$ as in previous section and using Theorem 4.4 Lund and Bowers [21], we have :

$$\begin{aligned} 0 = \Big\langle Lu_i - f_i(x), S_k\Big\rangle \quad &= \quad h\sum_{j=-N}^{N}\sum_{l=0}^{2}\frac{u_i(x_j)}{\phi'(x_j)h^l}\delta_{kj}^{(l)}g_{2,l}(x_j) + L_2^i N\exp\Big(-(\pi d\alpha N)^{1/2}\Big) \\ &+h\sum_{r=1}^{m}\frac{a_{ir}(x_k)u_r(x_k)\omega(x_k)}{\phi'(x_k)} + L_1^i N^{-1/2}\exp\Big(-(\pi d\alpha N)^{1/2}\Big) \\ &+h\frac{f_i(x_k)\omega(x_k)}{\phi'(x_k)}) + L_0^i N^{-1/2}\exp\Big(-(\pi d\alpha N)^{1/2}\Big) = \lambda_k^i \\ &+L_2^i N\exp\Big(-(\pi d\alpha N)^{1/2}\Big) + L_1^i N^{-1/2}\exp\Big(-(\pi d\alpha N)^{1/2}\Big) + L_0^i N^{-1/2}\exp\Big(-(\pi d\alpha N)^{1/2}\Big), \end{aligned}$$

where L_2^i, L_1^i and L_0^i are constants independent of N.
Thus,

$$|\,\lambda_k^i| \leq K_i N\exp\Big(-(\pi d\alpha N)^{1/2}\Big),$$

where

$$K_i = L_2^i + L_1^i + L_0^i.$$

Therefore, we have

$$\begin{aligned} \left\|(\mathcal{A}\widetilde{\mathbf{C}} - \mathbf{P})^i\right\|_2 = \left(\sum_{k=-N}^{N}|\lambda_k^i|^2\right)^{1/2} &\leq \left(\sum_{k=-N}^{N}\Big(K_i N\exp\Big(-(\pi d\alpha N)^{1/2}\Big)\Big)^2\right)^{1/2} \\ &\leq k_i N^{3/2}\exp\Big(-(\pi d\alpha N)^{1/2}\Big). \end{aligned}$$

□

Theorem 4.2. *Suppose $u_i(x), i = 1, 2, ..., m$ are the exact solutions of (12) and $\overline{u}_i(x)$ are their Sinc approximations defined by (14), then, under the assumptions of Theorem 1 and Lemma 2, there exists a constant c independent of N such that*

$$\sup_{x\in\Gamma}\left|u_i(x) - \overline{u}_i(x)\right| \leq cN^{\frac{3}{2}}\exp\Big(-(\pi\alpha d N)^{1/2}\Big), \quad i = 1, 2, ..., m. \tag{28}$$

Proof. Suppose analytic solutions of (12) at Sinc points x_j, $j = -N, ..., N$ are denoted by $\widehat{u_i}(x)$ and defined as

$$\widehat{u_i}(x) = \sum_{j=-N}^{N} u_i(x_j)S_j(x), \quad i = 1, 2, ..., m, \tag{29}$$

then by making use of the triangular inequality we have

$$\left|u_i(x) - \overline{u}_i(x)\right| \leq \left|u_i(x) - \widehat{u_i}(x)\right| + \left|\widehat{u_i}(x) - \overline{u}_i(x)\right|. \tag{30}$$

By using Theorem 1, there exists a constant c_2 independent of N such that

$$\sup_{x\in\Gamma}\left|u_i(x) - \widehat{u_i}(x)\right| \leq c_2 N^{\frac{1}{2}}\exp\Big(-(\pi\alpha d N)^{1/2}\Big). \tag{31}$$

Also by using Schwarz inequality, the second term in the right hand side of (30) satisfies

$$\left|\widehat{u}_i(x) - \overline{u}_i(x)\right| \leq \left|\sum_{j=-N}^{N} (u_i(x_j) - c^i_j) S_j(x)\right| \leq \left(\sum_{j=-N}^{N} \left|u_i(x_j) - c^i_j\right|^2\right)^{\frac{1}{2}} \left(\sum_{j=-N}^{N} \left|S_j(x)\right|^2\right)^{\frac{1}{2}} = R, \tag{32}$$

we know that $\left(\sum_{j=-N}^{N} \left|S_j(x)\right|^2\right)^{\frac{1}{2}} \leq c_3$ where c_3 is a constant, then by using lemma 4.1 and (24), we have

$$R \leq c_3 \left\|\widetilde{\mathbf{C}}^i - \mathbf{C}^i\right\|_2 \leq c_3 \left\|\mathcal{A}^{-1}\right\|_2 \left\|\mathcal{A}\widetilde{\mathbf{C}} - \mathbf{P}\right\|_2 \leq$$

$$\leq c_3 \left\|\mathcal{A}^{-1}\right\|_2 \sum_{i=1}^{m} \left\|(\mathcal{A}\widetilde{\mathbf{C}} - \mathbf{P})^i\right\|_2 \leq c_3 \widehat{k} \left\|\mathcal{A}^{-1}\right\|_2 N^{\frac{3}{2}} \exp\Big(-(\pi\alpha d N)^{1/2}\Big), \tag{33}$$

where, $\widehat{k} = k_1 + k_2 + ... + k_m$.
Finally, by applying relations (30-33) we can obtain

$$\sup_{x\in\Gamma} \left|u_i(x) - \overline{u}_i(x)\right| \leq c N^{\frac{3}{2}} \exp\Big(-(\pi\alpha d N)^{1/2}\Big), \tag{34}$$

where $c = \max\left\{c_2, c_3\widehat{k}\|\mathcal{A}^{-1}\|_2\right\}$ is a constant. □

5. Numerical experiments

In this section we illustrate the applications of the presented method on the following three test examples. In all of the examples considered in this paper, we choose $\alpha = 1$ and $d = \frac{\pi}{2}$ which yield $h = \frac{\pi}{\sqrt{2N}}$. Also the maximum pointwise errors are reported on uniform grids

$$U = \{z_0, z_1, ..., z_p\}, \quad z_r = \frac{r}{p}, \quad r = 0, 1, ..., p. \tag{35}$$

Example 1. Consider the following coupled system of reaction-diffusion:

$$\begin{cases} -\epsilon u_1''(x) + 2(x+1)^2 u_1(x) - (x^3+1) u_2(x) = 2e^x, \qquad 0 < x < 1, \\ -\epsilon u_2''(x) - 2\cos(\pi x/4) u_1(x) + 2.2 e^{(-x+1)} u_2(x) = 10x + 1, \end{cases} \tag{36}$$

with boundary conditions

$$u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0.$$

The exact solution to this problem is unknown. So the accuracy of its numerical solution will be computed using double mesh principle, therefore for each ϵ the maximum pointwise errors are estimated as

$$E^M_{i,\epsilon} = \max_r |\overline{u}^M_i(z_r) - \overline{u}^{2M}_i(z_r)|, \qquad i = 1, 2, \tag{37}$$

where $\overline{u}^M_i$ be the approximation solution by our numerical process for number of M Sinc points.
For this example, the maximum pointwise errors are tabulated in table 1 for various values of ϵ and N. These results verify efficiency and accuracy of the proposed method. It can be seen that the errors of proposed method are related to value of the parameter ϵ, of course, this dependency does not have a large impact on the performance of our method. From table 1, the monotonically decreasing behavior of errors

can be observed as N increases.
For this example, the graphs of the computed solutions u_1 and u_2 are given in Figs. 1 and 2 for different values of ϵ using $N = 64$. In these figures, it can be seen that the boundary layers are located at both boundaries $x = 0$ and $x = 1$ especially for small values of ϵ. Also, the convergence curves are plotted for various values of ϵ in Fig. 3. This figure shows that the treatment of maximum errors is exponential with increasing N and verifies the theoretical results. Of course, with decreasing perturbation parameter as $\epsilon = 2^{-30}$ this behavior is almost near to exponential.

Table 1: The maximum pointwise errors for example 1 for various values of ϵ and N with $p = 2N$.

	N=16		N=32		N=64		N=128	
ϵ	$E^M_{1,\epsilon}$	$E^M_{2,\epsilon}$	$E^M_{1,\epsilon}$	$E^M_{2,\epsilon}$	$E^M_{1,\epsilon}$	$E^M_{2,\epsilon}$	$E^M_{1,\epsilon}$	$E^M_{2,\epsilon}$
2^0	1.22e-2	3.22e-2	3.83e-4	1.03e-4	2.37e-6	8.78e-6	1.63e-9	7.34e-8
2^{-2}	2.13e-2	4.01e-2	4.21e-4	7.46e-4	3.41e-6	1.41e-5	2.21e-9	6.24e-8
2^{-6}	2.89e-2	5.12e-2	5.03e-4	7.90e-4	4.12e-6	7.14e-6	2.65e-8	7.43e-8
2^{-10}	3.01e-2	6.31e-2	6.21e-4	8.41e-4	4.21e-6	8.12e-6	3.01e-8	7.81e-8
2^{-14}	3.72e-2	7.10e-2	7.01e-4	8.81e-4	5.02e-6	8.62e-6	3.64e-8	8.14e-8
2^{-18}	4.21e-2	8.02e-2	8.12e-4	9.21e-4	6.42e-6	9.12e-6	4.63e-8	9.13e-8
2^{-22}	5.71e-2	8.60e-2	8.70e-4	9.55e-4	7.18e-6	1.21e-5	6.10e-8	1.04e-7
2^{-26}	6.16e-2	9.11e-2	9.13e-4	2.31e-3	8.41e-6	4.21e-5	8.31e-8	4.32e-6
2^{-30}	8.12e-2	2.31e-1	2.31e-3	5.28e-3	2.64e-5	8.52e-5	3.51e-7	7.22e-6

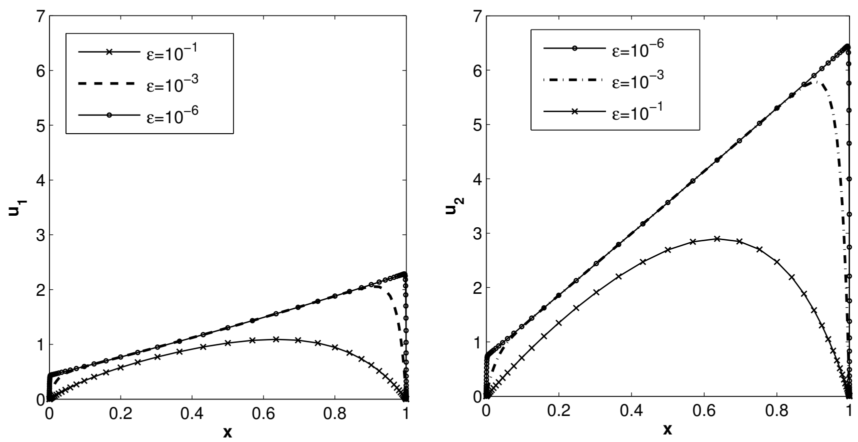

Figure 1: Numerical solution profiles of Example 1 for various values of ϵ with $N = 64$.

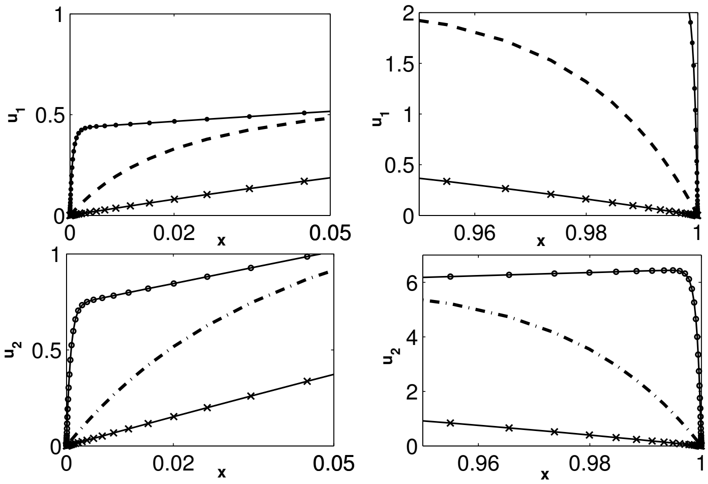

Figure 2: Numerical solutions of Example 1 near boundary layers regions

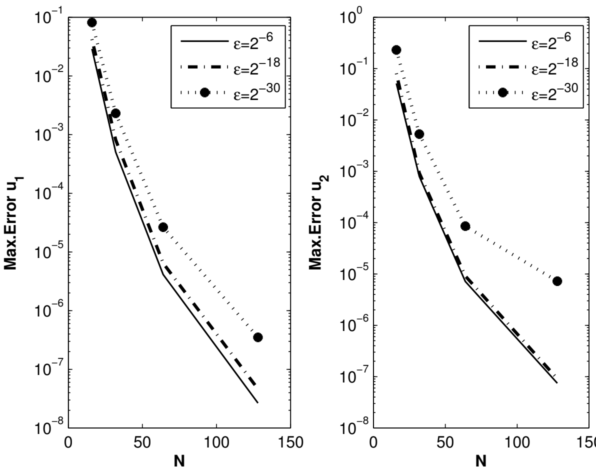

Figure 3: Convergence curves of the method for example 1 for various values of ϵ.

Example 2. Consider the following system of reaction-diffusion with constant coefficients:

$$\begin{cases} -\epsilon u_1''(x) + u_1(x) - 0.5u_2(x) = f_1(x), \qquad 0 < x < 1, \\ -\epsilon u_2''(x) - 2u_1(x) + 4u_2(x) = f_2(x), \end{cases} \tag{38}$$

The right-hand-side and the boundary conditions are such that the exact solution of problem is given by

$$u_1(x) = h_1(x)/k_1 + h_2(x)/k_2 - x + x^2 + \cos^2 \pi x,$$

$$u_2(x) = h_1(x)/k_1 - h_2(x)/k_2 + \sin \pi x,$$

where

$$h_1(x) = \exp\Big(\frac{-x}{\sqrt{\epsilon}}\Big) + \exp\Big(\frac{-(1-x)}{\sqrt{\epsilon}}\Big), \qquad h_2(x) = \exp\Big(\frac{-2x}{\sqrt{\epsilon}}\Big) + \exp\Big(\frac{-2(1-x)}{\sqrt{\epsilon}}\Big),$$

with

$$k_1 = \exp\Big(\frac{-1}{\sqrt{\epsilon}}\Big) + 1, \qquad k_2 = \exp\Big(\frac{-2}{\sqrt{\epsilon}}\Big) + 1.$$

Since we have an analytical solution for this problem the maximum pointwise errors can be calculated as

$$\overline{E}_{i,\epsilon}^{M} = \max_{r} \left| u_i(z_r) - \overline{u}_i^M(z_r) \right|, \qquad i = 1, 2, \tag{39}$$

where $\overline{u}_i^M$ be the computed solution by our scheme for number of M Sinc points.
The maximum pointwise errors for this example are given in table 2 for various values of ϵ and N. The obtained results show that the errors decrease with increasing N and the errors increase almost with decreasing perturbation parameter. Although, this table shows that the proposed method is applicable even for small perturbation parameter as $\epsilon = 2^{-30}$.
The graph of approximate solutions are represented in Fig. 4 for $\epsilon = 10^{-2}, 10^{-5}$ and $\epsilon = 10^{-8}$. This figure shows that there are no any the boundary layers for large value of ϵ as 10^{-2}, but boundary layers are located at both sides of domain for small values of ϵ, which validates the physical behavior of the solution. Also, for this problem the maximum errors for various values of ϵ are plotted in Fig. 5. This figure indicates that the maximum pointwise errors decrease at an exponential rate with respect to N especially for $\epsilon = 2^{-6}$.

Table 2: The maximum pointwise errors for example 2 for various values of ϵ and N with $p = 2N$.

	N=16		N=32		N=64		N=128	
ϵ	$\overline{E}_{1,\epsilon}^M$	$\overline{E}_{2,\epsilon}^M$	$\overline{E}_{1,\epsilon}^M$	$\overline{E}_{2,\epsilon}^M$	$\overline{E}_{1,\epsilon}^M$	$\overline{E}_{2,\epsilon}^M$	$\overline{E}_{1,\epsilon}^M$	$\overline{E}_{2,\epsilon}^M$
2^{0}	8.12e-3	4.40e-3	2.43e-5	7.23e-5	7.12e-7	6.31e-7	5.21e-8	4.34e-9
2^{-2}	9.03e-3	5.01e-3	6.21e-5	8.31e-5	3.41e-6	6.21e-7	8.76e-8	5.64e-9
2^{-6}	1.91e-2	7.02e-3	3.03e-4	9.52e-5	5.84e-6	8.41e-7	9.55e-8	7.12e-9
2^{-10}	4.01e-2	8.01e-3	5.26e-4	1.21e-4	6.71e-6	9.13e-7	1.73e-7	8.21e-9
2^{-14}	5.23e-2	9.10e-3	8.11e-4	3.01e-4	7.62e-6	1.22e-6	4.04e-7	9.64e-9
2^{-18}	7.51e-2	2.32e-2	9.42e-4	5.11e-4	9.02e-6	2.31e-6	6.13e-7	2.10e-8
2^{-22}	9.01e-2	4.20e-2	1.30e-3	7.63e-4	2.12e-5	4.21e-6	9.81e-7	6.01e-8
2^{-26}	1.26e-1	7.01e-2	4.43e-3	1.01e-3	5.21e-5	8.51e-6	3.21e-6	4.32e-7
2^{-30}	4.02e-1	2.10e-1	3.10e-2	6.18e-3	1.14e-4	3.12e-5	2.41e-5	6.14e-6

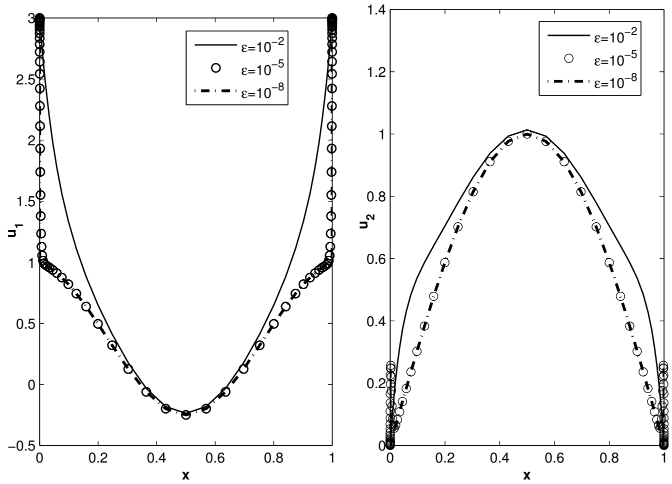

Figure 4: Numerical solution profiles of Example 2 for various values of ϵ with $N = 64$.

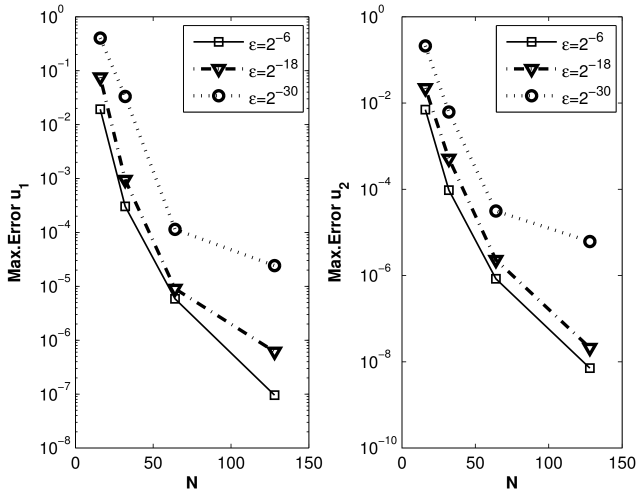

Figure 5: Convergence curves of the method for example 2 for various values of ϵ.

Example 3. Consider the following system of reaction-diffusion equations with variable coefficients:

$$\begin{cases} -\epsilon u_1'' + 2(x+1)^2 u_1 - (x^3+1)u_2 - 0.1u_3 - 0.2u_4 = 2 + x, \qquad 0 < x < 1, \\ -\epsilon u_2'' - 2\cos(\pi x/4)u_1 + (2 + \sqrt{2})e^{(-x+1)}u_2 - 0.2u_3 - 0.1u_4 = 1, \\ -\epsilon u_3'' - 2\cos(\pi x/4)u_1 - 0.5(x+1)^2 u_2 + 4.8e^{(-x+1)}u_3 - \cos(\pi/5)u_4 = 2e^x \\ -\epsilon u_4'' - (x^3+1)u_1 - 0.1u_2 - 0.2u_3 + 3(x+1)^3 u_4 = 0.1 \end{cases} \tag{40}$$

with boundary conditions

$$u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0, u_3(0) = u_3(1) = 1, u_4(0) = u_4(1) = 2$$

The exact solution to this problem is unknown, therefore the maximum pointwise errors are estimated as (37) in example 1. Tables 3 and 4 display the obtained results for this example, From these results we can see accuracy of method and the errors decrease with increasing N, for $N \geq 64$ the magnitude of errors decrease slightly especially for u_3 and u_4 which reason of it be computational complexity.
Fig. 6 shows the numerical solution profiles for $\epsilon = 10^{-2}$ and 10^{-6} with $N = 64$. This figure clearly indicates boundary layer is located at the both sides of the domain for small value of $\epsilon = 10^{-6}$. The convergence curve for the Sinc method is plotted for $\epsilon = 2^{-6}, 2^{-18}$ and 2^{-30} in Fig 7. Unlike the Figs 3 and 5 for examples 1 and 2, this figure state that the treatment of error is almost near to exponential and for $N \geq 64$ this behavior is not exponential especially for u_3 and u_4.

Table 3: The maximum pointwise errors for example 3 for various values of ϵ and N with $p = 2N$.

	N=16				N=32			
ϵ	$E^M_{1,\epsilon}$	$E^M_{2,\epsilon}$	$E^M_{3,\epsilon}$	$E^M_{4,\epsilon}$	$E^M_{1,\epsilon}$	$E^M_{2,\epsilon}$	$E^M_{3,\epsilon}$	$E^M_{4,\epsilon}$
2^{0}	3.02e-3	5.20e-3	2.12e-2	3.54e-2	2.61e-4	5.15e-4	8.11e-3	7.14e-3
2^{-2}	4.12e-3	6.01e-3	2.64e-2	3.86e-2	3.11e-4	5.51e-4	8.53e-3	7.89e-3
2^{-6}	5.01e-3	6.91e-3	3.01e-2	4.32e-2	3.74e-4	5.95e-4	8.85e-3	8.10e-3
2^{-10}	5.60e-3	7.25e-3	3.36e-2	4.70e-2	4.01e-4	6.20e-4	8.96e-3	8.21e-3
2^{-14}	5.81e-3	7.50e-3	3.71e-2	4.82e-2	4.42e-4	6.48e-4	9.04e-3	8.34e-3
2^{-18}	6.01e-3	7.82e-3	4.10e-2	5.00e-2	4.75e-4	6.90e-4	9.13e-3	8.48e-3
2^{-22}	7.21e-3	8.10e-3	4.71e-2	5.63e-2	5.23e-4	7.11e-4	9.20e-3	8.65e-3
2^{-26}	8.62e-3	8.82e-3	4.91e-2	6.31e-2	5.81e-4	7.75e-4	9.41e-3	8.92e-3
2^{-30}	1.95e-2	2.10e-2	7.21e-2	8.18e-2	7.24e-4	6.02e-3	1.41e-2	9.54e-3

Table 4: The maximum pointwise errors for example 3 for various values of ϵ and N with $p = 2N$.

	N=64				N=128			
ϵ	$E^M_{1,\epsilon}$	$E^M_{2,\epsilon}$	$E^M_{3,\epsilon}$	$E^M_{4,\epsilon}$	$E^M_{1,\epsilon}$	$E^M_{2,\epsilon}$	$E^M_{3,\epsilon}$	$E^M_{4,\epsilon}$
2^{0}	4.12e-6	5.11e-6	5.22e-4	6.12e-4	2.17e-7	1.10e-7	1.01e-4	2.14e-4
2^{-2}	4.78e-6	6.20e-6	5.74e-4	7.06e-4	5.61e-7	6.14e-7	2.13e-4	3.19e-4
2^{-6}	6.11e-6	8.21e-6	7.71e-4	8.17e-4	7.14e-7	8.15e-7	3.35e-4	3.71e-4
2^{-10}	7.11e-6	9.15e-6	8.31e-4	8.90e-4	8.11e-7	9.21e-7	4.86e-4	4.01e-4
2^{-14}	8.11e-6	1.80e-5	4.71e-3	3.82e-3	9.21e-7	3.11e-6	1.09e-3	1.01e-3
2^{-18}	1.21e-5	2.62e-5	4.80e-3	4.00e-3	2.15e-6	4.31e-6	1.17e-3	1.20e-3
2^{-22}	2.41e-5	3.17e-5	4.91e-3	4.43e-3	3.24e-6	5.63e-6	1.72e-3	1.64e-3
2^{-26}	4.12e-5	5.03e-5	5.01e-3	4.62e-3	4.55e-6	7.15e-6	1.86e-3	1.85e-3
2^{-30}	5.95e-5	6.64e-5	5.15e-3	4.77e-3	5.35e-6	8.11e-6	1.97e-3	2.03e-3

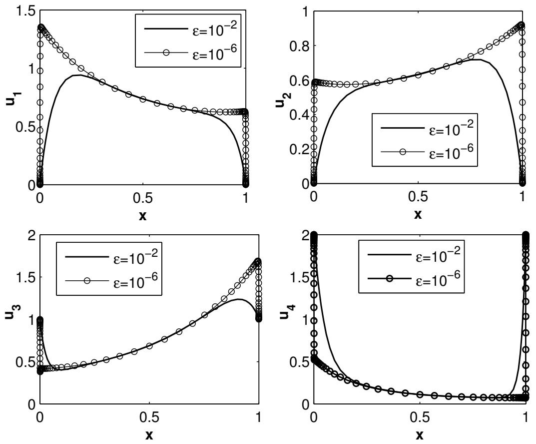

Figure 6: Numerical solution profiles of Example 3 for various values of ϵ with $N = 64$.

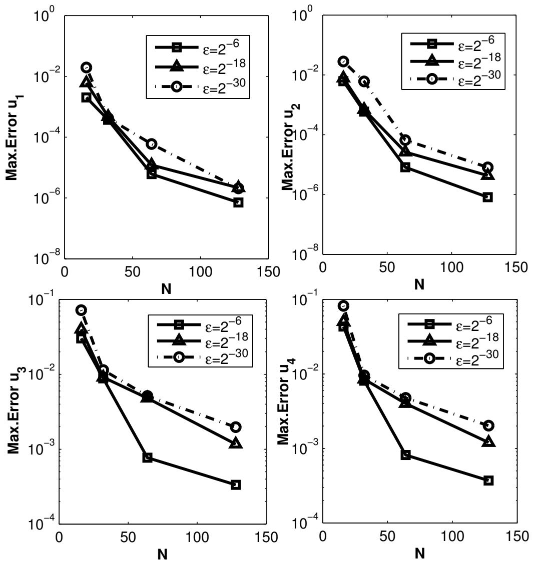

Figure 7: Convergence curves of the method for example 3 for various values of ϵ.

6. Conclusions

In this article, we developed a numerical method to approximate the solutions of a system of singularly perturbed reaction-diffusion equations (1), based on the Sinc-Galerkin method. The error analysis for the numerical solution is presented and an exponential convergence is carried out. To examine the accuracy and efficiency of the proposed algorithm, we give three numerical examples. The errors are summarized in tables and figures that verified the efficiency and validly of our presented scheme. Figs. 1,2,4 and 6 show that the boundary layers are appeared at both sides of domain for small values of perturbation parameter. Moreover, from the figures 3,5 and 7, we get some useful information about the convergence.

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