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Iterations for Systems of Variational Inequalities, Fixed Points of Pseudocontractions and Zeros of Accretive Operators

Lu-Chuan Ceng[a], Jen-Chih Yao[b], Yonghong Yao[c,d]

[a]*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*
[b]*Center for General Education, China Medical University, Taichung, 40402, Taiwan*
[c]*School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China*
[d]*The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China*

Abstract. In this paper, we introduce implicit composite three-step Mann iterations for finding a common solution of a general system of variational inequalities, a fixed point problem of a countable family of pseudocontractive mappings and a zero problem of an accretive operator in Banach spaces. Strong convergence of the suggested iterations are given.

1. Introduction

Let E be a smooth Banach space and C be a nonempty closed convex subset of E. Let $B_1, B_2 : C \to E$ be two nonlinear mappings. Recall that the general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho B_1 y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \eta B_2 x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1}$$

where ρ and η are two positive constants.

The variational inequality was first discussed by Lions and Stampacchia [15] and now is well-known. Variational inequality theory has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Some efficient methods have received great attention given by many authors, see e.g., [4–7, 11, 12, 21–28] and the references therein.

On the other hand, a great deal of effort has gone into the existence of zeros of accretive mappings or fixed points of pseudocontractive mappings (including nonexpansive mappings) and iterative construction of zeros of accretive mappings, and of fixed points of pseudocontractive mappings (including nonexpansive mappings); see, e.g., [1–5, 9–11, 17, 20].

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Email addresses: zenglc@hotmail.com (Lu-Chuan Ceng), yaojc@mail.cmu.edu.tw (Jen-Chih Yao), yaoyonghong@aliyun.com (Yonghong Yao)

Motivated and inspired by the research going on in this area, the main purpose of the paper is to introduce and analyze implicit composite three-step Mann iterations for finding a common solution of GSVI (1) and a fixed point problem (FPP) of a countable family of uniformly Lipschitzian pseudocontractive self-mappings on C and a zero problem (ZP) of an accretive operator in E. Under quite suitable assumptions, we derive some strong convergence results, which improve, extend, supplement and develop the corresponding ones announced in the earlier and very recent literature; see, e.g., [5].

2. Preliminaries

Let E be a real Banach space with the dual E^*. Let C be a nonempty closed convex subset of E. A mapping $f : C \to C$ is said to be k-Lipschitz if $k \in [0, +\infty)$ and $\|f(x) - f(y)\| \le k\|x - y\|$ for all $x, y \in C$. If $k < 1$, then f is called a k-contraction mapping. If $k = 1$, f is said to be nonexpansive. We use $F(f)$ to denote the set of fixed points of f.

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in E is said to be

(i) ([13]) pseudocontractive if

$$\|x - y\| \le \|x - y + r((I - T)x - (I - T)y)\|, \ \forall x, y \in D(T), \forall r > 0.$$

(ii) λ-strictly pseudocontractive if for each $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2, \text{ for some } \lambda \in (0, 1).$$

Recall that a mapping $T : C \to E$ is said to be

(i) accretive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \ge 0,$$

where J is the normalized duality mapping;

(ii) α-strongly accretive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \ge \alpha\|x - y\|^2, \quad \text{for some } \alpha \in (0, 1).$$

Let $\{T_n\}_{n=1}^\infty$ be a sequence of continuous pseudocontractive self-mappings on C. Then $\{T_n\}_{n=1}^\infty$ is said to be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C if there exists a constant $\ell > 0$ such that each T_n is ℓ-Lipschitz continuous.

Let q be a real number with $1 < q \le 2$ and let E be a Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c > 0$ such that

$$\|x + y\|^q + \|x - y\|^q \le 2(\|x\|^q + \|cy\|^q), \quad \forall x, y \in E.$$

Proposition 2.1. *[1] Let C be a nonempty closed convex subset of a Banach space E. Let $T_0, T_1, ...$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^\infty \sup\{\|T_n x - T_{n-1}x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Tx - T_n x\| : x \in C\} = 0$.*

Proposition 2.2. *[10] Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ be a continuous and strong pseudocontraction mapping. Then, T has a unique fixed point in C.*

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \ge 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Proposition 2.3. *[2, 14] Let C be a nonempty closed convex subset of a smooth Banach space E, D be a nonempty subset of C and Π be a retraction of C onto D. Then the following are equivalent:*

(i) *Π is sunny and nonexpansive;*

(ii) $\|\Pi(x)-\Pi(y)\|^2 \le \langle x-y, j(\Pi(x)-\Pi(y))\rangle, \forall x,y \in C;$

(iii) $\langle x-\Pi(x), j(y-\Pi(x))\rangle \le 0, \forall x \in C, y \in D.$

If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$, which is called the resolvent of A. It is well known that J_r is nonexpansive and $F(J_r) = A^{-1}0$ for all $r > 0$. If $A^{-1}0 \neq \emptyset$, then the inclusion $0 \in Ax$ is solvable.

Proposition 2.4. *[10, 16] For $\lambda, \mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda x\right).$$

Proposition 2.5. *[19] Given a number $r > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $\lambda \in [0,1]$ and $x,y \in E$ such that $\|x\| \le r$ and $\|y\| \le r$.

Proposition 2.6. *[12] Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0,2r] \to \mathbf{R}$, $g(0) = 0$ such that*

$$g(\|x-y\|) \le \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2, \quad \forall x,y \in B_r,$$

where $B_r = \{x \in E : \|x\| \le r\}$.

Lemma 2.7. *[19] Let E be a real Banach space and J be the normalized duality mapping on E. Then for any given $x,y \in E$, the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 2.8. *[4] Let C be a nonempty closed convex subset of a smooth Banach space E and $B_1, B_2 : C \to E$ be two nonlinear mappings. Let Π_C be a sunny nonexpansive retraction from E onto C. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the GSVI (1) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$ where $\mathrm{GSVI}(C, B_1, B_2)$ is the set of fixed points of the mapping $G := \Pi_C(I-\rho B_1)\Pi_C(I-\eta B_2)$ and $y^* = \Pi_C(I-\eta B_2)x^*$.*

Lemma 2.9. *[4] Let C be a nonempty closed convex subset of a smooth Banach space E, and let the mapping $B_i : C \to E$ be λ_i-strictly pseudocontractive and ζ_i-strongly accretive with $\lambda_i + \zeta_i \ge 1$ for $i = 1,2$. Then, for $\rho, \eta \in (0,1]$ we have*

$$\begin{cases} \|(I-\rho B_1)x - (I-\rho B_1)y\| \le [\sqrt{\frac{1-\zeta_1}{\lambda_1}} + (1-\rho)(1+\frac{1}{\lambda_1})]\|x-y\|, & \forall x,y \in C, \\ \|(I-\eta B_2)x - (I-\eta B_2)y\| \le [\sqrt{\frac{1-\zeta_2}{\lambda_2}} + (1-\eta)(1+\frac{1}{\lambda_2})]\|x-y\|, & \forall x,y \in C. \end{cases}$$

In particular, if $1-\frac{\lambda_1}{1+\lambda_1}(1-\sqrt{\frac{1-\zeta_1}{\lambda_1}}) \le \rho \le 1$ and $1-\frac{\lambda_2}{1+\lambda_2}(1-\sqrt{\frac{1-\zeta_2}{\lambda_2}}) \le \eta \le 1$, then $I-\rho B_1$ and $I-\eta B_2$ are nonexpansive mappings.

Lemma 2.10. *[4] Let C be a nonempty closed convex subset of a smooth Banach space E. Let Π_C be a sunny nonexpansive retraction from E onto C, and let the mapping $B_i : C \to E$ be λ_i-strictly pseudocontractive and ζ_i-strongly accretive with $\lambda_i+\zeta_i \ge 1$ for $i = 1,2$. Let $G : C \to C$ be the mapping defined by $G := \Pi_C(I-\rho B_1)\Pi_C(I-\eta B_2)$. If $1-\frac{\lambda_1}{1+\lambda_1}(1-\sqrt{\frac{1-\zeta_1}{\lambda_1}}) \le \rho \le 1$ and $1-\frac{\lambda_2}{1+\lambda_2}(1-\sqrt{\frac{1-\zeta_2}{\lambda_2}}) \le \eta \le 1$, then $G : C \to C$ is nonexpansive.*

Lemma 2.11. *[2] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let* $\{T_n\}_{n=0}^{\infty}$ *be a sequence of nonexpansive mappings on C. Suppose that* $\bigcap_{n=0}^{\infty} F(T_n)$ *is nonempty. Let* $\{\lambda_n\}$ *be a sequence of positive numbers with* $\sum_{n=0}^{\infty} \lambda_n = 1$. *Then a mapping S on C defined by* $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ *for* $x \in C$ *is defined well, nonexpansive and* $F(S) = \bigcap_{n=0}^{\infty} F(T_n)$ *holds.*

Lemma 2.12. *[20] Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E,* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$, *and* $f \in \Xi_C$. *Then the net* $\{x_t\}$ *defined by* $x_t = tf(x_t)+(1-t)Tx_t, \forall t \in (0,1)$, *converges strongly to a point in* $F(T)$. *If we define a mapping* $Q : \Xi_C \to F(T)$ *by* $Q(f) := s\text{-}\lim_{t\to 0} x_t, \forall f \in \Xi_C$, *then* $Q(f)$ *solves the VI:*

$$\langle (I - f)Q(f), j(Q(f) - x)\rangle \leq 0, \quad \forall x \in F(T).$$

In particular, if $f = u \in C$ *is a constant, then the map* $u \mapsto Q(u)$ *is reduced to the sunny nonexpansive retraction of Reich type from C onto* $F(T)$, *i.e.,*

$$\langle Q(u) - u, j(Q(u) - x)\rangle \leq 0, \quad \forall x \in F(T).$$

Lemma 2.13. *[8] Assume that E has a weakly continuous duality map* j_φ *with gauge* φ.

(i) For all $x, y \in E$, *the following inequality holds:*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y)\rangle.$$

(ii) Assume that a sequence $\{x_n\}$ *in E is weakly convergent to a point x. Then the following identity holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall y \in E.$$

Lemma 2.14. *[8] Let E be a reflexive Banach space and have a weakly continuous duality map* j_φ *with gauge* φ, *let C be a nonempty closed convex subset of E, let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$, *and let* $f \in \Xi_C$. *Then* $\{x_t\}$ *defined by* $x_t = tf(x_t) + (1 - t)Tx_t, \forall t \in (0, 1)$, *converges strongly to a point in* $F(T)$ *as* $t \to 0^+$. *Define* $Q : \Xi_C \to F(T)$ *by* $Q(f) := s\text{-}\lim_{t\to 0^+} x_t$. *Then* $Q(f)$ *solves the VI:*

$$\langle (I - f)Q(f), j_\varphi(Q(f) - x)\rangle \leq 0, \quad \forall x \in F(T).$$

In particular, if $f = u \in C$ *is a constant, then the map* $u \mapsto Q(u)$ *is reduced to the sunny nonexpansive retraction of Reich type from C onto* $F(T)$, *i.e.,* $\langle Q(u) - u, j_\varphi(Q(u) - x)\rangle \leq 0, \quad \forall x \in F(T)$.

Lemma 2.15. *[20] Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \varepsilon_n)a_n + \varepsilon_n \xi_n, \quad \forall n \geq 0,$$

where $\{\varepsilon_n\}_{n=0}^{\infty}$ *and* $\{\xi_n\}_{n=0}^{\infty}$ *are real sequences satisfying*

(i) $\{\varepsilon_n\}_{n=0}^{\infty} \subset (0, 1)$, $\sum_{n=0}^{\infty} \varepsilon_n = \infty$;

(ii) either $\limsup_{n\to\infty} \xi_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\varepsilon_n \xi_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.16. *[18] Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space E, and let* $\{\beta_n\}$ *be a sequence of nonnegative numbers in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which either is uniformly smooth or has a weakly continuous duality map j_φ with gauge φ. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $A \subset E \times E$ be an accretive operator in E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $B_i : C \to E$ be λ_i-strictly pseudocontractive and ζ_i-strongly accretive with $\lambda_i + \zeta_i \geq 1 (i = 1, 2)$, $1 - \frac{\lambda_1}{1+\lambda_1}(1 - \sqrt{\frac{1-\zeta_1}{\lambda_1}}) \leq \rho \leq 1$ and $1 - \frac{\lambda_2}{1+\lambda_2}(1 - \sqrt{\frac{1-\zeta_2}{\lambda_2}}) \leq \eta \leq 1$. Let $f : C \to C$ be a k-contraction, and $\{T_n\}_{n=0}^\infty$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C. Suppose that $\Omega := \bigcap_{n=0}^\infty F(T_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap A^{-1}0 \neq \emptyset$. For an arbitrary $x_0 \in C$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} z_n = \sigma_n x_n + (1 - \sigma_n)T_n z_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)J_{r_n} G z_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, \quad \forall n \geq 0, \end{cases} \tag{2}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\sigma_n\}$ are the sequences in $[0, 1]$ and $\{r_n\}$ is a sequence in $(0, \infty)$. Suppose that the following conditions hold:

(C1): $0 \leq \beta_n \leq 1 - k, \forall n \geq n_0$ for some $n_0 \geq 0$, and $\sum_{n=0}^\infty \beta_n = \infty$;

(C2): $\lim_{n\to\infty} |\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}| = 0$ and $\lim_{n\to\infty} |\sigma_{n+1} - \sigma_n| = 0$;

(C3): $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C4): $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$;

(C5): $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ and $r_n \geq \varepsilon > 0, \forall n \geq 0$.

Assume that $\sum_{n=0}^\infty \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C, and let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, and suppose that $F(T) = \bigcap_{n=0}^\infty F(T_n)$. Then $x_n \to x^* \in \Omega \Leftrightarrow \|\beta_n(f(x_n) - x_n)\| + \|Gx_n - x_n\| \to 0$. In this case, (x^*, y^*) is a solution of GSVI (1) with $y^* = \Pi_C(I - \eta B_2)x^*$, and we have

(i) if E is uniformly smooth, then $x^* \in \Omega$ solves the VI: $\langle x^* - f(x^*), j(x^* - x)\rangle \leq 0, \ \forall x \in \Omega$;

(ii) if E has a weakly continuous duality mapping j_φ with gauge φ then $x^* \in \Omega$ solves the VI: $\langle x^* - f(x^*), j_\varphi(x^* - x)\rangle \leq 0, \ \forall x \in \Omega$.

Proof. Without loss of generality, assume that $\{\sigma_n\} \subset [c, d] \subset (0, 1)$ for some $c, d \in (0, 1)$. Note that the mapping $G : C \to C$ is defined as $G := \Pi_C(I - \rho B_1)\Pi_C(I - \eta B_2)$, where $1 - \frac{\lambda_1}{1+\lambda_1}(1 - \sqrt{\frac{1-\zeta_1}{\lambda_1}}) \leq \rho \leq 1$ and $1 - \frac{\lambda_2}{1+\lambda_2}(1 - \sqrt{\frac{1-\zeta_2}{\lambda_2}}) \leq \eta \leq 1$. So, by Lemma 2.10, we obtain that G is nonexpansive. It is easy to see that for each $n \geq 0$ there exists a unique element $z_n \in C$ such that

$$z_n = \sigma_n x_n + (1 - \sigma_n)T_n z_n. \tag{3}$$

As a matter of fact, consider the mapping $F_n x = \sigma_n x_n + (1 - \sigma_n)T_n x, \ \forall x \in C$. Since $T_n : C \to C$ is a continuous pseudocontractive mapping, we deduce that all $x, y \in C$,

$$\langle F_n x - F_n y, j(x - y)\rangle = (1 - \sigma_n)\langle T_n x - T_n y, j(x - y)\rangle \leq (1 - \sigma_n)\|x - y\|^2.$$

Also, from $\{\sigma_n\} \subset [c, d] \subset (0, 1)$ we get $0 < 1 - \sigma_n < 1$ for all $n \geq 0$. Thus, F_n is a continuous and strong pseudocontraction mapping of C into itself. By Proposition 2.3, we know that for each $n \geq 0$ there exists a unique element $z_n \in C$, satisfying (3).

Now, let us show that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded. Indeed, take an element $p \in \Omega = \bigcap_{n=0}^{\infty} F(T_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap A^{-1}0$ arbitrarily. Then we have $Gp = p$, $J_{r_n}p = p$ and $T_n p = p$ for all $n \geq 0$. Since $T_n : C \to C$ is a continuous pseudocontractive mapping, it follows that

$$\begin{aligned} \|z_n - p\|^2 &= \langle z_n - p, j(z_n - p)\rangle \\ &= \sigma_n\langle x_n - p, j(z_n - p)\rangle + (1 - \sigma_n)\langle T_n z_n - p, j(z_n - p)\rangle \\ &\leq \sigma_n\|x_n - p\|\|z_n - p\| + (1 - \sigma_n)\|z_n - p\|^2, \end{aligned} \tag{4}$$

which hence yields

$$\|z_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0. \tag{5}$$

Since $G, J_{r_n} : C \to C$ are nonexpansive mappings, from (2) and (5) we have

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n x_n + (1 - \alpha_n)J_{r_n} G z_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|G z_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|z_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| \\ &= \|x_n - p\| \end{aligned} \tag{6}$$

and hence

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n\|f(x_n) - p\| + (1 - \beta_n)\|y_n - p\| \\ &\leq \beta_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1 - \beta_n)\|x_n - p\| \\ &\leq \beta_n(k\|x_n - p\| + \|f(p) - p\|) + (1 - \beta_n)\|x_n - p\| \\ &= (1 - (1 - k)\beta_n)\|x_n - p\| + (1 - k)\beta_n \frac{\|f(p) - p\|}{1 - k} \\ &\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - k}\}. \end{aligned}$$

By induction, we have

$$\|x_{n+1} - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1 - k}\}, \quad \forall n \geq 0.$$

Hence $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}, \{z_n\}, \{G z_n\}, \{J_{r_n} G z_n\}$ and $\{f(x_n)\}$ (due to (5), (6) and the nonexpansivity of J_{r_n}, G and f). In addition, since $\limsup_{n\to\infty} \sigma_n < 1$, we have $\liminf_{n\to\infty}(1 - \sigma_n) > 0$, which implies that there exist $\tilde{\epsilon} > 0$ and $\tilde{n} \geq 1$ such that $1 - \sigma_n \geq \tilde{\epsilon}$ for all $n \geq \tilde{n}$. So it follows that for all $n \geq \tilde{n}$,

$$\tilde{\epsilon}\|T_n z_n\| \leq (1 - \sigma_n)\|T_n z_n\| = \|z_n - \sigma_n x_n\| \leq \|z_n\| + \|x_n\|.$$

This means that $\{T_n z_n\}$ is bounded.

Suppose that $x_n \to x^* \in \Omega$ as $n \to \infty$. Then $x^* = Gx^*$, $x^* = J_{r_n}x^*$ and $x^* = T_n x^*$ for all $n \geq 0$. From (2) it follows that

$$\begin{aligned} \|y_n - x^*\| &\leq \alpha_n\|x_n - x^*\| + (1 - \alpha_n)\|G z_n - x^*\| \\ &\leq \alpha_n\|x_n - x^*\| + (1 - \alpha_n)\|z_n - x^*\| \\ &\leq \|x_n - x^*\| \to 0 \quad (n \to \infty), \end{aligned}$$

that is, $y_n \to x^*$. Again from (2) we obtain

$$\|\beta_n(f(x_n) - x_n)\| \leq \|x_{n+1} - x_n\| + (1 - \beta_n)\|y_n - x_n\| \leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \to 0.$$

In the meantime, from the nonexpansivity of G, it is easy to see that

$$\|Gx_n - x_n\| \leq \|Gx_n - x^*\| + \|x^* - x_n\| \leq \|x_n - x^*\| + \|x^* - x_n\| = 2\|x_n - x^*\| \to 0.$$

Consequently, we get

$$\|\beta_n(f(x_n) - x_n)\| + \|Gx_n - x_n\| \to 0 \quad (n \to \infty).$$

Conversely, suppose that $\|\beta_n(f(x_n) - x_n)\| + \|Gx_n - x_n\| \to 0$ $(n \to \infty)$. Put $\gamma_n = (1 - \beta_n)\alpha_n$ for each $n \geq 0$. Then it follows from conditions (C1) and (C3) that

$$\alpha_n \geq \gamma_n = (1 - \beta_n)\alpha_n \geq (1 - (1 - k))\alpha_n = k\alpha_n, \ \forall n \geq n_0,$$

and hence

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1. \tag{7}$$

Define $\hat{z}_n$ by

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n)\hat{z}_n, \quad \forall n \geq 0. \tag{8}$$

Observe that

$$\begin{aligned}\hat{z}_{n+1} - \hat{z}_n = &(\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n})(f(x_{n+1}) - J_{r_{n+1}}Gz_{n+1}) + \frac{\beta_n}{1-\gamma_n}(f(x_{n+1}) - f(x_n)) \\ &+ \frac{1-\gamma_n-\beta_n}{1-\gamma_n}(J_{r_{n+1}}Gz_{n+1} - J_{r_n}Gz_n).\end{aligned}$$

It follows that

$$\begin{aligned}\|\hat{z}_{n+1} - \hat{z}_n\| \leq &|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}|\|f(x_{n+1}) - J_{r_{n+1}}Gz_{n+1}\| + \frac{\beta_n}{1-\gamma_n}\|f(x_{n+1}) - f(x_n)\| \\ &+ \frac{1-\gamma_n-\beta_n}{1-\gamma_n}\|J_{r_{n+1}}Gz_{n+1} - J_{r_n}Gz_n\| \\ \leq &|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}|(\|f(x_{n+1})\| + \|J_{r_{n+1}}Gz_{n+1}\|) + \frac{k\beta_n}{1-\gamma_n}\|x_{n+1} - x_n\| \\ &+ \frac{1-\gamma_n-\beta_n}{1-\gamma_n}(\|J_{r_{n+1}}Gz_{n+1} - J_{r_n}Gz_n\|).\end{aligned} \tag{9}$$

On the other hand, if $r_n \leq r_{n+1}$, using the resolvent identity in Proposition 2.6,

$$J_{r_{n+1}}Gz_{n+1} = J_{r_n}(\frac{r_n}{r_{n+1}}Gz_{n+1} + (1 - \frac{r_n}{r_{n+1}})J_{r_{n+1}}Gz_{n+1}),$$

we get

$$\begin{aligned}\|J_{r_{n+1}}Gz_{n+1} - J_{r_n}Gz_n\| &\leq \frac{r_n}{r_{n+1}}\|Gz_{n+1} - Gz_n\| + (1 - \frac{r_n}{r_{n+1}})\|J_{r_{n+1}}Gz_Gz_{n+1} - Gz_n\| \\ &\leq \|z_{n+1} - z_n\| + \frac{r_{n+1} - r_n}{r_{n+1}}\|J_{r_{n+1}}Gz_{n+1} - Gz_n\| \\ &\leq \|z_{n+1} - z_n\| + \frac{1}{\varepsilon}|r_{n+1} - r_n|\|J_{r_{n+1}}Gz_{n+1} - Gz_n\|.\end{aligned}$$

If $r_{n+1} \leq r_n$, we derive in the similar way

$$\|J_{r_{n+1}}Gz_{n+1} - J_{r_n}Gz_n\| \leq \|z_n - z_{n+1}\| + \frac{1}{\varepsilon}|r_n - r_{n+1}|\|J_{r_n}Gz_n - Gz_{n+1}\|.$$

Thus, combining the above cases, we obtain

$$\|J_{r_{n+1}}Gz_{n+1} - J_{r_n}Gz_n\| \leq \|z_n - z_{n+1}\| + M_0|r_n - r_{n+1}|, \tag{10}$$

where $\sup_{n\geq 0}\{\frac{1}{\varepsilon}(\|J_{r_{n+1}}Gz_{n+1} - Gz_n\| + \|J_{r_n}Gz_n - Gz_{n+1}\|\} \leq M_0$ for some $M_0 > 0$. Substituting (10) into (9), we have

$$\begin{aligned}\|\hat{z}_{n+1} - \hat{z}_n\| \leq &|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}|(\|f(x_{n+1})\| + \|J_{r_{n+1}}Gz_{n+1}\|) + \frac{k\beta_n}{1-\gamma_n}\|x_{n+1} - x_n\| \\ &+ \frac{1-\gamma_n-\beta_n}{1-\gamma_n}(\|z_n - z_{n+1}\| + M_0|r_n - r_{n+1}|).\end{aligned} \tag{11}$$

Note that

$$z_{n+1} - z_n = \sigma_{n+1}(x_{n+1} - x_n) + (1-\sigma_{n+1})(T_{n+1}z_{n+1} - T_n z_n) + (\sigma_{n+1} - \sigma_n)(x_n - T_n z_n),$$

which hence yields

$$\begin{aligned}\|z_{n+1} - z_n\|^2 = &\sigma_{n+1}\langle x_{n+1} - x_n, j(z_{n+1} - z_n)\rangle + (1-\sigma_{n+1})[\langle T_{n+1}z_{n+1} - T_n z_{n+1}, j(z_{n+1} - z_n)\rangle \\ &+ \langle T_n z_{n+1} - T_n z_n, j(z_{n+1} - z_n)\rangle] + (\sigma_{n+1} - \sigma_n)\langle x_n - T_n z_n, j(z_{n+1} - z_n)\rangle \\ \leq &\sigma_{n+1}\|x_{n+1} - x_n\|\|z_{n+1} - z_n\| + (1-\sigma_{n+1})[\|T_{n+1}z_{n+1} - T_n z_{n+1}\|\|z_{n+1} - z_n\| + \|z_{n+1} - z_n\|^2] \\ &+ |\sigma_{n+1} - \sigma_n|\|x_n - T_n z_n\|\|z_{n+1} - z_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \sigma_{n+1}\|x_{n+1} - x_n\| + (1-\sigma_{n+1})[\|T_{n+1}z_{n+1} - T_n z_{n+1}\| + \|z_{n+1} - z_n\|] + |\sigma_{n+1} - \sigma_n|\|x_n - T_n z_n\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1-\sigma_{n+1}}{\sigma_{n+1}}\|T_{n+1}z_{n+1} - T_n z_{n+1}\| + |\sigma_{n+1} - \sigma_n|\frac{\|x_n - T_n z_n\|}{\sigma_{n+1}} \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}\|T_{n+1}z_{n+1} - T_n z_{n+1}\| + |\sigma_{n+1} - \sigma_n|\frac{\|x_n - T_n z_n\|}{c}.\end{aligned} \tag{12}$$

Putting $D = \{z_n : n \geq 0\}$, we know that D is a bounded subset of C. Then by the assumption we have $\sum_{n=0}^{\infty}\sup_{x\in D}\|T_{n+1}x - T_n x\| < \infty$, which hence implies $\|T_{n+1}z_{n+1} - T_n z_{n+1}\| \leq \sup_{x\in D}\|T_{n+1}x - T_n x\| \to 0 \quad (n \to \infty)$. That is,

$$\lim_{n\to\infty}\|T_{n+1}z_{n+1} - T_n z_{n+1}\| = 0. \tag{13}$$

Substituting (12) into (11), we get

$$\begin{aligned}\|\hat{z}_{n+1} - \hat{z}_n\| \leq &|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}|(\|f(x_{n+1})\| + \|J_{r_{n+1}}Gz_{n+1}\|) + \frac{k\beta_n}{1-\gamma_n}\|x_{n+1} - x_n\| + \frac{1-\gamma_n-\beta_n}{1-\gamma_n}(\|x_{n+1} - x_n\| \\ &+ \frac{1}{c}\|T_{n+1}z_{n+1} - T_n z_{n+1}\| + |\sigma_{n+1} - \sigma_n|\frac{\|x_n - T_n z_n\|}{c} + M_0|r_n - r_{n+1}|) \\ \leq &|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}|(\|f(x_{n+1})\| + \|J_{r_{n+1}}Gz_{n+1}\|) + \|x_{n+1} - x_n\| + \frac{1}{c}\|T_{n+1}z_{n+1} - T_n z_{n+1}\| \\ &+ |\sigma_{n+1} - \sigma_n|\frac{\|x_n - T_n z_n\|}{c} + M_0|r_n - r_{n+1}| \\ \leq &\|x_{n+1} - x_n\| + M(|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}| + \|T_{n+1}z_{n+1} - T_n z_{n+1}\| + |\sigma_{n+1} - \sigma_n| + |r_{n+1} - r_n|),\end{aligned}$$

where $\sup_{n\geq 0}\{\|f(x_n)\| + \|J_{r_n}Gz_n\| + \frac{1}{c} + \frac{\|x_n - T_n z_n\|}{c} + M_0\} \leq M$ for some $M > 0$. Then it immediately follows that

$$\|\hat{z}_{n+1} - \hat{z}_n\| - \|x_{n+1} - x_n\| \leq M(|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}| + \|T_{n+1}z_{n+1} - T_n z_{n+1}\| + |\sigma_{n+1} - \sigma_n| + |r_{n+1} - r_n|).$$

From (13) and conditions (C2), (C5), we deduce

$$\limsup_{n\to\infty}(\|\hat{z}_{n+1} - \hat{z}_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Thus by Lemma 2.16 we have

$$\lim_{n\to\infty} \|\hat{z}_n - x_n\| = 0.$$

It follows from (7) and (8) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \gamma_n)\|\hat{z}_n - x_n\| = 0. \tag{14}$$

From (2), we have $x_{n+1} - x_n = \beta_n(f(x_n) - x_n) + (1 - \beta_n)(y_n - x_n)$. This implies that

$$k\|y_n - x_n\| \le (1 - \beta_n)\|y_n - x_n\| \le \|x_{n+1} - x_n\| + \|\beta_n(f(x_n) - x_n)\|.$$

Since $x_{n+1} - x_n \to 0$ and $\beta_n(f(x_n) - x_n) \to 0$, we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{15}$$

Observe that

$$y_n - x_n = (1 - \alpha_n)(J_{r_n} G z_n - x_n). \tag{16}$$

It follows from condition (C3), (15) and (16) that

$$\lim_{n\to\infty} \|x_n - J_{r_n} G z_n\| = 0. \tag{17}$$

Also, according to (3) and Proposition 2.6, we have

$$\|z_n - p\|^2 \le \langle x_n - p, j(z_n - p)\rangle \le \frac{1}{2}[\|x_n - p\|^2 + \|z_n - p\|^2 - g(\|x_n - z_n\|)].$$

This immediately implies that $\|z_n - p\|^2 \le \|x_n - p\|^2 - g(\|x_n - z_n\|)$ which together with (6), leads to

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 - g(\|x_n - z_n\|)] \\
&= \|x_n - p\|^2 - (1 - \alpha_n)g(\|x_n - z_n\|).
\end{aligned}$$

So it follows from (15) that

$$\begin{aligned}
(1 - \alpha_n)g(\|x_n - z_n\|) &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\le (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|) \\
&\le (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \to 0 \quad (n \to \infty).
\end{aligned}$$

Again from condition (C3) we get

$$\lim_{n\to\infty} g(\|x_n - z_n\|) = 0.$$

In terms of the properties of g, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0, \tag{18}$$

which together with (3), implies that

$$\|T_n z_n - z_n\| = \frac{\sigma_n}{1 - \sigma_n}\|x_n - z_n\| \le \frac{d}{1 - d}\|x_n - z_n\| \to 0 \quad (n \to \infty).$$

That is,

$$\lim_{n\to\infty} \|T_n z_n - z_n\| = 0. \tag{19}$$

Further, since $\{T_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian on C, we deduce from (18) and (19) that

$$\begin{aligned}\|T_nx_n - x_n\| &\le \|T_nx_n - T_nz_n\| + \|T_nz_n - z_n\| + \|z_n - x_n\| \\ &\le \ell\|x_n - z_n\| + \|T_nz_n - z_n\| + \|z_n - x_n\| \\ &= (\ell+1)\|x_n - z_n\| + \|T_nz_n - z_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{20}$$

Next, let us show that $T : C \to C$ is pseudocontractive and ℓ-Lipschitzian such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ where $Tx = \lim_{n\to\infty} T_nx, \forall x \in C$. Observe that for all $x, y \in C$, $\lim_{n\to\infty} \|T_nx - Tx\| = 0$ and $\lim_{n\to\infty} \|T_ny - Ty\| = 0$. Since each T_n is pseudocontractive, we get

$$\langle Tx - Ty, j(x-y)\rangle = \lim_{n\to\infty} \langle T_nx - T_ny, j(x-y)\rangle \le \|x-y\|^2.$$

This means that T is pseudocontractive. Noting that $\{T_n\}_{n=0}^{\infty}$ is ℓ-uniformly Lipschitzian on C, we have

$$\|Tx - Ty\| = \lim_{n\to\infty} \|T_nx - T_ny\| \le \ell\|x - y\|, \quad \forall x, y \in C.$$

This means that T is ℓ-Lipschitzian. Taking into account the boundedness of $\{x_n\}$ and putting $D = \overline{\mathrm{conv}}\{x_n : n \ge 0\}$ (the closed convex hull of the set $\{x_n : n \ge 0\}$), by the assumption we have $\sum_{n=1}^{\infty} \sup_{x\in D} \|T_{n+1}x - T_nx\| < \infty$. Hence, by Proposition 2.1 we get

$$\lim_{n\to\infty} \sup_{x\in D} \|T_nx - Tx\| = 0,$$

which immediately yields

$$\lim_{n\to\infty} \|T_nx_n - Tx_n\| = 0. \tag{21}$$

Thus, combining (20) with (21) we have

$$\|x_n - Tx_n\| \le \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\| \to 0 \quad \text{as } n \to \infty. \tag{22}$$

We claim that if we define $\overline{T} := (2I - T)^{-1}$, then $\overline{T} : C \to C$ is nonexpansive, $F(\overline{T}) = F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ and $\lim_{n\to\infty} \|x_n - \overline{T}x_n\| = 0$. Indeed, put $\overline{T} := (2I - T)^{-1}$, where I is the identity mapping of E. Then it is known that $\overline{T}$ is nonexpansive and the fixed point set $F(\overline{T}) = F(T) = \bigcap_{n=0}^{\infty} F(T_n)$ as a consequence of Theorem 6 of [22]. From (22) it follows that

$$\begin{aligned}\|x_n - \overline{T}x_n\| &= \|\overline{T}\,\overline{T}^{-1}x_n - \overline{T}x_n\| \\ &\le \|\overline{T}^{-1}x_n - x_n\| \\ &= \|(2I - T)x_n - x_n\| = \|x_n - Tx_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{23}$$

In addition, let us show that $\lim_{n\to\infty} \|J_rx_n - x_n\| = 0$ for any given $r \in (0, \varepsilon)$. As a matter of fact, since $G, J_{r_n} : C \to C$ are nonexpansive mappings, from (17), (18) and $\|x_n - Gx_n\| \to 0$ we conclude that

$$\begin{aligned}\|J_{r_n}x_n - x_n\| &\le \|J_{r_n}x_n - J_{r_n}Gz_n\| + \|J_{r_n}Gz_n - x_n\| \\ &\le \|x_n - Gz_n\| + \|J_{r_n}Gz_n - x_n\| \\ &\le \|x_n - Gx_n\| + \|Gx_n - Gz_n\| + \|J_{r_n}Gz_n - x_n\| \\ &\le \|x_n - Gx_n\| + \|x_n - z_n\| + \|J_{r_n}Gz_n - x_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{24}$$

Also, taking into account the resolvent identity in Proposition 2.6, we have

$$\begin{aligned}\|J_{r_n}x_n - J_rx_n\| &= \|J_r(\frac{r}{r_n}x_n + (1 - \frac{r}{r_n})J_{r_n}x_n) - J_rx_n\| \\ &\le (1 - \frac{r}{r_n})\|J_{r_n}x_n - x_n\| \\ &\le \|x_n - J_{r_n}x_n\|,\end{aligned}$$

which together with (24) implies that

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - J_{r_n} x_n\| + \|J_{r_n} x_n - J_r x_n\| \\ &\le \|x_n - J_{r_n} x_n\| + \|x_n - J_{r_n} x_n\| \\ &= 2\|x_n - J_{r_n} x_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{25}$$

In the following, we claim that $\{x_n\}$ converges strongly to $x^* \in \Omega$. Indeed, it is sufficient for us to discuss two cases below.

(i) Firstly, suppose that E is uniformly smooth. Let us show that

$$\limsup_{n\to\infty} \langle x^* - f(x^*), j(x^* - x_n)\rangle \le 0, \quad x^* \in \Omega, \tag{26}$$

where z_t is the fixed point of the mapping $z \mapsto tf(z) + (1-t)(\theta_1 \overline{T} + \theta_2 G + (1-\theta_1-\theta_2)J_r)z$, $x^* = s\text{-}\lim_{t\to 0^+} z_t$ and x^* solves the VI: $\langle (I-f)x^*, j(x^* - x)\rangle \le 0$, $\forall x \in \Omega$.

Indeed, we define a mapping $\overline{W}x := \theta_1 \overline{T}x + \theta_2 Gx + (1-\theta_1-\theta_2)J_r x$ $\forall x \in C$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then from Lemma 2.11 it is easy to see that $\overline{W}$ is nonexpansive and

$$F(\overline{W}) = F(\overline{T}) \cap F(G) \cap F(J_r) = \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap A^{-1}0 (=: \Omega) \ne \emptyset.$$

Observe that

$$\begin{aligned}\|x_n - \overline{W}x_n\| &= \|\theta_1(\overline{T}x_n - x_n) + \theta_2(Gx_n - x_n) + (1-\theta_1-\theta_2)(J_r x_n - x_n)\| \\ &\le \theta_1\|\overline{T}x_n - x_n\| + \theta_2\|Gx_n - x_n\| + (1-\theta_1-\theta_2)\|J_r x_n - x_n\|.\end{aligned}$$

From (23), (25) and $\|Gx_n - x_n\| \to 0$, we obtain

$$\lim_{n\to\infty} \|x_n - \overline{W}x_n\| = 0. \tag{27}$$

It is clear that the mapping $z \mapsto tf(z) + (1-t)\overline{W}z$ is a contraction of C into itself for each $t \in (0,1)$. So, z_t solves the fixed point equation $z_t = tf(z_t) + (1-t)\overline{W}z_t$. Then we have

$$z_t - x_n = (1-t)(\overline{W}z_t - x_n) + t(f(z_t) - x_n). \tag{28}$$

Thus, from Lemma 2.7 and (28), we obtain

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1-t)^2\|\overline{W}z_t - x_n\|^2 + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\ &\le (1-t)^2[\|\overline{W}z_t - \overline{W}x_n\| + \|\overline{W}x_n - x_n\|]^2 + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\ &\le (1-t)^2[\|z_t - x_n\| + \|\overline{W}x_n - x_n\|]^2 + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\ &= (1-t)^2[\|z_t - x_n\|^2 + 2\|z_t - x_n\|\|\overline{W}x_n - x_n\| + \|\overline{W}x_n - x_n\|^2] + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle,\end{aligned}$$

that is,

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1-t)^2\|z_t - x_n\|^2 + \|\overline{W}x_n - x_n\|[2\|z_t - x_n\| + \|\overline{W}x_n - x_n\|] + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle \\ &\quad + 2t\|z_t - x_n\|^2 \\ &= (1+t^2)\|z_t - x_n\|^2 + \|\overline{W}x_n - x_n\|[2\|z_t - x_n\| + \|\overline{W}x_n - x_n\|] + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle.\end{aligned}$$

It follows that

$$\langle z_t - f(z_t), j(z_t - x_n)\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}\|\overline{W}x_n - x_n\|[2\|z_t - x_n\| + \|\overline{W}x_n - x_n\|]. \tag{29}$$

Letting $n \to \infty$ in (3.27) and noticing $\|\overline{W}x_n - x_n\| \to 0$ $(n \to \infty)$, we have

$$\limsup_{n\to\infty} \langle z_t - f(z_t), j(z_t - x_n)\rangle \le \frac{t}{2}M_1, \tag{30}$$

where M_1 is a constant such that $\|z_t - x_n\|^2 \le M_1$ for all $n \ge 0$ and $t \in (0,1)$. Utilizing Lemma 2.12 we deduce that $\{z_t\}$ converges strongly to a fixed point $x^* \in F(\overline{W}) = F(\overline{T}) \cap F(G) \cap F(J_r) = \Omega$, which solves the variational inequality:

$$\langle (I-f)x^*, j(x^* - x)\rangle \le 0, \quad \forall x \in \Omega.$$

Since the duality mapping $j(\cdot)$ is norm-to-norm uniformly continuous on bounded subsets of E, by letting $t \to 0^+$ in (30), we know that (26) holds.

Now, let us show that $x_n \to x^*$ as $n \to \infty$. Utilizing Lemma 2.7, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\beta_n(f(x_n) - f(x^*)) + (1-\beta_n)(y_n - x^*) + \beta_n(f(x^*) - x^*)\|^2 \\ &\le \|\beta_n(f(x_n) - f(x^*)) + (1-\beta_n)(y_n - x^*)\|^2 + 2\beta_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\ &\le \beta_n\|f(x_n) - f(x^*)\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + 2\beta_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\ &\le \beta_n k\|x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\ &= (1-(1-k)\beta_n)\|x_n - x^*\|^2 + 2\beta_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\ &= (1-(1-k)\beta_n)\|x_n - x^*\|^2 + (1-k)\beta_n\frac{2\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle}{1-k}. \end{aligned} \tag{31}$$

Therefore, applying Lemma 2.15 to (31), we conclude from (26) and condition (C1) that $x_n \to x^*$ as $n \to \infty$.

(ii) Secondly, suppose that E has a weakly continuous duality mapping j_φ with gauge φ. Let z_t be the unique fixed point of the contraction mapping T_t given by

$$T_t x := tf(x) + (1-t)\overline{W}x, \quad \forall t \in (0,1),$$

where $\overline{W} := \theta_1\overline{T} + \theta_2 G + (1-\theta_1-\theta_2)J_r$ with $\theta_1, \theta_2 \in (0,1)$ being two constants satisfying $\theta_1 + \theta_2 < 1$. By Lemma 2.14, we can define $x^* := s\text{-}\lim_{t\to 0^+} z_t$, and $x^* \in F(\overline{W}) = \Omega$ solves the VI:

$$\langle x^* - f(x^*), j_\varphi(x^* - x)\rangle \le 0, \quad \forall x \in \Omega. \tag{32}$$

Let us show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle \le 0. \tag{33}$$

We take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle = \lim_{k\to\infty}\langle f(x^*) - x^*, j_\varphi(x_{n_k} - x^*)\rangle. \tag{34}$$

Since E is reflexive and $\{x_n\}$ is bounded, we may further assume that $x_{n_k} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. Since j_φ is weakly continuous, utilizing Lemma 2.13, we have

$$\limsup_{k\to\infty}\Phi(\|x_{n_k} - x\|) = \limsup_{k\to\infty}\Phi(\|x_{n_k} - \bar{x}\|) + \Phi(\|x - \bar{x}\|), \quad \forall x \in E.$$

Put $\Gamma(x) = \limsup_{k\to\infty}\Phi(\|x_{n_k} - x\|), \forall x \in E$. It follows that $\Gamma(x) = \Gamma(\bar{x}) + \Phi(\|x - \bar{x}\|)$, $\forall x \in E$. From (27), we have

$$\begin{aligned} \Gamma(\overline{W}\bar{x}) &= \limsup_{k\to\infty}\Phi(\|x_{n_k} - \overline{W}\bar{x}\|) \\ &= \limsup_{k\to\infty}\Phi(\|\overline{W}x_{n_k} - \overline{W}\bar{x}\|) \\ &\le \limsup_{k\to\infty}\Phi(\|x_{n_k} - \bar{x}\|) = \Gamma(\bar{x}). \end{aligned} \tag{35}$$

Furthermore, observe that

$$\Gamma(\overline{W}\bar{x}) = \Gamma(\bar{x}) + \Phi(\|\overline{W}\bar{x} - \bar{x}\|). \tag{36}$$

Combining (35) with (36), we obtain $\Phi(\|\overline{W}\bar{x}-\bar{x}\|)\le 0$. Hence $\overline{W}\bar{x}=\bar{x}$ and $\bar{x}\in F(\overline{W})=\Omega$. Thus, from (32) and (34), it is easy to see that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, j_\varphi(x_n-x^*)\rangle=\lim_{k\to\infty}\langle f(x^*)-x^*, j_\varphi(\bar{x}-x^*)\rangle\le 0.$$

Therefore, we conclude that (33) holds.

Next, let us show that $x_n\to x^*$ as $n\to\infty$. Utilizing Lemma 2.13, we obtain

$$\begin{aligned}\Phi(\|x_{n+1}-x^*\|)&=\Phi(\|\beta_n(f(x_n)-x^*)+(1-\beta_n)(y_n-x^*)\|)\\&=\Phi(\|\beta_n(f(x_n)-f(x^*))+(1-\beta_n)(y_n-x^*)+\beta_n(f(x^*)-x^*)\|)\\&\le\Phi(\|\beta_n(f(x_n)-f(x^*))+(1-\beta_n)(y_n-x^*)\|)+\beta_n\langle f(x^*)-x^*, j_\varphi(x_{n+1}-x^*)\rangle\\&\le\Phi(\beta_n\|f(x_n)-f(x^*)\|+(1-\beta_n)\|y_n-x^*\|)+\beta_n\langle f(x^*)-x^*, j_\varphi(x_{n+1}-x^*)\rangle\\&\le\Phi(\beta_nk\|x_n-x^*\|+(1-\beta_n)\|x_n-x^*\|)+\beta_n\langle f(x^*)-x^*, j_\varphi(x_{n+1}-x^*)\rangle\\&\le(1-(1-k)\beta_n)\Phi(\|x_n-x^*\|)+\beta_n\langle f(x^*)-x^*, j_\varphi(x_{n+1}-x^*)\rangle\\&\le(1-(1-k)\beta_n)\Phi(\|x_n-x^*\|)+(1-k)\beta_n\frac{\langle f(x^*)-x^*, j_\varphi(x_{n+1}-x^*)\rangle}{1-k}.\end{aligned}\tag{37}$$

Applying Lemma 2.15 to (37), we conclude from (33) and condition (C1) that $\Phi(\|x_n-x^*\|)\to 0\quad(n\to\infty)$ which implies that $\|x_n-x^*\|\to 0$ $(n\to\infty)$, i.e., $x_n\to x^*$ $(n\to\infty)$. This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Assume, in addition, that E either is uniformly smooth or has a weakly continuous duality map j_φ with gauge φ. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $B_i : C\to E$ be λ_i-strictly pseudocontractive and ζ_i-strongly accretive with $\lambda_i+\zeta_i\ge 1$ for $i=1,2$. Let $f: C\to C$ be a fixed contraction mapping with coefficient $k\in(0,1)$, and $\{T_n\}_{n=0}^\infty$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C such that $\Omega:=\bigcap_{n=0}^\infty F(T_n)\cap\mathrm{GSVI}(C,B_1,B_2)\ne\emptyset$, where $\mathrm{GSVI}(C,B_1,B_2)$ is the fixed point set of the mapping $G:=\Pi_C(I-\rho B_1)\Pi_C(I-\eta B_2)$ with $1-\frac{\lambda_1}{1+\lambda_1}(1-\sqrt{\frac{1-\zeta_1}{\lambda_1}})\le\rho\le 1$ and $1-\frac{\lambda_2}{1+\lambda_2}(1-\sqrt{\frac{1-\zeta_2}{\lambda_2}})\le\eta\le 1$. For an arbitrary $x_0\in C$, let $\{x_n\}$ be generated by*

$$\begin{cases}z_n=\sigma_nx_n+(1-\sigma_n)T_nz_n,\\y_n=\alpha_nx_n+(1-\alpha_n)Gz_n,\\x_{n+1}=\beta_nf(x_n)+(1-\beta_n)y_n,\quad\forall n\ge 0,\end{cases}\tag{38}$$

where $\{\alpha_n\},\{\beta_n\}$ and $\{\sigma_n\}$ are the sequences in $[0,1]$ such that the following conditions hold:

(C1): $0\le\beta_n\le 1-k,\forall n\ge n_0$ for some $n_0\ge 0$, and $\sum_{n=0}^\infty\beta_n=\infty$;

(C2): $\lim_{n\to\infty}|\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}}-\frac{\beta_n}{1-(1-\beta_n)\alpha_n}|=0$ and $\lim_{n\to\infty}|\sigma_{n+1}-\sigma_n|=0$;

(C3): $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$;

(C4): $0<\liminf_{n\to\infty}\sigma_n\le\limsup_{n\to\infty}\sigma_n<1$.

Assume that $\sum_{n=0}^\infty\sup_{x\in D}\|T_{n+1}x-T_nx\|<\infty$ for any bounded subset D of C, and let T be a mapping of C into itself defined by $Tx=\lim_{n\to\infty}T_nx$ for all $x\in C$, and suppose that $F(T)=\bigcap_{n=0}^\infty F(T_n)$. Then $x_n\to x^*\in\Omega\Leftrightarrow\beta_n(f(x_n)-x_n)\to 0$. In this case, (x^*,y^*) is a solution of GSVI (1) with $y^*=\Pi_C(I-\eta B_2)x^*$, and we have

(a) if E is uniformly smooth, then $x^*\in\Omega$ solves the VI: $\langle x^*-f(x^*), j(x^*-x)\rangle\le 0$, $\forall x\in\Omega$;

(b) if E has a weakly continuous duality mapping j_φ with gauge φ then $x^*\in\Omega$ solves the VI: $\langle x^*-f(x^*), j_\varphi(x^*-x)\rangle\le 0$, $\forall x\in\Omega$.

Proof. In Theorem 3.1, we put $Ax = 0$ for all $x \in E$. Then for any positive sequence $\{r_n\}$, we have $J_{r_n} = I$ the identity mapping of E. Hence the iterative scheme (2) reduces to (38). Repeating the same arguments as those of (17) and (18) we derive

$$\lim_{n\to\infty} \|x_n - Gz_n\| = \lim_{n\to\infty} \|x_n - J_{r_n} Gz_n\| = 0,$$

and

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Combining two limit equalities, we get

$$\|x_n - Gx_n\| \le \|x_n - Gz_n\| + \|Gz_n - Gx_n\| \le \|x_n - Gz_n\| + \|z_n - x_n\| \to 0 \quad (n \to \infty).$$

That is,

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0.$$

So it follows that

$$\|\beta_n(f(x_n) - x_n)\| \to 0 \;\Leftrightarrow\; \|\beta_n(f(x_n) - x_n)\| + \|x_n - Gx_n\| \to 0.$$

Therefore, utilizing Theorem 3.1, we obtain the desired result. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Assume, in addition, that E either is uniformly smooth or has a weakly continuous duality map j_φ with gauge φ. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $A \subset E \times E$ be an accretive operator in E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$, and let $B : C \to E$ be λ-strictly pseudocontractive and ζ-strongly accretive with $\lambda + \zeta \ge 1$. Let $f : C \to C$ be a fixed contraction mapping with coefficient $k \in (0, 1)$, and $\{T_n\}_{n=0}^\infty$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C such that $\Omega := \bigcap_{n=0}^\infty F(T_n) \cap F(G) \cap A^{-1}0 \ne \emptyset$, where $F(G)$ is the fixed point set of the mapping $G := \Pi_C(I - \rho B)\Pi_C(I - \eta B)$ with $1 - \frac{\lambda}{1+\lambda}(1 - \sqrt{\frac{1-\zeta}{\lambda}}) \le \rho \le 1$ and $1 - \frac{\lambda}{1+\lambda}(1 - \sqrt{\frac{1-\zeta}{\lambda}}) \le \eta \le 1$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by*

$$\begin{cases} z_n = \sigma_n x_n + (1 - \sigma_n)T_n z_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)J_{r_n} Gz_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\sigma_n\}$ are the sequences in $[0, 1]$ and $\{r_n\}$ is a sequence in $(0, \infty)$. Suppose that the following conditions hold:

(C1): $0 \le \beta_n \le 1 - k, \forall n \ge n_0$ *for some* $n_0 \ge 0$, *and* $\sum_{n=0}^\infty \beta_n = \infty$;

(C2): $\lim_{n\to\infty} |\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}| = 0$ *and* $\lim_{n\to\infty} |\sigma_{n+1} - \sigma_n| = 0$;

(C3): $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(C4): $0 < \liminf_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} \sigma_n < 1$;

(C5): $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $r_n \ge \varepsilon > 0 \; \forall n \ge 0$.

Assume that $\sum_{n=0}^\infty \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C, and let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, and suppose that $F(T) = \bigcap_{n=0}^\infty F(T_n)$. Then $x_n \to x^* \in \Omega \Leftrightarrow \|\beta_n(f(x_n) - x_n)\| + \|Gx_n - x_n\| \to 0$. In this case, (x^*, y^*) is a solution of SVI with $y^* = \Pi_C(I - \eta B)x^*$, and we have

(a) if E is uniformly smooth, then $x^* \in \Omega$ solves the VI: $\langle x^* - f(x^*), j(x^* - x)\rangle \le 0, \; \forall x \in \Omega$;

(b) if E has a weakly continuous duality mapping j_φ with gauge φ then $x^* \in \Omega$ solves the VI: $\langle x^* - f(x^*), j_\varphi(x^* - x)\rangle \le 0, \; \forall x \in \Omega$.

Corollary 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Assume, in addition, that E either is uniformly smooth or has a weakly continuous duality map j_φ with gauge φ. Let Π_C be a sunny nonexpansive retraction from E onto C. Let $A \subset E \times E$ be an accretive operator in E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$, and let $B : C \to E$ be λ-strictly pseudocontractive and ζ-strongly accretive with $\lambda + \zeta \geq 1$. Let $f : C \to C$ be a fixed contraction mapping with coefficient $k \in (0, 1)$, and $\{T_n\}_{n=0}^{\infty}$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C such that $\Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap \mathrm{VI}(C, B) \cap A^{-1}0 \neq \emptyset$. For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by*

$$\begin{cases} z_n = \sigma_n x_n + (1 - \sigma_n) T_n z_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n) J_{r_n} G z_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, \quad \forall n \geq 0, \end{cases}$$

where $G := \Pi_C(I - \rho B)$ with $1 - \frac{\lambda}{1+\lambda}(1 - \sqrt{\frac{1-\zeta}{\lambda}}) \leq \rho \leq 1$, $\{\alpha_n\}, \{\beta_n\}$ and $\{\sigma_n\}$ are the sequences in $[0, 1]$ and $\{r_n\}$ is a sequence in $(0, \infty)$. Suppose that the following conditions hold:

(C1): $0 \leq \beta_n \leq 1 - k, \forall n \geq n_0$ *for some* $n_0 \geq 0$, *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(C2): $\lim_{n\to\infty} |\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}| = 0$ *and* $\lim_{n\to\infty} |\sigma_{n+1} - \sigma_n| = 0$;

(C3): $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C4): $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$;

(C5): $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ *and* $r_n \geq \varepsilon > 0\ \forall n \geq 0$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C, and let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, and suppose that $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then $x_n \to x^* \in \Omega \Leftrightarrow \|\beta_n(f(x_n) - x_n)\| + \|Gx_n - x_n\| \to 0$. In this case, x^* is a solution of VI with $x^* = \Pi_C(I - \rho B)x^*$, and we have

(a) if E is uniformly smooth, then $x^* \in \Omega$ solves the VI: $\langle x^* - f(x^*), j(x^* - x)\rangle \leq 0$, $\forall x \in \Omega$;

(b) if E has a weakly continuous duality mapping j_φ with gauge φ then $x^* \in \Omega$ solves the VI: $\langle x^* - f(x^*), j_\varphi(x^* - x)\rangle \leq 0$, $\forall x \in \Omega$.

4. Conclusions

In this paper, we introduce and analyze implicit composite three-step Mann iterations for finding a common solution of GSVI (1) and a fixed point problem (FPP) of a countable family of uniformly Lipschitzian pseudocontractive self-mappings and a zero problem (ZP) of an accretive operator in a uniformly convex Banach space E which either is uniformly smooth or has a weakly continuous duality mapping. Here, implicit composite three-step Mann iterations are based on, the Mann iteration method, the viscosity approximation method and the Korpelevich extragradient method. Under quite suitable assumptions, we derive some strong convergence results. Noting that in our suggested iterative sequence (Equation (2)), the involved operators A, $B_i (i = 1, 2)$ and $\{T_n\}_{n=0}^{\infty}$ require some additional assumptions. A natural question arises, i.e., how to weaken these assumptions?

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