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On the Logarithmic Mean of Accretive Matrices

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Abstract. In this paper, we define the logarithmic mean of two accretive matrices and study its basic properties. Among other results, we show that if A, B are accretive matrices, then

$$\Re L(A, B) \geq L(\Re A, \Re B),$$

where $L(A, B)$ is the logarithmic mean of A and B, and $\Re A$ means the real part of A. This complements a recent result of Lin and Sun.

1. Introduction

The logarithmic mean of two positive numbers a and b, which is of interest in geometry, statistics, and thermodynamics, is defined as

$$L(a, b) = \frac{a - b}{\log a - \log b} = \int_0^1 a^{1-t} b^t \mathrm{d}t.$$

It is well known that

$$\sqrt{ab} \leq L(a, b) \leq \frac{a + b}{2}. \tag{1}$$

The logarithmic mean has also been defined for positive definite matrices or operators; see for example [6], in which comparison with various other means are studied. In the sequel, we let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. The conjugate transpose of $A \in \mathbb{M}_n$ is denoted by A^*. Every $A \in \mathbb{M}_n$ has a unique Cartesian decomposition

$$A = \Re A + i\Im A,$$

where $\Re A = \frac{A+A^*}{2}$ and $\Im A = \frac{A-A^*}{2i}$ are called the real and imaginary part of A, respectively. If $\Re A$ is positive definite, then we say A is accretive. This class of matrices and its subclass, viz, accretive-dissipative matrices, are receiving much attention over the past few years; see [4, 11–16, 19].

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The geometric mean of two accretive matrices $A, B \in \mathbb{M}_n$ was first brought in by Drury [3], who defined

$$A\sharp B = \left(\frac{2}{\pi}\int_0^\infty \left(sA + s^{-1}B\right)^{-1} \frac{\mathrm{d}s}{s}\right)^{-1}.$$

However, to define the logarithmic mean of accretive matrices, a weighted geometric mean seems essential. Raissouli, Moslehian and Furuichi [17] recently defined the following weighted geometric mean of two accretive matrices $A, B \in \mathbb{M}_n$,

$$A\sharp_t B = \frac{\sin t\pi}{\pi}\int_0^\infty s^{t-1}\left(A^{-1} + sB^{-1}\right)^{-1} \mathrm{d}s,$$

where $t \in [0, 1]$. It could be verified that $A\sharp_{1/2}B = A\sharp B$. We summarize some basic properties of the weighted geometric mean in the following proposition.

Proposition 1.1. *[17] Let $A, B \in \mathbb{M}_n$ be accretive. Then*

1. *$A\sharp_t B$ is accretive;*
2. *$A\sharp_t B = B\sharp_{1-t}A$;*
3. *for any nonsingular $P \in \mathbb{M}_n$, $(PAP^*)\sharp(PBP^*) = P(A\sharp_t B)P^*$;*
4. *in particular, the definition of $A\sharp_t B$ coincides with the regular definition of weighted geometric mean when A and B are positive definite.*

With the weighted geometric mean of two accretive matrices, we are able to define the logarithmic mean of accretive matrices $A, B \in \mathbb{M}_n$ as

$$L(A, B) = \int_0^1 A\sharp_t B \,\mathrm{d}t. \tag{2}$$

In this paper, we intend to study some basic properties of the logarithmic mean (2) and compare it with other matrix means. To enrich our study, we need to define a sector S_θ on the complex plane

$$S_\theta = \{z \in \mathbb{C} : \Re z > 0, \ |\Im z| \le (\Re z)\tan\theta\},$$

where $\theta \in [0, \pi/2)$ is fixed.

Recall that the numerical range (see, e.g., [5]) of $A \in \mathbb{M}_n$ is defined as the set on the complex plane

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, \ x^*x = 1\}.$$

In [9], if $W(A) \subset S_\theta$, then A is called a sector matrix. Clearly, if $W(A) \subset S_\theta$, then $\Re A$ is positive definite. Some recent studies of sector matrices can be found in [2, 9, 18, 20].

2. Auxiliary Results

In this section, we present some auxiliary results which motivate and facilitate the proofs of the main results in the next section.

For two Hermitian matrices A, B, we write $A \ge B$ to mean that $A - B$ is positive semidefinite. The following remarkable property about the geometric mean of accretive matrices was proved by Raissouli, Moslehian and Furuichi.

Proposition 2.1. *[17, Theorem 2.4] Let $A, B \in \mathbb{M}_n$ be accretive and let $t \in [0, 1]$. Then*

$$\Re(A\sharp_t B) \ge (\Re A)\sharp_t(\Re B). \tag{3}$$

We remark that when $t = 1/2$, the previous result was observed by Lin and Sun in [10]. Our Proposition 3.2 in the next section complements Lin and Sun's result.

Proposition 2.2. *Let $A, B \in \mathbb{M}_n$ be positive definite. Then*

$$A\sharp B \le L(A,B) \le \frac{A+B}{2}. \tag{4}$$

Proof. This is a known result (e.g. [1, Eq. (17)]), but we mention a simple proof here. The key observation is the simultaneous diagonalization of two positive definite matrices, that is, there is a nonsingular $P \in \mathbb{M}_n$ such that PAP^* and PBP^* are diagonal; see [7, Theorem 7.6.1]. Then (4) reduces to the case where the underlying matrices are positive diagonal, which is essentially the scalar inequality (1). □

Lemma 2.3. *[8, Lemma 2.4] Let $A \in \mathbb{M}_n$ be accretive. Then*

$$(\Re A)^{-1} \ge \Re A^{-1}.$$

A reverse inequality of Lemma 2.3 is as follows.

Lemma 2.4. *[9, Lemma 3] Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\theta$. Then*

$$(\Re A)^{-1} \le (\sec\theta)^2 \Re A^{-1}.$$

The next lemma is known as the Ostrowski-Taussky inequality.

Lemma 2.5. *[7, p. 510] If $A \in \mathbb{M}_n$ is accretive, then it holds*

$$\det(\Re A) \le |\det A|.$$

The following lemma gives a reverse of the Ostrowski-Taussky inequality.

Lemma 2.6. *[8, Lemma 2.6] If $A \in \mathbb{M}_n$ such that $W(A) \subset \mathcal{S}_\theta$, then it holds*

$$|\det A| \le \sec^n(\theta)\det(\Re A).$$

3. Main Results

Some basic properties about the logarithmic mean are included in the following proposition.

Proposition 3.1. *Let $A, B \in \mathbb{M}_n$ be accretive. Then*

1. *$L(A,B)$ is accretive;*
2. *$L(A,B) = L(B,A)$;*
3. *for any nonsingular $P \in \mathbb{M}_n$, $L(PAP^*, PBP^*) = PL(A,B)P^*$.*

Proof. Since we know from [17] that $A\sharp_t B$ is accretive for all $t \in [0,1]$, it follows

$$\Re L(A,B) = \Re \int_0^1 A\sharp_t B \,\mathrm{d}t = \int_0^1 \Re(A\sharp_t B)\mathrm{d}t$$

is positive definite. That is, $L(A,B)$ is accretive. To show the second item, notice that $A\sharp_t B = B\sharp_{1-t} A$, then

$$L(A,B) = \int_0^1 A\sharp_t B \,\mathrm{d}t = \int_0^1 B\sharp_{1-t} A \,\mathrm{d}t = \int_0^1 B\sharp_s A \,\mathrm{d}s = L(B,A),$$

in which the third equality by change of variable. To show the third item, notice that $(PAP^*)\sharp_t(PBP^*) = P(A\sharp_t B)P^*$, then

$$\begin{aligned} L(PAP^*, PBP^*) &= \int_0^1 (PAP^*)\sharp_t(PBP^*)\mathrm{d}t \\ &= \int_0^1 P(A\sharp_t B)P^*\mathrm{d}t = PL(A,B)P^*. \end{aligned}$$

This completes the proof. □

The next result provides an analogue of Proposition 2.1.

Proposition 3.2. *Let $A, B \in \mathbb{M}_n$ be accretive. Then*

$$\Re L(A,B) \geq L(\Re A, \Re B).$$

Proof. We compute

$$\begin{aligned} \Re L(A,B) &= \int_0^1 \Re(A\sharp_t B)\mathrm{d}t \\ &\geq \int_0^1 (\Re A)\sharp_t(\Re B)\mathrm{d}t \\ &= L(\Re A, \Re B), \end{aligned}$$

in which the inequality is by Proposition 2.1. □

Under the assumption that A, B are sector matrices, we could derive a reverse inequality. We need a new lemma.

Lemma 3.3. *Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\theta$. Then*

$$\Re(A\sharp_t B) \leq (\sec\theta)^2 \left((\Re A)\sharp_t(\Re B)\right)$$

Proof. First of all, by Lemma 2.3 we have

$$\Re\left(A^{-1} + tB^{-1}\right)^{-1} \leq \left(\Re A^{-1} + t\Re B^{-1}\right)^{-1}.$$

On the other hand, by Lemma 2.4 we have

$$\Re A^{-1} + t\Re B^{-1} \geq (\cos\theta)^2 \left((\Re A)^{-1} + t(\Re B)^{-1}\right).$$

Thus

$$\Re\left(A^{-1} + tB^{-1}\right)^{-1} \leq (\sec\theta)^2 \left((\Re A)^{-1} + t(\Re B)^{-1}\right)^{-1}.$$

Combining previous two inequalities gives

$$\begin{aligned} \Re(A\sharp_t B) &= \frac{\sin t\pi}{\pi} \int_0^\infty s^{t-1} \Re\left(A^{-1} + sB^{-1}\right)^{-1} \mathrm{d}s \\ &\leq \frac{\sin t\pi}{\pi} \int_0^\infty s^{t-1} (\sec\theta)^2 \left((\Re A)^{-1} + s(\Re B)^{-1}\right)^{-1} \mathrm{d}s \\ &= (\sec\theta)^2((\Re A)\sharp_t(\Re B)). \end{aligned}$$

The proof is complete. □

Proposition 3.4. *Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\theta$. Then*

$$\Re L(A,B) \leq (\sec\theta)^2 L(\Re A, \Re B).$$

Proof. By Lemma 3.3, we could estimate

$$\begin{aligned} \Re L(A,B) &= \int_0^1 \Re(A\sharp_t B)\mathrm{d}t \\ &\leq (\sec\theta)^2 \int_0^1 (\Re A)\sharp_t(\Re B)\mathrm{d}t \\ &= (\sec\theta)^2 L(\Re A, \Re B). \end{aligned}$$

This completes the proof. □

In the next theorem, we establish an analogue of Proposition 2.2.

Theorem 3.5. *Let* $A, B \in \mathbb{M}_n$ *with* $W(A), W(B) \subset S_\theta$. *Then*

$$(\cos\theta)^2 \Re(A\sharp B) \le \Re L(A,B) \le (\sec\theta)^2 \Re\frac{A+B}{2}. \tag{5}$$

Proof. By Lemma 3.3,

$$\Re(A\sharp B) \le (\sec\theta)^2((\Re A)\sharp(\Re B)).$$

Then by the first inequality of (4), we have

$$(\Re A)\sharp(\Re B) \le L(\Re A, \Re B).$$

Combing with Proposition 3.2 gives

$$\Re(A\sharp B) \le (\sec\theta)^2 \Re L(A,B),$$

which is the first inequality of (5). To show the second inequality of (5), we estimate

$$\begin{aligned} \Re L(A,B) &\le (\sec\theta)^2 L(\Re A, \Re B) \\ &\le (\sec\theta)^2 \frac{\Re A + \Re B}{2} \\ &= (\sec\theta)^2 \Re\frac{A+B}{2}, \end{aligned}$$

in which the first inequality is by Proposition 3.4 and the second inequality is by (4). □

Note that if $A \ge B \ge 0$, then $\det A \ge \det B \ge 0$. Thus we have an immediate corollary of Theorem 3.5.

Corollary 3.6. *Let* $A, B \in \mathbb{M}_n$ *with* $W(A), W(B) \subset S_\theta$. *Then*

$$(\cos\theta)^{2n} \det \Re(A\sharp B) \le \det \Re L(A,B) \le (\sec\theta)^{2n} \det \Re\frac{A+B}{2}. \tag{6}$$

The next result shows the first inequality of (6) could be considerably improved.

Proposition 3.7. *Let* $A, B \in \mathbb{M}_n$ *with* $W(A), W(B) \subset S_\theta$. *Then*

$$(\cos\theta)^n \det \Re(A\sharp B) \le \det \Re L(A,B).$$

Proof. By Lemma 2.5,

$$\det \Re(A\sharp B) \le |\det(A\sharp B)| = \sqrt{|\det A||\det B|},$$

in which the equality is by [3, Theorem 3.4] since $A\sharp B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}$. Then by Lemma 2.6,

$$\sqrt{|\det A||\det B|} \le (\sec\theta)^n \sqrt{(\det \Re A)(\det \Re B)} = (\sec\theta)^n \det(\Re A)\sharp(\Re B).$$

It follows by the first inequality of (4) and Proposition 3.2 that

$$\begin{aligned} \det \Re(A\sharp B) &\le (\sec\theta)^n \det(\Re A)\sharp(\Re B) \\ &\le (\sec\theta)^n \det L(\Re A, \Re B) \\ &\le (\sec\theta)^n \det \Re L(A,B). \end{aligned}$$

This proves the assertion. □

It would be interesting to know whether the second inequality of (6) could be similarly improved. We leave it as a question for future research.

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