



On Hermite-Hadamard type inequalities of coordinated (p_1, h_1) - (p_2, h_2) -convex functions via Katugampola Fractional Integrals

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Abstract. The present article establishes some results on Hermite-Hadamard type inequalities of coordinated (p_1, h_1) - (p_2, h_2) -convex functions by using Katugampola fractional integrals. The results are in generalized form and deduced for various special cases like coordinated pq -convex functions, coordinated (p, s) -convex functions, coordinated s -convex functions and classical case of coordinated convex functions.

1. Introduction

During the last three decades, the theory of convex functions has been extensively studied due to its vast applications in optimization theory and biological system [1, 2]. Important inequalities are introduced for this class of functions. In recent years many generalizations of convexity such as s -convexity, h -convexity, m -convexity and some combinations of these concepts shows the impact to the community of investigators. These notions modified and generalized those inequalities which were found for classical convexity (see for example [3]-[13]). Due to geometrical interpretation of Hermite-Hadamard type inequalities, significant importance is given to them. Dragomir [14] introduced the notion of convex functions on the coordinates in a rectangle from the plane \mathbb{R}^2 . He considered a bi-dimensional interval $\Delta : [a, b] \times [c, d]$ with $0 \leq a < b < \infty, 0 \leq c < d < \infty$ and defined the concept as follows: A function $f : \Delta \rightarrow \mathbb{R}$, will be called convex on the coordinates on Δ , if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v)$ are convex for all $y, v \in [c, d]$ and for all $x, u \in [a, b]$ respectively. Recall that convex functions on Δ are satisfying the inequality:

$$f(\lambda x + (1 - \lambda)u, \lambda y + (1 - \lambda)v) \leq \lambda f(x, y) + (1 - \lambda)f(u, v) \quad (1)$$

for all $(x, y), (u, v) \in \Delta$ and $\lambda \in [0, 1]$. Every convex function is coordinated convex but the converse is not true (see[14]). Dragomir [14] established Hadamard type inequalities similar to one dimensional case. Later on many generalizations of the coordinated convex functions are made and new inequalities are established by different authors. We refer interested readers to [15]-[20].

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Noor et al. [21] defined a new class of functions named as two dimensional(coordinated) pq -convex functions. They generalize some inequalities for this class of functions. Yang [22] extend this notion for a more general class termed as coordinated (p_1, h_1) - (p_2, h_2) -convex functions. This is a very large class as it includes coordinated (p_1, s_1) - (p_2, s_2) -convex functions, coordinated pq -convex functions, coordinated (h_1, h_2) -convex functions, coordinated (s_1, s_2) -convex functions, coordinated h -convex functions, coordinated s -convex functions and coordinated convex functions as special cases. Yang established inequalities of Hermite-Hadamard type for this class of functions and connected the results to the previously known results for classical integrals. Sarikaya [23] extended the Hadamard type inequalities by using coordinated convex functions via Riemann-Liouville Fractional integrals. Chen [24] established Hadamard type inequalities for coordinated s -convex functions. Set et al. [26] considered coordinated h -convex functions and generalized some inequalities of Hadamard type.

Motivated by these papers, we present the following work whose purpose is to establish the Hadamard type inequalities for coordinated (p_1, h_1) - (p_2, h_2) -convex functions via Katugampola fractional integral. We presented our results for several special cases like coordinated (p_1, s_1) - (p_2, s_2) -convex functions, coordinated pq -convex functions, coordinated (h_1, h_2) -convex functions, coordinated (s_1, s_2) -convex functions. Results proved in this study are continue to hold for established results for various kinds of convexities like coordinated h -convex functions, coordinated s -convex functions and coordinated convex functions.

2. Preliminaries

In this section we give some definitions and properties of convexity and fractional integral operators. Recall that a real valued function f defined in an interval J is called convex function if for all x, y in J and for any t in $[0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y). \tag{2}$$

Let $f : J = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function with $a \leq b$, then the following double inequality which is called Hermite-Hadamard’s inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \tag{3}$$

Let us consider a bi-dimensional interval $\Delta : [a, b] \times [c, d]$ in \mathbb{R}^2 such that $0 \leq a < b < \infty, 0 \leq c < d < \infty$.

Theorem 2.1. [14] Suppose that $f : \Delta \rightarrow \mathbb{R}$ is convex on the coordinates on Δ . Then following inequalities hold:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy \\ &\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{d-c} \int_c^d [f(a, y) + f(b, y)] dy \right] \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned}$$

A formal definition of coordinated convex function may be stated as follows:

Definition 2.2. A function $f : \Delta \rightarrow \mathbb{R}$ will be called coordinated convex on Δ , if the following inequality holds:

$$f(tx + (1-t)u, sy + (1-s)v) \leq tsf(x, y) + t(1-s)f(x, v) + (1-t)sf(u, y) + (1-t)(1-s)f(u, v)$$

for all $(x, y), (x, v), (u, y), (u, v) \in \Delta$ and $s, t \in [0, 1]$.

In the following we present some definitions and results which are helpful in the further study. Details can be found in [23],[27]-[34].

Definition 2.3. Let $f \in L_1[a, b]$. The left and right Riemann- Liouville integrals of order $\alpha > 0$ with $a \geq 0$ are denoted and defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau)^{\alpha-1} f(\tau) d\tau, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} f(\tau) d\tau, \quad x < b,$$

respectively. Where Γ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Definition 2.4. Let $f \in L_1([a, b] \times [c, d])$. The Riemann- Liouville integrals $J_{a+,c+}^{\alpha,\beta}$, $J_{a+,d-}^{\alpha,\beta}$, $J_{b-,c+}^{\alpha,\beta}$ and $J_{b-,d-}^{\alpha,\beta}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by

$$J_{a+,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x - \tau_1)^{\alpha-1} (y - \tau_2)^{\beta-1} f(\tau_1, \tau_2) d\tau_2 d\tau_1, \quad x > a, \quad y > c,$$

$$J_{a+,d-}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x - \tau_1)^{\alpha-1} (\tau_2 - y)^{\beta-1} f(\tau_1, \tau_2) d\tau_2 d\tau_1, \quad x > a, \quad y < d,$$

$$J_{b-,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (\tau_1 - x)^{\alpha-1} (y - \tau_2)^{\beta-1} f(\tau_1, \tau_2) d\tau_2 d\tau_1, \quad x < b, \quad y > c,$$

and

$$J_{b-,d-}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (\tau_1 - x)^{\alpha-1} (\tau_2 - y)^{\beta-1} f(\tau_1, \tau_2) d\tau_2 d\tau_1, \quad x < b, \quad y < d,$$

respectively. Here Γ is the Gamma function,

$$J_{a+,c+}^{0,0} f(x, y) = J_{a+,d-}^{0,0} f(x, y) = J_{b-,c+}^{0,0} f(x, y) = J_{b-,d-}^{0,0} f(x, y) = f(x, y)$$

and

$$J_{a+,d-}^{1,1} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d f(\tau_1, \tau_2) d\tau_2 d\tau_1.$$

Similar to Definition 2.3, Sarikaya [23] introduced the following fractional integrals:

$$J_{a+}^{\alpha} f\left(x, \frac{c+d}{2}\right) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau_1)^{\alpha-1} f\left(\tau_1, \frac{c+d}{2}\right) d\tau_1, \quad x > a,$$

$$J_{b-}^{\alpha} f\left(x, \frac{c+d}{2}\right) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau_1 - x)^{\alpha-1} f\left(\tau_1, \frac{c+d}{2}\right) d\tau_1, \quad x < b,$$

$$J_{c^+}^\alpha f\left(\frac{a+b}{2}, y\right) = \frac{1}{\Gamma(\beta)} \int_c^y (y - \tau_2)^{\beta-1} f\left(\frac{a+b}{2}, \tau_2\right) d\tau_2, \quad y > c,$$

$$J_{d^-}^\alpha f\left(\frac{a+b}{2}, y\right) = \frac{1}{\Gamma(\beta)} \int_x^b (\tau_2 - y)^{\beta-1} f\left(\frac{a+b}{2}, \tau_2\right) d\tau_2, \quad y < d.$$

It is worth noting that Sarikaya gave the following remarkable results in [23].

Theorem 2.5. Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a coordinated convex function on $\Delta := [a, b] \times [c, d]$ in \mathbb{R}^2 with $0 \leq a < b$, $0 \leq c < d$ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned}$$

Theorem 2.6. Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a coordinated convex function on $\Delta := [a, b] \times [c, d]$ in \mathbb{R}^2 with $0 \leq a < b$, $0 \leq c < d$ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[J_{a^+}^\alpha f\left(b, \frac{c+d}{2}\right) + J_{b^-}^\alpha f\left(a, \frac{c+d}{2}\right) \right] + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[J_{c^+}^\beta f\left(\frac{a+b}{2}, d\right) + J_{d^-}^\beta f\left(\frac{a+b}{2}, c\right) \right] \\ &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] \\ &\leq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[J_{a^+}^\alpha f(b,c) + J_{a^+}^\alpha f(b,d) + J_{b^-}^\alpha f(a,c) + J_{b^-}^\alpha f(a,d) \right] \\ &\quad + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[J_{c^+}^\beta f(a,d) + J_{c^+}^\beta f(b,d) + J_{d^-}^\beta f(a,c) + J_{d^-}^\beta f(b,c) \right] \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned}$$

Noor et. al. [21] introduced the notion of coordinated pq -convex functions to generalize the p -convex functions as follows:

Definition 2.7. Let $\Delta := [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle. A function $f : \Delta \rightarrow \mathbb{R}$ is said to be two dimensional (coordinated) pq -convex function, if

$$f\left([tx^p + (1-t)u^p]^{\frac{1}{p}}, [ry^q + (1-r)v^q]^{\frac{1}{q}}\right) \leq tf(x, y) + t(1-r)f(x, v) + (1-t)rf(u, y) + (1-t)(1-r)f(u, v) \quad (4)$$

for all $(x, y), (x, v), (u, y), (u, v) \in \Delta$ and $r, t \in [0, 1]$.

They established the following result.

Theorem 2.8. Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a pq -convex function on the coordinates on Δ , then following inequalities hold:

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}, \left[\frac{c^q + d^q}{2}\right]^{\frac{1}{q}}\right) \leq \frac{pq}{(b^p - a^p)(d^q - c^q)} \int_a^b \int_c^d x^{p-1} y^{q-1} f(x, y) dy dx \leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.$$

Yang [22] generalized this concept by defining a larger class of coordinated convex functions termed as coordinated (p_1, h_1) - (p_2, h_2) -convex function as follows:

Definition 2.9. Let $h_1, h_2 : J \rightarrow \mathbb{R}$ be two non-negative and non-zero functions. A mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be (p_1, h_1) - (p_2, h_2) -convex function on the coordinates on Δ , if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v)$ are (p_1, h_1) -convex with respect to u on $[a, b]$ and (p_2, h_2) -convex with respect to v on $[c, d]$ respectively for all $y \in [c, d]$ and $x \in [a, b]$.

From the above definition, we can say that if f is coordinated (p_1, h_1) - (p_2, h_2) -convex function, then the following inequality holds:

$$f\left(\left[tx^{p_1} + (1-t)u^{p_1}\right]^{\frac{1}{p_1}}, \left[ry^{p_2} + (1-r)v^{p_2}\right]^{\frac{1}{p_2}}\right) \leq h_1(t)h_2(r)f(x, y) + h_1(t)h_2(1-r)f(x, v) + h_1(1-t)h_2(r)f(u, y) + h_1(1-t)h_2(1-r)f(u, v). \tag{5}$$

Remark 2.10. If $p_1 = p_2 = 1$, then the function f defined in inequality (5) will be reduced to coordinated (h_1, h_2) -convex function.

Remark 2.11. If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$, then the function f defined in inequality (5) will be called coordinated (p_1, s_1) - (p_2, s_2) -convex function.

Remark 2.12. If $h_1(t) = t^{s_1}, h_2(t) = t^{s_2}$ and $p_1 = p_2 = 1$, then the function f defined in inequality (5) will be called coordinated (s_1, s_2) -convex function.

Yang [22] gave following two results along with many other results.

Theorem 2.13. Let $f : \Delta \rightarrow \mathbb{R}$ be a (p_1, h_1) - (p_2, h_2) -convex function on the coordinates on Δ . Then one has the inequalities:

$$\begin{aligned} \frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) &\leq \frac{p_1p_2}{(b^{p_1} - a^{p_1})(d^{p_2} - c^{p_2})} \int_a^b \int_c^d x^{p_1-1}y^{p_2-1}f(x, y)dydx \\ &\leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \int_0^1 h_1(t)dt \int_0^1 h_2(t)dt. \end{aligned}$$

Theorem 2.14. Let $f : \Delta \rightarrow \mathbb{R}$ be a (p_1, h_1) - (p_2, h_2) -convex function on the coordinates on Δ . Then one has the inequalities:

$$\begin{aligned} &\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\ &\leq \frac{p_1}{4h_1\left(\frac{1}{2}\right)(b^{p_1} - a^{p_1})} \int_a^b x^{p_1-1}f\left(x, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)dx + \frac{p_2}{4h_2\left(\frac{1}{2}\right)(d^{p_2} - c^{p_2})} \int_c^d y^{p_2-1}f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, y\right)dy \\ &\leq \frac{p_1p_2}{(b^{p_1} - a^{p_1})(d^{p_2} - c^{p_2})} \int_a^b \int_c^d x^{p_1-1}y^{p_2-1}f(x, y)dydx \\ &\leq \frac{p_1}{2(b^{p_1} - a^{p_1})} \left[\int_a^b x^{p_1-1}f(x, c)dx + \int_a^b x^{p_1-1}f(x, d)dx \right] \int_0^1 h_2(t)dt \\ &\quad + \frac{p_2}{2(d^{p_2} - c^{p_2})} \left[\int_c^d y^{p_2-1}f(a, y)dy + \int_c^d y^{p_2-1}f(b, y)dy \right] \int_0^1 h_1(t)dt \\ &\leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \int_0^1 h_1(t)dt \int_0^1 h_2(t)dt. \end{aligned}$$

Definition 2.15. [27] $X_c^p(a, b) (c \in \mathbb{R}, 1 \leq p \leq \infty)$ is the set of those complex valued Lebesgue measurable functions f of $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by $\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty$ for $1 \leq p < \infty, c \in \mathbb{R}$ and for the case $p = \infty, \|f\|_{X_c^p} = \text{ess sup}_{a \leq t \leq b} [t^c |f(t)|], c \in \mathbb{R}$.

Katugampola introduced a new fractional integral which generalizes the Riemann-Liouville and Hadamard fractional integrals into single form as follows (see for example [29-32]).

Definition 2.16. Let $[a, b] \subseteq \mathbb{R}$ be a finite interval. Then, the left and right-sided Katugampola fractional integrals of order $\alpha (> 0)$ of $f \in X_c^p(a, b)$ with $a \geq 0$ are defined by:

$${}^\rho I_{a+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\alpha-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t) dt$$

and

$${}^\rho I_{b-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\alpha-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t) dt$$

with $a < x < b$ and $\rho > 0$, provided the integrals exist.

Katugampola fractional integrals into two dimensional case may be given as follows:

Definition 2.17. Let $f \in X_c^p(\Delta)$. The Katugampola fractional integrals ${}^{p_1, p_2} I_{a+, c+}^{\alpha, \beta}, {}^{p_1, p_2} I_{a+, d-}^{\alpha, \beta}, {}^{p_1, p_2} I_{b-, c+}^{\alpha, \beta}$ and ${}^{p_1, p_2} I_{b-, d-}^{\alpha, \beta}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by

$${}^{p_1, p_2} I_{a+, c+}^{\alpha, \beta} f(x, y) = \frac{p_1^{1-\alpha} p_2^{1-\beta}}{\Gamma(\alpha) \Gamma(\beta)} \int_a^x \int_c^y \frac{t^{p_1-1} s^{p_2-1}}{(x^{p_1} - t^{p_1})^{1-\alpha} (y^{p_2} - s^{p_2})^{1-\beta}} f(t, s) ds dt, \quad x > a, y > c,$$

$${}^{p_1, p_2} I_{a+, d-}^{\alpha, \beta} f(x, y) = \frac{p_1^{1-\alpha} p_2^{1-\beta}}{\Gamma(\alpha) \Gamma(\beta)} \int_a^x \int_y^d \frac{t^{p_1-1} s^{p_2-1}}{(x^{p_1} - t^{p_1})^{1-\alpha} (s^{p_2} - y^{p_2})^{1-\beta}} f(t, s) ds dt, \quad x > a, y < d,$$

$${}^{p_1, p_2} I_{b-, c+}^{\alpha, \beta} f(x, y) = \frac{p_1^{1-\alpha} p_2^{1-\beta}}{\Gamma(\alpha) \Gamma(\beta)} \int_x^b \int_c^y \frac{t^{p_1-1} s^{p_2-1}}{(t^{p_1} - x^{p_1})^{1-\alpha} (y^{p_2} - s^{p_2})^{1-\beta}} f(t, s) ds dt, \quad x < b, y > c,$$

and

$${}^{p_1, p_2} I_{b-, d-}^{\alpha, \beta} f(x, y) = \frac{p_1^{1-\alpha} p_2^{1-\beta}}{\Gamma(\alpha) \Gamma(\beta)} \int_x^b \int_y^d \frac{t^{p_1-1} s^{p_2-1}}{(t^{p_1} - x^{p_1})^{1-\alpha} (s^{p_2} - y^{p_2})^{1-\beta}} f(t, s) ds dt, \quad x < b, y < d,$$

respectively and $p_1, p_2 > 0$. Here Γ is the Gamma function. Moreover,

$${}^{p_1, p_2} I_{a+, c+}^{0, 0} f(x, y) = {}^{p_1, p_2} I_{a+, d-}^{0, 0} f(x, y) = {}^{p_1, p_2} I_{b-, c+}^{0, 0} f(x, y) = {}^{p_1, p_2} I_{b-, d-}^{0, 0} f(x, y) = f(x, y)$$

and

$${}^{p_1, p_2} I_{a+, d-}^{1, 1} f(x, y) = \int_a^x \int_y^d t^{p_1-1} s^{p_2-1} f(t, s) ds dt.$$

Similar to Definition 2.16, we introduce the following fractional integrals:

$$\begin{aligned}
 {}^{p_1}I_{a+}^\alpha f \left(x, \left[\frac{c^{p_2} + d^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) &= \frac{p_1^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{p_1-1}}{(x^{p_1} - t^{p_1})^{1-\alpha}} f \left(t, \left[\frac{c^{p_2} + d^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) dt, \quad x > a, \\
 {}^{p_1}I_{b-}^\alpha f \left(x, \left[\frac{c^{p_2} + d^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) &= \frac{p_1^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{p_1-1}}{(t^{p_1} - x^{p_1})^{1-\alpha}} f \left(t, \left[\frac{c^{p_2} + d^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) dt, \quad x < b, \\
 {}^{p_1}I_{c+}^\beta f \left(\left[\frac{a^{p_1} + b^{p_1}}{2} \right]^{\frac{1}{p_1}}, y \right) &= \frac{p_2^{1-\beta}}{\Gamma(\beta)} \int_c^y \frac{s^{p_2-1}}{(y^{p_2} - s^{p_2})^{1-\beta}} f \left(\left[\frac{a^{p_1} + b^{p_1}}{2} \right]^{\frac{1}{p_1}}, s \right) ds, \quad y > c, \\
 {}^{p_2}I_{d-}^\beta f \left(\left[\frac{a^{p_1} + b^{p_1}}{2} \right]^{\frac{1}{p_1}}, y \right) &= \frac{p_2^{1-\beta}}{\Gamma(\beta)} \int_y^d \frac{s^{p_2-1}}{(s^{p_2} - y^{p_2})^{1-\beta}} f \left(\left[\frac{a^{p_1} + b^{p_1}}{2} \right]^{\frac{1}{p_1}}, s \right) ds, \quad y < d.
 \end{aligned}$$

It is important to notice that if $p_1 = p_2 = 1$, then Katugampola fractional integrals reduces to Riemann Liouville fractional integrals given in Definition 2.4.

3. Main Results

In this section we give the Hadamard type inequalities by using (p_1, h_1) - (p_2, h_2) -convex functions of two variables on $\Delta = [a, b] \times [c, d]$.

Theorem 3.1. Suppose that $f : \Delta \rightarrow \mathbb{R}$ is a (p_1, h_1) - (p_2, h_2) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned}
 &\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f \left(\left[\frac{a^{p_1} + b^{p_1}}{2} \right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) \\
 &\leq \frac{p_1^\alpha p_2^\beta \Gamma(\alpha + 1) \Gamma(\beta + 1)}{4(b^{p_1} - a^{p_1})^\alpha (d^{p_2} - c^{p_2})^\beta} \left[{}^{p_1, p_2}I_{a+, c+}^{\alpha, \beta} f(b, d) + {}^{p_1, p_2}I_{a+, d-}^{\alpha, \beta} f(b, c) + {}^{p_1, p_2}I_{b-, c+}^{\alpha, \beta} f(a, d) + {}^{p_1, p_2}I_{b-, d-}^{\alpha, \beta} f(a, c) \right] \quad (6) \\
 &\leq \frac{\alpha\beta}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \int_0^1 \int_0^1 t_1^{\alpha-1} t_2^{\beta-1} [h_1(t_1) + h_1(1-t_1)] [h_2(t_2) + h_2(1-t_2)] dt_1 dt_2.
 \end{aligned}$$

Proof. Let $x^{p_1} = t_1 a^{p_1} + (1 - t_1) b^{p_1}$, $y^{p_1} = (1 - t_1) a^{p_1} + t_1 b^{p_1}$ and $u^{p_2} = t_2 c^{p_2} + (1 - t_2) d^{p_2}$, $v^{p_2} = (1 - t_2) c^{p_2} + t_2 d^{p_2}$, then by coordinated (p_1, h_1) - (p_2, h_2) -convexity of f , we have,

$$\begin{aligned}
 f \left(\left[\frac{a^{p_1} + b^{p_1}}{2} \right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) &= f \left(\left[\frac{x^{p_1} + y^{p_1}}{2} \right]^{\frac{1}{p_1}}, \left[\frac{u^{p_2} + v^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) \\
 &\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) [f(x, u) + f(x, v) + f(y, u) + f(y, v)]. \quad (7)
 \end{aligned}$$

Multiply by $\frac{\alpha\beta}{4} t_1^{\alpha-1} t_2^{\beta-1}$ and integrating over $([0, 1] \times [0, 1])$, one has

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f \left(\left[\frac{a^{p_1} + b^{p_1}}{2} \right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2} \right]^{\frac{1}{p_2}} \right) \leq \frac{\alpha\beta}{4} \int_0^1 \int_0^1 t_1^{\alpha-1} t_2^{\beta-1} [f(x, u) + f(x, v) + f(y, u) + f(y, v)] dt_1 dt_2. \quad (8)$$

Note that by the change of variable, we have on the right-hand side of the inequality (8):

$$\begin{aligned} & \int_0^1 \int_0^1 t_1^{\alpha-1} t_2^{\beta-1} [f(x, u) + f(x, v) + f(y, u) + f(y, v)] dt_1 dt_2 \\ &= \frac{p_1 p_2}{(b^{p_1} - a^{p_1})^\alpha (d^{p_2} - c^{p_2})^\beta} \left[\int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1}}{(b^{p_1} - x^{p_1})^{1-\alpha} (d^{p_2} - y^{p_2})^{1-\beta}} f(x, y) dy dx \right. \\ &+ \int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1}}{(b^{p_1} - x^{p_1})^{1-\alpha} (y^{p_2} - c^{p_2})^{1-\beta}} f(x, y) dy dx + \int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1}}{(x^{p_1} - a^{p_1})^{1-\alpha} (d^{p_2} - y^{p_2})^{1-\beta}} f(x, y) dy dx \\ &\left. + \int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1}}{(x^{p_1} - a^{p_1})^{1-\alpha} (y^{p_2} - c^{p_2})^{1-\beta}} f(x, y) dy dx \right]. \end{aligned}$$

Now applying the Definition 2.17 of Katugampola fractional integral, the first inequality of (6) is obtained. For the second inequality on the right hand side of (6), we use the coordinated (p_1, h_1) - (p_2, h_2) -convexity of f as follows:

$$\begin{aligned} f(x, u) &= f\left([t_1 a^{p_1} + (1 - t_1) b^{p_1}]^{\frac{1}{p_1}}, [t_2 c^{p_2} + (1 - t_2) d^{p_2}]^{\frac{1}{p_2}}\right) \\ &\leq h_1(t_1) h_2(t_2) f(a, c) + h_1(t_1) h_2(1 - t_2) f(a, d) + h_1(1 - t_1) h_2(t_2) f(b, c) \\ &\quad + h_1(1 - t_1) h_2(1 - t_2) f(b, d), \end{aligned} \tag{9}$$

$$\begin{aligned} f(x, v) &= f\left([t_1 a^{p_1} + (1 - t_1) b^{p_1}]^{\frac{1}{p_1}}, [(1 - t_2) c^{p_2} + t_2 d^{p_2}]^{\frac{1}{p_2}}\right) \\ &\leq h_1(t_1) h_2(1 - t_2) f(a, c) + h_1(t_1) h_2(t_2) f(a, d) + h_1(1 - t_1) h_2(1 - t_2) f(b, c) \\ &\quad + h_1(1 - t_1) h_2(t_2) f(b, d), \end{aligned} \tag{10}$$

$$\begin{aligned} f(y, u) &= f\left([(1 - t_1) a^{p_1} + t_1 b^{p_1}]^{\frac{1}{p_1}}, [t_2 c^{p_2} + (1 - t_2) d^{p_2}]^{\frac{1}{p_2}}\right) \\ &\leq h_1(1 - t_1) h_2(t_2) f(a, c) + h_1(1 - t_1) h_2(1 - t_2) f(a, d) + h_1(t_1) h_2(t_2) f(b, c) \\ &\quad + h_1(t_1) h_2(1 - t_2) f(b, d) \end{aligned} \tag{11}$$

and

$$\begin{aligned} f(y, v) &= f\left([(1 - t_1) a^{p_1} + t_1 b^{p_1}]^{\frac{1}{p_1}}, [(1 - t_2) c^{p_2} + t_2 d^{p_2}]^{\frac{1}{p_2}}\right) \\ &\leq h_1(1 - t_1) h_2(1 - t_2) f(a, c) + h_1(1 - t_1) h_2(t_2) f(a, d) + h_1(t_1) h_2(1 - t_2) f(b, c) \\ &\quad + h_1(t_1) h_2(t_2) f(b, d). \end{aligned} \tag{12}$$

Adding inequalities (9), (10), (11), and (12), we come to the result:

$$\begin{aligned} f(x, u) + f(x, v) + f(y, u) + f(y, v) &\leq [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \{h_1(t_1) h_2(t_2) + h_1(t_1) h_2(1 - t_2) \\ &\quad + h_1(1 - t_1) h_2(t_2) + h_1(1 - t_1) h_2(1 - t_2)\}. \end{aligned} \tag{13}$$

Multiplying (13) by $\frac{\alpha\beta}{4} t_1^{\alpha-1} t_2^{\beta-1}$ and integrating over $([0, 1] \times [0, 1])$, one has the second inequality of (6) by applying Definition 2.17, which then completes the proof. \square

Remark 3.2. If $\alpha = 1 = \beta$, then above result become Theorem 2.13 which was proved in [22].

Remark 3.3. If $h_1(t) = t = h_2(t)$ and $\alpha = \beta = 1$, then above result coincide to Theorem 2.8 which was proved in [21].

Remark 3.4. If $p_1 = p_2 = 1$, then inequality (6) reduced to:

$$\begin{aligned} & \frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \left[I_{a+,c+}^{\alpha,\beta} f(b,d) + I_{a+,d-}^{\alpha,\beta} f(b,c) + I_{b-,c+}^{\alpha,\beta} f(a,d) + I_{b-,d-}^{\alpha,\beta} f(a,c) \right] \\ \leq & \frac{\alpha\beta}{4} [f(a,c) + f(a,d) + f(b,c) + f(b,d)] \int_0^1 \int_0^1 t_1^{\alpha-1} t_2^{\beta-1} [h_1(t_1) + h_1(1-t_1)] [h_2(t_2) + h_2(1-t_2)] dt_1 dt_2. \end{aligned}$$

This result generalizes Theorem 2.1 of [26]. It coincide with Theorem 2.1 of [26], if $h_1(t) = h_2(t) = h(t)$. Furthermore, if $\alpha = \beta = 1$, it reduced to Theorem 7 of [16].

Remark 3.5. If $h_1(t) = h_2(t) = t$ and $p_1 = p_2 = 1$, then our result coincide with Theorem 2.5.

Corollary 3.6. Suppose that $f : \Delta \rightarrow \mathbb{R}$ is (p_1, s_1) - (p_2, s_2) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} & 2^{s_1+s_2-2} f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\ \leq & \frac{p_1^\alpha p_2^\beta \Gamma(\alpha+1)\Gamma(\beta+1)}{4(b^{p_1} - a^{p_1})^\alpha (d^{p_2} - c^{p_2})^\beta} \left[p_1 p_2 I_{a+,c+}^{\alpha,\beta} f(b,d) + p_1 p_2 I_{a+,d-}^{\alpha,\beta} f(b,c) + p_1 p_2 I_{b-,c+}^{\alpha,\beta} f(a,d) + p_1 p_2 I_{b-,d-}^{\alpha,\beta} f(a,c) \right] \\ \leq & \frac{\alpha\beta}{4} [f(a,c) + f(a,d) + f(b,c) + f(b,d)] \left\{ \frac{1}{(\alpha+s_1)(\beta+s_2)} + \frac{B(\beta, s_2+1)}{(\alpha+s_1)} + \frac{B(\alpha, s_1+1)}{(\beta+s_2)} \right. \\ & \left. + B(\beta, s_2+1)B(\alpha, s_1+1) \right\}, \end{aligned}$$

where $B(x, y) = \int_0^1 \tau^{x-1} (1-\tau)^{y-1} d\tau$, for all $x, y > 0$ is the Beta function.

Proof. If one chooses $h_1(t) = t^{s_1}, h_2(t) = t^{s_2}$ for $s_1, s_2 \in (0, 1]$, then calculation of integrals involved in inequality (6) leads to the required result. \square

Corollary 3.7. Suppose that $f : \Delta \rightarrow \mathbb{R}$ is an (s_1, s_2) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} & 2^{s_1+s_2-2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \left[I_{a+,c+}^{\alpha,\beta} f(b,d) + I_{a+,d-}^{\alpha,\beta} f(b,c) + I_{b-,c+}^{\alpha,\beta} f(a,d) + I_{b-,d-}^{\alpha,\beta} f(a,c) \right] \\ \leq & \frac{\alpha\beta}{4} [f(a,c) + f(a,d) + f(b,c) + f(b,d)] \left\{ \frac{1}{(\alpha+s_1)(\beta+s_2)} + \frac{B(\beta, s_2+1)}{(\alpha+s_1)} + \frac{B(\alpha, s_1+1)}{(\beta+s_2)} \right. \\ & \left. + B(\beta, s_2+1)B(\alpha, s_1+1) \right\}, \end{aligned}$$

where $B(x, y)$ is the Beta function as defined in Corollary 3.6. Above result extends Theorem 10 of [24].

Corollary 3.8. Suppose that $f : \Delta \rightarrow \mathbb{R}$ is a (p_1, s) - (p_2, s) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} & 4^{s-1} f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}}, \left[\frac{c^p + d^p}{2} \right]^{\frac{1}{p}} \right) \\ & \leq \frac{p_1^\alpha p_2^\beta \Gamma(\alpha + 1) \Gamma(\beta + 1)}{4 (b^{p_1} - a^{p_1})^\alpha (d^{p_2} - c^{p_2})^\beta} \left[{}^{p_1, p_2} I_{a+, c+}^{\alpha, \beta} f(b, d) + {}^{p_1, p_2} I_{a+, d-}^{\alpha, \beta} f(b, c) + {}^{p_1, p_2} I_{b-, c+}^{\alpha, \beta} f(a, d) + {}^{p_1, p_2} I_{b-, d-}^{\alpha, \beta} f(a, c) \right] \\ & \leq \frac{\alpha \beta}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \left\{ \frac{1}{(\alpha + s)(\beta + s)} + \frac{B(\beta, s + 1)}{(\alpha + s)} + \frac{B(\alpha, s + 1)}{(\beta + s)} + B(\beta, s + 1)B(\alpha, s + 1) \right\}. \end{aligned}$$

Corollary 3.9. Suppose that $f : \Delta \rightarrow \mathbb{R}$ is (p, s) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} & 4^{s-1} f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}}, \left[\frac{c^p + d^p}{2} \right]^{\frac{1}{p}} \right) \\ & \leq \frac{p^{\alpha+\beta} \Gamma(\alpha + 1) \Gamma(\beta + 1)}{4 (b^p - a^p)^\alpha (d^p - c^p)^\beta} \left[{}^{p, p} I_{a+, c+}^{\alpha, \beta} f(b, d) + {}^{p, p} I_{a+, d-}^{\alpha, \beta} f(b, c) + {}^{p, p} I_{b-, c+}^{\alpha, \beta} f(a, d) + {}^{p, p} I_{b-, d-}^{\alpha, \beta} f(a, c) \right] \\ & \leq \frac{\alpha \beta}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \left\{ \frac{1}{(\alpha + s)(\beta + s)} + \frac{B(\beta, s + 1)}{(\alpha + s)} + \frac{B(\alpha, s + 1)}{(\beta + s)} + B(\beta, s + 1)B(\alpha, s + 1) \right\}. \end{aligned}$$

This result also gives a generalization of Theorem 10 of [24].

To prove the next result, we need the following motivation.

Proposition 3.10. Let $f : I = [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a (p, h) -convex function and $f \in L_1[a, b]$. Then following double inequality holds:

$$\frac{1}{h\left(\frac{1}{2}\right)} f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \leq \frac{p^\alpha \Gamma(\alpha + 1)}{(b^p - a^p)^\alpha} \left[{}^p I_{a+}^\alpha f(b) + {}^p I_{b-}^\alpha f(a) \right] \leq \alpha [f(a) + f(b)] \int_0^1 t^{\alpha-1} [h(t) + h(1-t)] dt. \tag{14}$$

Proof. Since f is a (p, h) -convex function on $[a, b]$, so by taking $x^p = ta^p + (1-t)b^p, y^p = (1-t)a^p + tb^p$ and for all $t \in [0, 1]$,

$$\frac{1}{h\left(\frac{1}{2}\right)} f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \leq f \left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) + f \left([(1-t)a^p + tb^p]^{\frac{1}{p}} \right). \tag{15}$$

Multiplying both sides of (15) by $t^{\alpha-1}$ and integrating w.r.t. t over $[0, 1]$,

$$\frac{1}{\alpha h\left(\frac{1}{2}\right)} f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \leq \int_0^1 t^{\alpha-1} f \left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) dt + \int_0^1 t^{\alpha-1} f \left([(1-t)a^p + tb^p]^{\frac{1}{p}} \right) dt. \tag{16}$$

By change of variable in (16), we have

$$\frac{1}{\alpha h\left(\frac{1}{2}\right)} f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \leq \frac{p}{(b^p - a^p)^\alpha} \left[\int_a^b \frac{x^{p-1}}{(b^p - x^p)^\alpha} f(x) dx + \int_a^b \frac{x^{p-1}}{(x^p - a^p)^\alpha} f(x) dx \right].$$

Applying the Definition 2.16 of Katugampola fractional integrals, one has the first inequality of (14).

For the second inequality on the right hand side of (14), by using the (p, h) -convexity of f , we have

$$f(x) + f(y) = f \left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) + f \left([(1-t)a^p + tb^p]^{\frac{1}{p}} \right) \leq [f(a) + f(b)] (h(t) + h(1-t)).$$

Multiplying by $t^{\alpha-1}$ on both sides and integrating over $[0, 1]$, we obtained the second inequality of (14). \square

Remark 3.11. If $\alpha = 1$, then above result coincide to Theorem 5 of [7].

Remark 3.12. If $p = 1$ and $h(t) = t$, then we get Theorem 2 of [10].

Now we give our next main result.

Theorem 3.13. Let $f : \Delta \rightarrow \mathbb{R}$ be a coordinated (p_1, h_1) - (p_2, h_2) -convex function and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\ & \leq \frac{p_1^\alpha \Gamma(\alpha + 1)}{2h_2\left(\frac{1}{2}\right)(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f\left(b, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) + {}^{p_1}I_{b^-}^\alpha f\left(a, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \right] \\ & \quad + \frac{p_2^\beta \Gamma(\beta + 1)}{2h_1\left(\frac{1}{2}\right)(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, d\right) + {}^{p_2}I_{d^-}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, c\right) \right] \\ & \leq \frac{p_1^\alpha p_2^\beta \Gamma(\alpha + 1) \Gamma(\beta + 1)}{(b^{p_1} - a^{p_1})^\alpha (d^{p_2} - c^{p_2})^\beta} \left[{}^{p_1, p_2}I_{a^+, c^+}^{\alpha, \beta} f(b, d) + {}^{p_1, p_2}I_{a^+, d^-}^{\alpha, \beta} f(b, c) + {}^{p_1, p_2}I_{b^-, c^+}^{\alpha, \beta} f(a, d) + {}^{p_1, p_2}I_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] \quad (17) \\ & \leq \frac{\beta p_1^\alpha \Gamma(\alpha + 1)}{2(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f(b, c) + {}^{p_1}I_{a^+}^\alpha f(b, d) + {}^{p_1}I_{b^-}^\alpha f(a, c) + {}^{p_1}I_{b^-}^\alpha f(a, d) \right] \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1 - t_2)] dt_2 \\ & \quad + \frac{\alpha p_2^\beta \Gamma(\beta + 1)}{2(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f(a, d) + {}^{p_2}I_{c^+}^\beta f(b, d) + {}^{p_2}I_{d^-}^\beta f(a, c) + {}^{p_2}I_{d^-}^\beta f(b, c) \right] \int_0^1 t_1^{\alpha-1} [h_1(t_1) + h_1(1 - t_1)] dt_1 \\ & \leq \alpha \beta [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \int_0^1 \int_0^1 t_1^{\alpha-1} t_2^{\beta-1} [h_2(t_2) + h_2(1 - t_2)] [h_1(t_1) + h_1(1 - t_1)] dt_2 dt_1. \end{aligned}$$

Proof. Since $f : \Delta \rightarrow \mathbb{R}$ is a (p_1, h_1) - (p_2, h_2) -convex function, so the partial mapping $f_x : [c, d] \rightarrow \mathbb{R}$ defined by $f_x(v) = f(x, v)$ for all $x \in [a, b]$ is (p_2, h_2) -convex on $[c, d]$. Similarly $f_y : [a, b] \rightarrow \mathbb{R}$ defined by $f_y(u) = f(u, y)$ for all $y \in [c, d]$ is (p_1, h_1) -convex on $[a, b]$. Then by Proposition 3.10 and applying the (p_2, h_2) -convexity of f_x , we have

$$\frac{1}{h_2\left(\frac{1}{2}\right)} f_x\left(\left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \leq \frac{p_2^\beta \Gamma(\beta + 1)}{(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f_x(d) + {}^{p_2}I_{d^-}^\beta f_x(c) \right] \leq \beta [f_x(c) + f_x(d)] \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1 - t_2)] dt_2.$$

Or

$$\begin{aligned} \frac{1}{h_2\left(\frac{1}{2}\right)} f\left(x, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) & \leq \frac{p_2^\beta \Gamma(\beta + 1)}{(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f(x, d) + {}^{p_2}I_{d^-}^\beta f(x, c) \right] \\ & \leq \beta [f(x, c) + f(x, d)] \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1 - t_2)] dt_2. \quad (18) \end{aligned}$$

Integrating inequality (18) w.r.t. x over $[a, b]$ after multiplying by $\frac{\alpha p_1 x^{p_1-1}}{2(b^{p_1}-a^{p_1})^\alpha (b^{p_1}-x^{p_1})^{1-\alpha}}$ and $\frac{\alpha p_1 x^{p_1-1}}{2(b^{p_1}-a^{p_1})^\alpha (x^{p_1}-a^{p_1})^{1-\alpha}}$,

$$\begin{aligned} & \frac{\alpha p_1}{2h_2\left(\frac{1}{2}\right)(b^{p_1}-a^{p_1})^\alpha} \int_a^b \frac{x^{p_1-1}}{(b^{p_1}-x^{p_1})^{1-\alpha}} f\left(x, \left[\frac{c^{p_2}+d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) dx \\ \leq & \frac{\alpha\beta p_1 p_2}{2(d^{p_2}-c^{p_2})^\beta (b^{p_1}-a^{p_1})^\alpha} \left[\int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1} f(x, y)}{(b^{p_1}-x^{p_1})^{1-\alpha} (d^{p_2}-y^{p_2})^{1-\beta}} dy dx + \int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1} f(x, y)}{(b^{p_1}-x^{p_1})^{1-\alpha} (y^{p_2}-c^{p_2})^{1-\beta}} dy dx \right] \\ \leq & \frac{\alpha\beta p_1}{2(b^{p_1}-a^{p_1})^\alpha} \left[\int_a^b \frac{x^{p_1-1} f(x, c)}{(b^{p_1}-x^{p_1})^{1-\alpha}} dx + \int_a^b \frac{x^{p_1-1} f(x, d)}{(b^{p_1}-x^{p_1})^{1-\alpha}} dx \right] \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1-t_2)] dt_2, \end{aligned} \tag{19}$$

and

$$\begin{aligned} & \frac{\alpha p_1}{2h_2\left(\frac{1}{2}\right)(b^{p_1}-a^{p_1})^\alpha} \int_a^b \frac{x^{p_1-1}}{(x^{p_1}-a^{p_1})^{1-\alpha}} f\left(x, \left[\frac{c^{p_2}+d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) dx \\ \leq & \frac{\alpha\beta p_1 p_2}{2(d^{p_2}-c^{p_2})^\beta (b^{p_1}-a^{p_1})^\alpha} \left[\int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1} f(x, y)}{(x^{p_1}-a^{p_1})^{1-\alpha} (d^{p_2}-y^{p_2})^{1-\beta}} dy dx + \int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1} f(x, y)}{(x^{p_1}-a^{p_1})^{1-\alpha} (y^{p_2}-c^{p_2})^{1-\beta}} dy dx \right] \\ \leq & \frac{\alpha\beta p_1}{2(b^{p_1}-a^{p_1})^\alpha} \left[\int_a^b \frac{x^{p_1-1} f(x, c)}{(x^{p_1}-a^{p_1})^{1-\alpha}} dx + \int_a^b \frac{x^{p_1-1} f(x, d)}{(x^{p_1}-a^{p_1})^{1-\alpha}} dx \right] \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1-t_2)] dt_2. \end{aligned} \tag{20}$$

Now again by Proposition 3.10 and applying (p_1, h_1) -convexity of f_y , we have

$$\begin{aligned} \frac{1}{h_1\left(\frac{1}{2}\right)} f_y\left(\left[\frac{a^{p_1}+b^{p_1}}{2}\right]^{\frac{1}{p_1}}\right) & \leq \frac{p_1^\alpha \Gamma(\alpha+1)}{(b^{p_1}-a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f_y(b) + {}^{p_1}I_{b^-}^\alpha f_y(a) \right] \\ & \leq \alpha [f_y(a) + f_y(b)] \int_0^1 t_1^{\alpha-1} [h_1(t_1) + h_1(1-t_1)] dt_1. \end{aligned} \tag{21}$$

Integrating (21) w.r.t. y over $[c, d]$ after multiplying by $\frac{\beta p_2 y^{p_2-1}}{2(d^{p_2}-c^{p_2})^\beta (d^{p_2}-y^{p_2})^{1-\beta}}$ and $\frac{\beta p_2 y^{p_2-1}}{2(d^{p_2}-c^{p_2})^\beta (y^{p_2}-c^{p_2})^{1-\beta}}$, we have

$$\begin{aligned} & \frac{\beta p_2}{2h_1\left(\frac{1}{2}\right)(d^{p_2}-c^{p_2})^\beta} \int_c^d \frac{y^{p_2-1}}{(d^{p_2}-y^{p_2})^{1-\beta}} f\left(\left[\frac{a^{p_1}+b^{p_1}}{2}\right]^{\frac{1}{p_1}}, y\right) dy \\ \leq & \frac{\alpha\beta p_1 p_2}{2(d^{p_2}-c^{p_2})^\beta (b^{p_1}-a^{p_1})^\alpha} \left[\int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1} f(x, y)}{(b^{p_1}-x^{p_1})^{1-\alpha} (d^{p_2}-y^{p_2})^{1-\beta}} dy dx + \int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1} f(x, y)}{(x^{p_1}-a^{p_1})^{1-\alpha} (d^{p_2}-y^{p_2})^{1-\beta}} dy dx \right] \\ \leq & \frac{\alpha\beta p_2}{2(d^{p_2}-c^{p_2})^\beta} \left[\int_c^d \frac{y^{p_2-1} f(a, y)}{(d^{p_2}-y^{p_2})^{1-\beta}} dy + \int_c^d \frac{y^{p_2-1} f(b, y)}{(d^{p_2}-y^{p_2})^{1-\beta}} dy \right] \int_0^1 t_1^{\alpha-1} [h_1(t_1) + h_2(1-t_1)] dt_1. \end{aligned} \tag{22}$$

and

$$\begin{aligned} & \frac{\beta p_2}{2h_1\left(\frac{1}{2}\right)(d^{p_2} - c^{p_2})^\beta} \int_c^d \frac{y^{p_2-1}}{(y^{p_2} - c^{p_2})^{1-\beta}} f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, y\right) dy \\ \leq & \frac{\alpha\beta p_1 p_2}{2(d^{p_2} - c^{p_2})^\beta (b^{p_1} - a^{p_1})^\alpha} \left[\int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1} f(x, y)}{(b^{p_1} - x^{p_1})^{1-\alpha} (y^{p_2} - c^{p_2})^{1-\beta}} dy dx + \int_a^b \int_c^d \frac{x^{p_1-1} y^{p_2-1} f(x, y)}{(x^{p_1} - a^{p_1})^{1-\alpha} (y^{p_2} - c^{p_2})^{1-\beta}} dy dx \right] \\ \leq & \frac{\alpha\beta p_2}{2(d^{p_2} - c^{p_2})^\beta} \left[\int_c^d \frac{y^{p_2-1} f(a, y)}{(y^{p_2} - c^{p_2})^{1-\beta}} dy + \int_c^d \frac{y^{p_2-1} f(b, y)}{(y^{p_2} - c^{p_2})^{1-\beta}} dy \right] \int_0^1 t_1^{\alpha-1} [h_1(t_1) + h_2(1 - t_1)] dt_1. \end{aligned} \tag{23}$$

Adding inequalities (19),(20),(22),(23) and applying Definition 2.17, one obtained

$$\begin{aligned} & \frac{p_1^\alpha \Gamma(\alpha + 1)}{2h_2\left(\frac{1}{2}\right)(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f\left(b, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) + {}^{p_1}I_{b^-}^\alpha f\left(a, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \right] \\ & + \frac{p_2^\beta \Gamma(\beta + 1)}{2h_1\left(\frac{1}{2}\right)(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, d\right) + {}^{p_2}I_{d^-}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, c\right) \right] \\ \leq & \frac{p_1^\alpha p_2^\beta \Gamma(\alpha + 1) \Gamma(\beta + 1)}{(b^{p_1} - a^{p_1})^\alpha (d^{p_2} - c^{p_2})^\beta} \left[{}^{p_1, p_2}I_{a^+, c^+}^{\alpha, \beta} f(b, d) + {}^{p_1, p_2}I_{a^+, d^-}^{\alpha, \beta} f(b, c) + {}^{p_1, p_2}I_{b^-, c^+}^{\alpha, \beta} f(a, d) + {}^{p_1, p_2}I_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] \\ \leq & \frac{\beta p_1^\alpha \Gamma(\alpha + 1)}{2(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f(b, c) + {}^{p_1}I_{a^+}^\alpha f(b, d) + {}^{p_1}I_{b^-}^\alpha f(a, c) + {}^{p_1}I_{b^-}^\alpha f(a, d) \right] \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1 - t_2)] dt_2 \\ & + \frac{\alpha p_2^\beta \Gamma(\beta + 1)}{2(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f(a, d) + {}^{p_2}I_{c^+}^\beta f(b, d) + {}^{p_2}I_{d^-}^\beta f(a, c) + {}^{p_2}I_{d^-}^\beta f(b, c) \right] \int_0^1 t_1^{\alpha-1} [h_1(t_1) + h_1(1 - t_1)] dt_1. \end{aligned}$$

Which are the second and third inequalities of (17).

For the last inequality of (17), applying Proposition 3.10 to the last part of above inequality, we have

$$\begin{aligned} & \frac{\beta p_1^\alpha \Gamma(\alpha + 1)}{2(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f(b, c) + {}^{p_1}I_{a^+}^\alpha f(b, d) + {}^{p_1}I_{b^-}^\alpha f(a, c) + {}^{p_1}I_{b^-}^\alpha f(a, d) \right] \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1 - t_2)] dt_2 \\ & + \frac{\alpha p_2^\beta \Gamma(\beta + 1)}{2(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f(a, d) + {}^{p_2}I_{c^+}^\beta f(b, d) + {}^{p_2}I_{d^-}^\beta f(a, c) + {}^{p_2}I_{d^-}^\beta f(b, c) \right] \int_0^1 t_1^{\alpha-1} [h_1(t_1) + h_1(1 - t_1)] dt_1 \\ \leq & \alpha\beta [f(b, c) + f(a, c) + f(b, d) + f(a, d)] \int_0^1 t_1^{\alpha-1} [h_1(t_1) + h_1(1 - t_1)] dt_1 \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1 - t_2)] dt_2. \end{aligned}$$

For the first inequality of (17), we again use Proposition 3.10 and get

$$f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \leq \frac{h_1\left(\frac{1}{2}\right) p_1^\alpha \Gamma(\alpha + 1)}{(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f\left(b, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) + {}^{p_1}I_{b^-}^\alpha f\left(a, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \right]. \tag{24}$$

Similarly,

$$f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \leq \frac{h_2\left(\frac{1}{2}\right) p_2^\beta \Gamma(\beta + 1)}{(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, d\right) + {}^{p_2}I_{d^-}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, c\right) \right]. \tag{25}$$

Adding (24) and (25), then dividing by $2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)$, one has the first inequality of (17), which then completes the proof. \square

Remark 3.14. If $\alpha = 1 = \beta$, then above result gives Theorem 2.14, which was proved in [22].

Remark 3.15. If $p_1 = p_2 = 1$ and $h_1(t) = h_2(t) = t$, our result reduced to Theorem 2.6, which was proved in [23].

Corollary 3.16. Let f be an (h_1, h_2) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$, then one has following inequalities:

$$\begin{aligned} & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq & \frac{\Gamma(\alpha + 1)}{2h_2\left(\frac{1}{2}\right)(b-a)^\alpha} \left[I_{a^+}^\alpha f\left(b, \frac{c+d}{2}\right) + I_{b^-}^\alpha f\left(a, \frac{c+d}{2}\right) \right] + \frac{\Gamma(\beta + 1)}{2h_1\left(\frac{1}{2}\right)(d-c)^\beta} \left[I_{c^+}^\beta f\left(\frac{a+b}{2}, d\right) + I_{d^-}^\beta f\left(\frac{a+b}{2}, c\right) \right] \\ \leq & \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{(b-a)^\alpha(d-c)^\beta} \left[I_{a^+,c^+}^{\alpha,\beta} f(b, d) + I_{a^+,d^-}^{\alpha,\beta} f(b, c) + I_{b^-,c^+}^{\alpha,\beta} f(a, d) + I_{b^-,d^-}^{\alpha,\beta} f(a, c) \right] \\ \leq & \frac{\beta\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[I_{a^+}^\alpha f(b, c) + I_{a^+}^\alpha f(b, d) + I_{b^-}^\alpha f(a, c) + I_{b^-}^\alpha f(a, d) \right] \int_0^1 t_2^{\beta-1} [h_2(t_2) + h_2(1-t_2)] dt_2 \\ & + \frac{\alpha\Gamma(\beta + 1)}{2(d-c)^\beta} \left[I_{c^+}^\beta f(a, d) + I_{c^+}^\beta f(b, d) + I_{d^-}^\beta f(a, c) + I_{d^-}^\beta f(b, c) \right] \int_0^1 t_1^{\alpha-1} [h_1(t_1) + h_1(1-t_1)] dt_1 \\ \leq & \alpha\beta [f(a, c) + f(a, c) + f(a, c) + f(a, c)] \int_0^1 \int_0^1 t_1^{\alpha-1} t_2^{\beta-1} [h_2(t_2) + h_2(1-t_2)] [h_1(t_1) + h_1(1-t_1)] dt_2 dt_1. \end{aligned}$$

Remark 3.17. Corollary 3.16 gives the classical version of Hadamard type inequalities for coordinated (h_1, h_2) -convex functions, if $\alpha = \beta = 1$.

Corollary 3.18. Let $f : \Delta \rightarrow \mathbb{R}$ be a (p_1, s_1) - (p_2, s_2) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} & 2^{s_1+s_2} f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\ \leq & \frac{2^{s_2-1} p_1^\alpha \Gamma(\alpha + 1)}{(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f\left(b, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) + {}^{p_1}I_{b^-}^\alpha f\left(a, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \right] \\ & + \frac{2^{s_1-1} p_2^\beta \Gamma(\beta + 1)}{(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, d\right) + {}^{p_2}I_{d^-}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, c\right) \right] \\ \leq & \frac{p_1^\alpha p_2^\beta \Gamma(\alpha + 1)\Gamma(\beta + 1)}{(b^{p_1} - a^{p_1})^\alpha (d^{p_2} - c^{p_2})^\beta} \left[{}^{p_1,p_2}I_{a^+,c^+}^{\alpha,\beta} f(b, d) + {}^{p_1,p_2}I_{a^+,d^-}^{\alpha,\beta} f(b, c) + {}^{p_1,p_2}I_{b^-,c^+}^{\alpha,\beta} f(a, d) + {}^{p_1,p_2}I_{b^-,d^-}^{\alpha,\beta} f(a, c) \right] \end{aligned}$$

$$\begin{aligned} &\leq \frac{\beta p_1^\alpha \Gamma(\alpha + 1)}{2(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f(b, c) + {}^{p_1}I_{a^+}^\alpha f(b, d) + {}^{p_1}I_{b^-}^\alpha f(a, c) + {}^{p_1}I_{b^-}^\alpha f(a, d) \right] \left\{ \frac{1}{\beta + s_2} + B(\beta, s_2 + 1) \right\} \\ &\quad + \frac{\alpha p_2^\beta \Gamma(\beta + 1)}{2(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f(a, d) + {}^{p_2}I_{c^+}^\beta f(b, d) + {}^{p_2}I_{d^-}^\beta f(a, c) + {}^{p_2}I_{d^-}^\beta f(b, c) \right] \left\{ \frac{1}{\alpha + s_1} + B(\alpha, s_1 + 1) \right\} \\ &\leq \alpha\beta [f(a, c) + f(a, c) + f(a, c) + f(a, c)] \left\{ \frac{1}{(\alpha + s_1)(\beta + s_2)} + \frac{B(\alpha, s_1 + 1)}{\beta + s_2} + \frac{B(\beta, s_2 + 1)}{\alpha + s_1} \right. \\ &\quad \left. + B(\alpha, s_1 + 1)B(\beta, s_2 + 1) \right\}. \end{aligned}$$

Remark 3.19. If $\alpha = 1 = \beta$, then Corollary 3.18 reduces to a new result for (p_1, s_1) - (p_2, s_2) -convex functions on the coordinates on Δ via classical integrals as follows:

$$\begin{aligned} &2^{s_1+s_2} f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\ &\leq \frac{2^{s_2} p_1}{(b^{p_1} - a^{p_1})} \int_a^b x^{p_1-1} f\left(x, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) dx + \frac{2^{s_1} p_2}{(d^{p_2} - c^{p_2})} \int_c^d y^{p_2-1} f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, y\right) dy \\ &\leq \frac{4p_1 p_2}{(b^{p_1} - a^{p_1})(d^{p_2} - c^{p_2})} \int_a^b \int_c^d x^{p_1-1} y^{p_2-1} f(x, y) dy dx \\ &\leq \frac{2p_1}{(b^{p_1} - a^{p_1})(s_2 + 1)} \int_a^b x^{p_1-1} [f(x, c) + f(x, d)] dx + \frac{2p_2}{(d^{p_2} - c^{p_2})(s_1 + 1)} \int_c^d y^{p_2-1} [f(a, y) + f(b, y)] dy \\ &\leq 4 \frac{[f(a, c) + f(a, d) + f(b, c) + f(b, d)]}{(s_1 + 1)(s_2 + 1)}. \end{aligned}$$

Corollary 3.20. Let $f : \Delta \rightarrow \mathbb{R}$ be a (p_1, s) - (p_2, s) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} &4^s f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\ &\leq \frac{2^{s-1} p_1^\alpha \Gamma(\alpha + 1)}{(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f\left(b, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) + {}^{p_1}I_{b^-}^\alpha f\left(a, \left[\frac{c^{p_2} + d^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \right] \\ &\quad + \frac{2^{s-1} p_2^\beta \Gamma(\beta + 1)}{(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, d\right) + {}^{p_2}I_{d^-}^\beta f\left(\left[\frac{a^{p_1} + b^{p_1}}{2}\right]^{\frac{1}{p_1}}, c\right) \right] \\ &\leq \frac{p_1^\alpha p_2^\beta \Gamma(\alpha + 1) \Gamma(\beta + 1)}{(b^{p_1} - a^{p_1})^\alpha (d^{p_2} - c^{p_2})^\beta} \left[{}^{p_1, p_2}I_{a^+, c^+}^{\alpha, \beta} f(b, d) + {}^{p_1, p_2}I_{a^+, d^-}^{\alpha, \beta} f(b, c) + {}^{p_1, p_2}I_{b^-, c^+}^{\alpha, \beta} f(a, d) + {}^{p_1, p_2}I_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] \\ &\leq \frac{\beta p_1^\alpha \Gamma(\alpha + 1)}{2(b^{p_1} - a^{p_1})^\alpha} \left[{}^{p_1}I_{a^+}^\alpha f(b, c) + {}^{p_1}I_{a^+}^\alpha f(b, d) + {}^{p_1}I_{b^-}^\alpha f(a, c) + {}^{p_1}I_{b^-}^\alpha f(a, d) \right] \left\{ \frac{1}{\beta + s} + B(\beta, s + 1) \right\} \\ &\quad + \frac{\alpha p_2^\beta \Gamma(\beta + 1)}{2(d^{p_2} - c^{p_2})^\beta} \left[{}^{p_2}I_{c^+}^\beta f(a, d) + {}^{p_2}I_{c^+}^\beta f(b, d) + {}^{p_2}I_{d^-}^\beta f(a, c) + {}^{p_2}I_{d^-}^\beta f(b, c) \right] \left\{ \frac{1}{\alpha + s} + B(\alpha, s + 1) \right\} \\ &\leq \alpha\beta [f(a, c) + f(a, c) + f(a, c) + f(a, c)] \left\{ \frac{1}{(\alpha + s)(\beta + s)} + \frac{B(\alpha, s + 1)}{\beta + s} + \frac{B(\beta, s + 1)}{\alpha + s} + B(\alpha, s + 1)B(\beta, s + 1) \right\}. \end{aligned}$$

Remark 3.21. If one use $\alpha = 1 = \beta$, then inequalities in Corollary 3.20 will present the classical version of Hadamard type inequalities for coordinated (p_1, s) - (p_2, s) -convex functions.

Corollary 3.22. Let $f : \Delta \rightarrow \mathbb{R}$ be a (p, s) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} & 4^s f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}}, \left[\frac{c^p + d^p}{2} \right]^{\frac{1}{p}} \right) \\ & \leq \frac{2^{s-1} p^\alpha \Gamma(\alpha + 1)}{(b^p - a^p)^\alpha} \left[{}^p I_{a^+}^\alpha f \left(b, \left[\frac{c^p + d^p}{2} \right]^{\frac{1}{p}} \right) + {}^p I_{b^-}^\alpha f \left(a, \left[\frac{c^p + d^p}{2} \right]^{\frac{1}{p}} \right) \right] \\ & \quad + \frac{2^{s-1} p^\beta \Gamma(\beta + 1)}{(d^p - c^p)^\beta} \left[{}^p I_{c^+}^\beta f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}}, d \right) + {}^p I_{d^-}^\beta f \left(\left[\frac{a^p + b^p}{2} \right]^{\frac{1}{p}}, c \right) \right] \\ & \leq \frac{p^{\alpha+\beta} \Gamma(\alpha + 1) \Gamma(\beta + 1)}{(b^p - a^p)^\alpha (d^p - c^p)^\beta} \left[{}^{p,p} I_{a^+, c^+}^{\alpha, \beta} f(b, d) + {}^{p,p} I_{a^+, d^-}^{\alpha, \beta} f(b, c) + {}^{p,p} I_{b^-, c^+}^{\alpha, \beta} f(a, d) + {}^{p,p} I_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] \\ & \leq \frac{\beta p^\alpha \Gamma(\alpha + 1)}{2(b^p - a^p)^\alpha} \left[{}^p I_{a^+}^\alpha f(b, c) + {}^p I_{a^+}^\alpha f(b, d) + {}^p I_{b^-}^\alpha f(a, c) + {}^p I_{b^-}^\alpha f(a, d) \right] \left\{ \frac{1}{\beta + s} + B(\beta, s + 1) \right\} \\ & \quad + \frac{\alpha p^\beta \Gamma(\beta + 1)}{2(d^p - c^p)^\beta} \left[{}^p I_{c^+}^\beta f(a, d) + {}^p I_{c^+}^\beta f(b, d) + {}^p I_{d^-}^\beta f(a, c) + {}^p I_{d^-}^\beta f(b, c) \right] \left\{ \frac{1}{\alpha + s} + B(\alpha, s + 1) \right\} \\ & \leq \alpha \beta [f(a, c) + f(a, c) + f(a, c) + f(a, c)] \left\{ \frac{1}{(\alpha + s)(\beta + s)} + \frac{B(\alpha, s + 1)}{\beta + s} + \frac{B(\beta, s + 1)}{\alpha + s} + B(\alpha, s + 1)B(\beta, s + 1) \right\}. \end{aligned}$$

Corollary 3.23. Let $f : \Delta \rightarrow \mathbb{R}$ be an (s_1, s_2) -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the inequalities:

$$\begin{aligned} & 2^{s_1+s_2} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \\ & \leq \frac{2^{s_2-1} \Gamma(\alpha + 1)}{(b-a)^\alpha} \left[I_{a^+}^\alpha f \left(b, \frac{c+d}{2} \right) + I_{b^-}^\alpha f \left(a, \frac{c+d}{2} \right) \right] + \frac{2^{s_1-1} \Gamma(\beta + 1)}{(d-c)^\beta} \left[I_{c^+}^\beta f \left(\frac{a+b}{2}, d \right) + I_{d^-}^\beta f \left(\frac{a+b}{2}, c \right) \right] \\ & \leq \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{(b-a)^\alpha (d-c)^\beta} \left[I_{a^+, c^+}^{\alpha, \beta} f(b, d) + I_{a^+, d^-}^{\alpha, \beta} f(b, c) + I_{b^-, c^+}^{\alpha, \beta} f(a, d) + I_{b^-, d^-}^{\alpha, \beta} f(a, c) \right] \\ & \leq \frac{\beta \Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[I_{a^+}^\alpha f(b, c) + I_{a^+}^\alpha f(b, d) + I_{b^-}^\alpha f(a, c) + I_{b^-}^\alpha f(a, d) \right] \left\{ \frac{1}{\beta + s_2} + B(\beta, s_2 + 1) \right\} \\ & \quad + \frac{\alpha \Gamma(\beta + 1)}{2(d-c)^\beta} \left[I_{c^+}^\beta f(a, d) + I_{c^+}^\beta f(b, d) + I_{d^-}^\beta f(a, c) + I_{d^-}^\beta f(b, c) \right] \left\{ \frac{1}{\alpha + s_1} + B(\alpha, s_1 + 1) \right\} \\ & \leq \alpha \beta [f(a, c) + f(a, c) + f(a, c) + f(a, c)] \left\{ \frac{1}{(\alpha + s_1)(\beta + s_2)} + \frac{B(\alpha, s_1 + 1)}{\beta + s_2} + \frac{B(\beta, s_2 + 1)}{\alpha + s_1} \right. \\ & \quad \left. + B(\alpha, s_1 + 1)B(\beta, s_2 + 1) \right\}. \end{aligned}$$

Remark 3.24. If $\alpha = 1 = \beta$, then the inequalities in Corollary 3.23 extends Theorem 2.1 of [20]. It will coincide to the Theorem 2.1 of [20] if $s_1 = s_2 = s$.

Corollary 3.25. Let $f : \Delta \rightarrow \mathbb{R}$ be an s -convex function on the coordinates on Δ and $f \in L_1(\Delta)$. Then one has the

inequalities:

$$\begin{aligned}
& 4^s f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \leq \frac{2^{s-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[I_{a^+}^\alpha f\left(b, \frac{c+d}{2}\right) + I_{b^-}^\alpha f\left(a, \frac{c+d}{2}\right) \right] + \frac{2^{s-1}\Gamma(\beta+1)}{(d-c)^\beta} \left[I_{c^+}^\beta f\left(\frac{a+b}{2}, d\right) + I_{d^-}^\beta f\left(\frac{a+b}{2}, c\right) \right] \\
& \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta} \left[I_{a^+,c^+}^{\alpha,\beta} f(b,d) + I_{a^+,d^-}^{\alpha,\beta} f(b,c) + I_{b^-,c^+}^{\alpha,\beta} f(a,d) + I_{b^-,d^-}^{\alpha,\beta} f(a,c) \right] \\
& \leq \frac{\beta\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[I_{a^+}^\alpha f(b,c) + I_{a^+}^\alpha f(b,d) + I_{b^-}^\alpha f(a,c) + I_{b^-}^\alpha f(a,d) \right] \left\{ \frac{1}{\beta+s} + B(\beta, s+1) \right\} \\
& \quad + \frac{\alpha\Gamma(\beta+1)}{2(d-c)^\beta} \left[I_{c^+}^\beta f(a,d) + I_{c^+}^\beta f(b,d) + I_{d^-}^\beta f(a,c) + I_{d^-}^\beta f(b,c) \right] \left\{ \frac{1}{\alpha+s} + B(\alpha, s+1) \right\} \\
& \leq \alpha\beta [f(a,c) + f(a,c) + f(a,c) + f(a,c)] \left\{ \frac{1}{(\alpha+s)(\beta+s)} + \frac{B(\alpha, s+1)}{\beta+s} + \frac{B(\beta, s+1)}{\alpha+s} + B(\alpha, s+1)B(\beta, s+1) \right\}.
\end{aligned}$$

Remark 3.26. Inequalities in Corollary 3.25, will be reduced to special case of classical integrals if $\alpha = 1 = \beta$. In that case it will coincide to Theorem 2.1 of [20].

Conclusion 3.27. In this paper two inequalities of Hadamard type are presented for the Katugampola fractional integrals keeping coordinated (p_1, h_1) - (p_2, h_2) -convexity into account. The special cases are discussed to see the compatibility with the previously known results. It is found that the results are highly compatible and they can be extend for other types of convexities.

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