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Generalizations of Killing Vector Fields in Sol Space

Zlatko Erjavec[a]

[a]*Faculty of Organization and Informatics, University of Zagreb, Pavlinska 2, HR-42000, Varaždin, Croatia*

Abstract. We consider two generalizations of the Killing vector fields in the 3D Sol space. Conformal Killing vector fields are the first generalization, 2-Killing vector fields are the second. We characterize proper conformal Killing vector fields and determine some proper 2-Killing vector fields in Sol space.

1. Introduction

Killing vector field on Riemannian manifold (M, g) is a vector field X which satisfies the Killing equation $\mathfrak{L}_X g = 0$, where $\mathfrak{L}$ denotes a Lie derivative. The Killing equation expresses that a metric of Riemannian manifold is invariant under the vector field X. Killing vector field flows preserve shapes and sizes and they are manifestations of symmetries in the context of general relativity. Also, conformal Killing vector fields have been relevant in many problems in space time geometry and homothetic vector fields especially so.

This paper studies the generalizations of the Killing vector fields in the 3D Sol space. Conformal Killing vector fields are the first generalization. They are defined by the conformal Killing equation $\mathfrak{L}_X g = \lambda g$, where λ is a smooth function on M. 2-Killing vector fields defined by the 2-Killing equation $\mathfrak{L}_X(\mathfrak{L}_X g) = 0$ are the second generalization.

In this paper we characterize proper conformal Killing vector fields and determine some proper 2-Killing vector fields in Sol space. It seems that there are no proper conformal Killing vector field in Sol space. We suppose that this is related to the absence of rotational isometry in Sol space. On the other hand, we explore the proper 2-Killing vector fields in Sol space and it seems that the approach used here can be starting point for classification of 2-Killing vector fields in other 3D homogeneous Riemannian geometries. Our choice of Sol as the ambient space is not arbitrary because among all eight 3D homogeneous geometries Sol has lowest, three dimensional, isometry group. Hence only three basic Killing vector fields exist.

2. Preliminaries

2.1. *Sol space*

We recall some relevant facts on Sol_3 space.

The model space Sol in the sense of W. Thurston [18] is the Cartesian space $\mathbb{R}^3(x, y, z)$ equipped with a homogeneous Riemannian metric (see [17])

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2. \tag{1}$$

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Email address: zlatko.erjavec@foi.hr (Zlatko Erjavec)

The Sol space is a Lie group G with respect to the multiplication law

$$(x, y, z) * (a, b, c) = (x + e^{-z}a, y + e^{z}b, z + c).$$

The left invariant orthonormal frame field, i.e. the basis of the Sol space, is given by

$$e_1 = e^{-z}\frac{\partial}{\partial x},\;\; e_2 = e^{z}\frac{\partial}{\partial y},\;\; e_3 = \frac{\partial}{\partial z}. \tag{2}$$

The Levi-Civita connection ∇ of Sol space is given by

$$\begin{array}{lll} \nabla_{e_1}e_1 = -e_3, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = e_3, & \nabla_{e_2}e_3 = -e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = 0. \end{array} \tag{3}$$

The non-vanishing components of the Riemannian curvature tensor are

$$R_{1212} = 1, \quad R_{1313} = -1, \quad R_{2323} = -1. \tag{4}$$

The Ricci tensor field is given by

$$R_{11} = R_{22} = 0, \quad R_{33} = -2. \tag{5}$$

The scalar curvature ρ is -2.

More details on curves, geodesics, minimal and constant angle surfaces in Sol space can be found in [2, 6, 8, 9, 11–13].

2.2. *Killing vector fields in Sol space*

As mentioned in Introduction the Killing vector field is defined by the Killing equation

$$\mathfrak{L}_X g = 0. \tag{6}$$

The Killing equation (6) is usually given in the coordinate form (for derivation see e.g. [15])

$$g(\nabla_U X, V) + g(\nabla_V X, U) = 0, \quad \forall\, U, V \in \mathfrak{X}(M), \tag{7}$$

where ∇ is Levi-Civita connection compatible with the metric g.

The basic Killing vector fields in Sol_3 space (see [19]) are

$$X_1 = \partial_x,\;\; X_2 = \partial_y,\;\; X_3 = x\partial_x - y\partial_y - \partial_z. \tag{8}$$

These three vector fields form a basis of a Lie algebra of Killing vector fields. The X_1 and X_2 represent conservation of linear momentum along x and y direction. X_3 represents conservation of linear momentum in the direction of fibre in line bundle over the Euclidean plane.

General properties of the Killing vector fields in Riemannian and pseudo-Riemannian spaces are reviewed in [10]. Applications of proper conformal Killing vector fields in Einstein spaces and generalized Sasakian spaces are studied in [3] and [16], respectively.

Recently, Killing vector fields have been used in study of magnetic curves. Killing magnetic curves in Euclidean space, Minkowski spacetime, Sol space, $\mathbb{S}^2 \times \mathbb{R}$ space and Walker manifolds were studied in [4, 5], [7], [14] and [1], respectively.

3. Conformal Killing vector fields in Sol space

Definition 3.1. *Let (M, g) be a Riemannian manifold. A vector field $X \in \mathfrak{X}(M)$ is called conformal Killing vector field if*

$$\mathfrak{L}_X g = \lambda g, \quad \lambda \in C^\infty(M), \tag{9}$$

where $\mathfrak{L}$ denotes Lie derivative.

The conformal Killing vector fields are vector fields whose flow preserves the conformal structures of the manifold. The set of conformal Killing vector fields is a Lie algebra using the well known property $\mathfrak{L}_{[U,V]} = \mathfrak{L}_U \circ \mathfrak{L}_V - \mathfrak{L}_V \circ \mathfrak{L}_U$ and those of homothetic and Killing vector fields are its subalgebras.

The goal of this section is to find a conformal Killing vector field X in Sol space. Let's assume that the conformal Killing vector field X is given by

$$X = a(x, y, z)\, e_1 + b(x, y, z)\, e_2 + c(x, y, z)\, e_3.$$

Then, taking $U = e_i, V = e_j$, for $i, j \in \{1, 2, 3\}$, the equation (9) implies the following nonlinear system of PDE's:

$$\begin{array}{ll} (i) & 2c + 2e^{-z} a_x = \lambda, \\ (ii) & e^z a_y + e^{-z} b_x = 0, \\ (iii) & -a + a_z + e^{-z} c_x = 0, \\ (iv) & -2c + 2e^z b_y = \lambda, \\ (v) & b + b_z + e^z c_y = 0, \\ (vi) & 2c_z = \lambda. \end{array} \tag{10}$$

For clarity, we show how the second equation is obtained.

From (9), using (7), for $U = e_1, V = e_2$, it follows

$$g(\nabla_{e_1}(a\, e_1 + b\, e_2 + c\, e_3), e_2) + g(\nabla_{e_2}(a\, e_1 + b\, e_2 + c\, e_3), e_1) = \lambda g(e_1, e_2).$$

By direct computation, using (3) and (2), we get $e_1(b) + e_2(b) = 0$ and hence the equation $e^{-z}\partial_x b + e^z \partial_y a = 0$.

Let us assume that $\lambda = 0$. Then system (10) determines Killing vector fields in Sol space. We shortly present how to obtain the Killing vector fields (8).

From the last equation of the system (10) we have $c = c(x, y)$. Then, the equations (i) and (iv) of the system (10) imply $a = f(x, y)e^z$ and $b = g(x, y)e^{-z}$, respectively. Substituting obtained functions in (ii), it follows $a = f(x)e^z$ and $b = g(y)e^{-z}$. Further, from (iii) and (v), it follows $c_x = 0$ and $c_y = 0$, and hence $c = const$. Now, we consider two cases: $c = 0$ and $c = const \neq 0$.

If $c = 0$, then (i) and (iv) imply $a = e^z$ and $b = e^{-z}$, respectively. Hence, we obtain the Killing vector fields $X_1 = e^z e_1 = \partial_x$ and $X_2 = e^{-z} e_2 = \partial_y$, respectively.

If $c = const \neq 0$, then (i) and (iv) imply $f_x = -c$ and $g_y = c$, respectively. Particularly for $c = -1$, we obtain the Killing vector field $X_3 = x\partial_x - y\partial_y - \partial z$.

3.1. *Homothetic Killing vector fields in Sol space*

The homothetic Killing vector field is a special case of the conformal Killing vector field.

Definition 3.2. *Let (M, g) be a Riemannian manifold. A vector field $X \in \mathfrak{X}(M)$ is called homothetic Killing vector field if*

$$\mathfrak{L}_X g = \lambda g, \quad \lambda \in \mathbb{R}. \tag{11}$$

where $\mathfrak{L}$ denotes Lie derivative.

Definition 3.3. *Proper homothetic Killing vector field is Killing vector field such that the homothetic factor $\lambda \neq 0$.*

We solve the system (10) for $\lambda \in \mathbb{R}$. From the last equation of the system (10) we get $c = \frac{1}{2}\lambda z + h(x, y)$. Substituting this in (*i*), it follows

$$a = \frac{\lambda}{2} e^{z}(1 - z)x - e^{z} \int h(x, y)dx \tag{12}$$

Analogously, from (*iv*) follows

$$b = \frac{\lambda}{2} e^{-z}(1 + z)y + e^{-z} \int h(x, y)dy. \tag{13}$$

Substituting the equations (12) and (13) in (*ii*), it follows $h_x = h_y = 0$ i.e. $h = const$. Further, from (*iii*) it follows $h = \frac{\lambda}{4}$ and from (v) finally $\lambda = 0$. The further consideration coincides with the already given for $\lambda = 0$.

We prove the following proposition.

Proposition 3.4. *There is no proper homothetic Killing vector field in Sol space.*

3.2. Proper conformal Killing vector fields in Sol space

In this subsection we try to solve the system (10) for an arbitrary conformal function $\lambda = \lambda(x, y, z)$ which may not be constant.

Definition 3.5. *Proper conformal Killing vector field is Killing vector field such that the conformal function λ is non constant.*

In general case when conformal function $\lambda = \lambda(x, y, z)$, we faced with complicated system of differential equations and it seems that the solution doesn't exist. However, in the following theorem we give a characterization of the proper conformal Killing vector fields in Sol space.

Proposition 3.6. *The vector field $X = a\, e_1 + b\, e_2 + c\, e_3$ given by*

$$a = \frac{e^{z}}{2} \int \lambda dx - \frac{e^{z}}{2} \iint \lambda dz dx - e^{z} \int h dx, \tag{14}$$

$$b = \frac{e^{-z}}{2} \int \lambda dy + \frac{e^{-z}}{2} \iint \lambda dz dy + e^{-z} \int h dy, \tag{15}$$

$$c = \frac{1}{2} \int \lambda dz + h(x, y), \tag{16}$$

is a conformal Killing vector field in Sol space if the smooth conformal function $\lambda = \lambda(x, y, z)$ exists and satisfies the following integrability conditions

$$e^{z} \int (\lambda_z - \lambda)\, dx + e^{-z} \int \lambda_x dz + 2e^{-z} h_x = 0, \tag{17}$$

$$e^{-z} \int (\lambda_z + \lambda)\, dy + e^{z} \int \lambda_y dz + 2e^{z} h_y = 0, \tag{18}$$

$$e^{2z} \int \left(\lambda_y - \int \lambda_y dz\right) dx - 2e^{z} \int h_y dx + e^{-2z} \int \left(\lambda_x - \int \lambda_x dz\right) dy + 2e^{-z} \int h_x dy = 0, \tag{19}$$

where $h = h(x, y)$ is an arbitrary smooth function.

Proof. We already explained how the definition (9) implies the system (10). Next, from the last equation of the system (10) we get (16). Substituting (16) in (*i*) and (*iv*), it follows (14) and (15), respectively. Further, substituting the equations (14), (15) and (16) in (*iii*), (*v*) and (*ii*), we obtain (17), (18) and (19), respectively. □

Corollary 3.7. *There is no proper conformal Killing vector field in Sol space such that the conformal function* $\lambda = \lambda(z)$.

Proof. Substituting $\lambda = \lambda(z)$ in (19), we have $h = const$. Then, from (17) and (18), it follows $\lambda = 0$. □

Corollary 3.8. *There is no proper conformal Killing vector field in Sol space such that the conformal function* $\lambda = \lambda(x, y)$.

Proof. Substituting $\lambda = \lambda(x, y)$ in (14) and (15), it follows $a = \frac{e^z}{2}(1-z)\int \lambda dx - e^z \int h dx$ and $b = \frac{e^{-z}}{2}(1+z)\int \lambda dy + e^{-z}\int h dy$. Substituting the obtained expressions in (*ii*), we first obtain $\lambda_y = h_y = 0$ and $\lambda_x = h_x = 0$ and finally $\lambda = const$. □

Considered particular cases encourage our impression about non-existence of proper conformal Killing vector field in Sol space.

4. 2-Killing vector fields in Sol space

Definition 4.1. *Let* (M, g) *be a Riemannian manifold. A vector field* $X \in \mathfrak{X}(M)$ *is called 2-Killing if* $\mathfrak{L}_X\mathfrak{L}_X g = 0$, *where* $\mathfrak{L}$ *denotes Lie derivative.*

Obviously, every Killing vector field is 2-Killing vector field.

Definition 4.2. *Proper 2-Killing vector field is a 2-Killing vector field which is not a Killing vector field.*

T. Oprea in [15] proved the following characterization of the 2-Killing vector field in Riemannian manifold.

Theorem 4.3 ([15], Theorem 2.1). *A vector field* $X \in \mathfrak{X}(M)$ *is 2-Killing if and only if*

$$R(X, U, X, U) = g(\nabla_U\nabla_X X, U) + g(\nabla_U X, \nabla_U X), \quad \forall\, U \in \mathfrak{X}(M), \tag{20}$$

where R is the curvature tensor of Riemannian manifold (M, g).

Particularly, for $X = X^1(x, y, z)\,\partial_x + X^2(x, y, z)\,\partial_y + X^3(x, y, z)\,\partial_z$ and $U = e_j$, for $j \in \{1, 2, 3\}$ we obtain the following characterization of the 2-Killing vector fields in 3D Euclidean space.

Corollary 4.4. *The vector field* $X = X^1\,\partial_x + X^2\,\partial_y + X^3\,\partial_z$ *is 2-Killing vector field in* E^3 *if its components fulfill the following system of PDE's*

$$(X^i_{,j})^2 + X^i_{,j}X^j_{,i} + X^iX^j_{,ij} = 0 \tag{21}$$

where $j \in \{1, 2, 3\}$, *i is summation index and* $X^j_{,i} = \frac{\partial X^j}{\partial x^i}$.

We use this formula later for comparing results in Sol and Euclidean space.

Further, we use the formula (20) to explore 2-Killing vector fields in Sol space.
Let assume that the 2-Killing vector field V is given by

$$V = a(x, y, z)\,e_1 + b(x, y, z)\,e_2 + c(x, y, z)\,e_3.$$

After long but straightforward computation the equation (20) for $U = e_i$, $\forall\, i \in \{1, 2, 3\}$, using (2), (3) and (4) implies the following system of PDE's (further called 2-KVF):

$$(2c^2 + cc_z + a_y b_x + a_{xy} b) + e^{-z}(a_z c_x + 3a_x c + a_{xz} c) + \\ + e^z bc_y + e^{-2z}(2a_x^2 + aa_{xx} + b_x^2 + c_x^2) = 0, \tag{22}$$

$$(2c^2 - cc_z + a_y b_x + ab_{xy}) + e^z(b_z c_y + 3b_y c - b_{yz} c) + \\ - e^{-z} ac_x + e^{2z}(a_y^2 + bb_{yy} + 2b_y^2 + c_y^2) = 0, \tag{23}$$

$$\Big((a - a_z)^2 + (b + b_z)^2 + 2c_z^2 + cc_{zz}\Big) + e^z(bc_y + b_z c_y + bc_{yz}) + \\ + e^{-z}(-ac_x + a_z c_x + ac_{xz}) = 0. \tag{24}$$

Unfortunately, the 2-KVF is complicated nonlinear second order system of PDE's. Although we can't find exact solutions, we can determine 2-Killing vector fields that are generalizations of the Killing vector fields ∂_x and ∂_y.

Let assume that $V = a(x, y, z)\, e_1$, i.e. $b = c = 0$. Then the 2-KVF system became

$$e^{-2z}(2a_x^2 + aa_{xx}) = 0, \quad e^{2z}(a_y^2) = 0, \quad (a - a_z)^2 = 0. \tag{25}$$

From the second and the third equation of (25) it follows $a(x, z) = f(x)e^z$.

From the first equation of (25) we get the differential equation $2f_x^2 + f f_{xx} = 0$ whose solution is a function $f(x) = c_1 \sqrt[3]{3x - c_2},\ c_1, c_2 \in \mathbb{R}$. Hence the 2-Killing vector field is given by

$$V_1 = c_1 \sqrt[3]{3x - c_2} \cdot e^z \cdot e_1, \quad c_1, c_2 \in \mathbb{R}. \tag{26}$$

Particulary for $x = 0$, $c_1 = 1$ and $c_2 = -1$ we get the Killing vector field

$$X_1 = e^z \cdot e_1 = \partial_x.$$

Thus V_1 is a generalization of X_1.

Quit analogously, assuming that $V = b(x, y, z)\, e_2$, i.e. $a = c = 0$, the 2-KVF system became

$$e^{-2z}(b_x^2) = 0, \quad e^{-2z}(2b_y^2 + bb_{yy}) = 0, \quad (b + b_z)^2 = 0.$$

The solution of this system is a function $b(y, z) = c_3 \sqrt[3]{3y - c_4} \cdot e^{-z}$ and hence the 2-Killing vector field is

$$V_2 = c_3 \sqrt[3]{3y - c_4} \cdot e^{-z} \cdot e_2, \quad c_3, c_4 \in \mathbb{R}. \tag{27}$$

Particulary for $y = 0$, $c_3 = 1$ and $c_4 = -1$ we get the Killing vector field

$$X_2 = e^{-z}\, e_2 = \partial_y.$$

If we assume that $V = c(x, y, z)\, e_3$, i.e. $a = b = 0$, then the 2-KVF system became

$$2c^2 + cc_z + e^{-2z}c_x^2 = 0, \quad 2c^2 - cc_z + e^{2z}c_y^2 = 0, \quad 2c_z^2 + cc_{zz} = 0.$$

The only solution of this system is $c = 0$. Therefore, there is no 2-Killing vector field of the form $V = c(x, y, z)\, e_3$.

Remark 4.5. *The obtained results are according to the final remark in [15] which says that the proper 2-Killing vector field on $\mathbb{R}$ of the form $X = f\frac{d}{dt}$ is given by $X = (at - b)^{\frac{1}{3}} \frac{d}{dt}$. Moreover, it seems that in all 3D homogeneous geometries where the Killing vector field generate translation (i.e. $X = \partial_{x_i}$), the corresponding 2-Killing vector field has the form $V = c_1 \sqrt[3]{3x_i - c_2}\, \partial_{x_i}$.*

Next, we try to determine a 2-Killing vector field which is a generalization of the Killing vector filed $X_3 = x\partial_x - y\partial_y - \partial_z$ and such that its coordinate functions are

$$a = f(x)e^z, \quad b = g(y)e^{-z}, \quad c = const = k.$$

Substituting these in (22) and (23) ((24) is identically zero) we have equations

$$2k^2 + 4kf_x + 2f_x^2 + ff_{xx} = 0, \qquad 2k^2 - 4kg_y + 2g_y^2 + gg_{yy} = 0.$$

The solutions of these equations are linear functions

$$f(x) = -kx + c_5 \quad \text{and} \quad g(y) = ky + c_6.$$

Therefore we obtain 2-Killing vector field $X = (-kx + c_5)\partial_x + (ky + c_6)\partial_y + k\partial z$. Although it seems that we obtain a generalization of the Killing vector field $X_3 = x\partial_x - y\partial_y - \partial_z$, unfortunately this is not true. The obtained vector field can be given as a linear combination of basic Killing vector fields, i.e. $X = -kX_3 + c_5X_1 + c_6X_2$ and hence it is not proper 2-Killing vector field.

In the sequelae we check some other potential generalizations of Killing vector field X_3. As far as we were able to check, it seems that there is no proper 2-Killing vector field which is generalization of X_3.

Case 1

Let assume $a = f(x)e^z$, $b = g(y)e^{-z}$, $c = c(z) \neq const$. Adding (22) and (23) we have

$$4c^2 + 4c(f_x + g_y) + 2f_x^2 + f_{xx}f + 2g_y^2 + g_{yy}g = 0. \tag{28}$$

Further, from (24) it follows $c(z) = k_1\sqrt[3]{3z - k_2}$. Substitution of this function c in (28) leads to the contradiction.

Case 2

Let assume $a = f(x)e^z$, $b = g(y)e^{-z}$, $c = c(x, y, z) \neq const$. Adding (22) and (23) we have

$$4c^2 + 4c(f_x + g_y) + 2f_x^2 + f_{xx}f + 2g_y^2 + g_{yy}g + e^{-2z}c_x^2 + e^{2z}c_y^2 = 0. \tag{29}$$

Further, (24) became

$$2c_z^2 + c_{zz}c + gc_{yz} + fc_{xz} = 0. \tag{30}$$

The equation (30) implies that function $c = c(x, y, z)$ exponentially depends on variable z which in combination with (29) gives contradiction.

Case 3

Let assume $a = f(x)e^{kz}$, $b = g(y)e^{-kz}$, $c = c(x, y, z) \neq const$ and $k \in \mathbb{R}$.

From (22) and (23) we have

$$\begin{aligned} 2c^2 + cc_z + ke^{(k-1)z}c_x + e^{(1-k)z}c_y + e^{-2z}c_x^2 &= 0, \\ 2c^2 - cc_z - e^{(k-1)z}c_x - ke^{(1-k)z}c_y + e^{2z}c_y^2 &= 0. \end{aligned} \tag{31}$$

Notice that the reasonable assumption is that function $c = c(x, y, z)$ has the same functional dependence regard to x and y. If this is the case, then the only solution of (31) is $c = const$ which contradict to our assumption.

Proposition 4.6. *The vector fields*

$$V_1 = c_1 \sqrt[3]{3x - c_2}\, \partial_x \text{ and } V_2 = c_3 \sqrt[3]{3y - c_4}\, \partial_y, \quad c_1, c_2, c_3, c_4 \in \mathbb{R},$$

are proper 2-Killing vector fields in Sol space.

Furthermore, there is no proper 2-Killing vector field of the form

$$X = f(x)e^{kz}e_1 + g(y)e^{-kz}e_2 + c(x, y, z)e_3,$$

where $f, g, c \in C^\infty$ and $k \in \mathbb{R}$.

Remark 4.7. *In Corollary 4.4 a characterization of 2-Killing vector fields in E^3 is given. It seems that a proper 2-Killing vector field which generalizes the Killing vector field related to conservation of angular momentum in E^3 does not exist. On the other hand, proper 2-Killing vector fields which generalize the Killing vector fields related to conservation of linear momentum in E^3 coincide with vector fields given in Proposition 4.6.*

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