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Fixed Point under Set-Valued Relation-Theoretic Nonlinear Contractions and Application

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Abstract. We establish a relation theoretic version of the main result of Aydi et al. [H. Aydi, M. Abbas, C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric space, Topol. Appl. (159), 2012, 3234–3242] and extend the results of Alam and Imdad [A. Alam, M. Imdad, Relation-theoretic contraction priciple, J. Fixed Point Theory Appl., 17(4), 2015, 693–702.] for a set-valued map in a partial Pompeiu-Hausdorff metric space. Numerical examples are presented to validate the theoretical finding and to demonstrate that our results generalize, improve and extend the recent results in different spaces equipped with binary relations to their set-valued variant exploiting weaker conditions. Our results provide a new answer, in the setting of relation theoretic contractions, to the open question posed by Rhoades on continuity at fixed point. We also show that, under the assumption of k-continuity, the solution to the Rhoades' problem given by Bisht and Rakočević characterizes completeness of the metric space. As an application of our main result, we solve an integral inclusion of Fredholm type.

1. Introduction

The distance between two closed sets is currently a vital tool in mathematics, computer science and numerous other areas of research. More than one hundred years ago D. Pompeiu [16] (1873−1954) stated this idea in the context of complex analysis. Infact, Pompeiu needed this distance to define the distance between two curves in the complex plane and also to introduce the notion of a limit of a sequence of sets by means of this distance. Let $\mathcal{U}, \mathcal{V}$ be two bounded and closed sets. If $u \in \mathcal{U}$, then $d(u, \mathcal{V}) = \min\{d(u, v) : v \in \mathcal{V}\}$, is the distance between the points u and the set $\mathcal{V}$ and $d(u, v)$ is the distance between the points u and v. Further, Pompeiu noted that $D(\mathcal{U}, \mathcal{V})$ is not symmetric and consequently, he defined the asymmetric distance between the sets $\mathcal{U}$ and $\mathcal{V}$ as: $D(\mathcal{U}, \mathcal{V}) = \max\{d(u, \mathcal{V}) : u \in \mathcal{U}\}$ and between the sets $\mathcal{V}$ and $\mathcal{U}$ as: $D(\mathcal{V}, \mathcal{U}) = \max\{d(v, \mathcal{U}) : v \in \mathcal{V}\}$. He pointed out that $D(\mathcal{U}, \mathcal{V}) = 0$ iff $\mathcal{U} \subset \mathcal{V}$ and $D(\mathcal{V}, \mathcal{U}) = 0$ iff $\mathcal{V} \subset \mathcal{U}$. In order to endow the distance between two sets with its most natural property (symmetry), Pompeiu considered a natural way to symmetrize his notion, by defining the distance between the sets $\mathcal{U}$ and $\mathcal{V}$ as $P(\mathcal{U}, \mathcal{V}) = D(\mathcal{U}, \mathcal{V}) + D(\mathcal{V}, \mathcal{U})$ and concluded that $P(\mathcal{U}, \mathcal{V}) = 0$ iff $D(\mathcal{U}, \mathcal{V}) = 0$ and $D(\mathcal{V}, \mathcal{U}) = 0$, i.e., iff $\mathcal{U} = \mathcal{V}$.

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On the other hand, in 1914, F. Hausdorff (1868 – 1942) [5] considered all the basic concepts introduced by Pompeiu, in the general framework of a metric space and adopted an alternative way to symmetrize the asymmetric distances $D(\mathcal{U},\mathcal{V})$ and $D(\mathcal{V},\mathcal{U})$, which is commonly named as Hausdorff metric. Both the definitions are equivalent, by virtue of the double inequality which yields $\frac{1}{2}(u+v) \leq \max\{u,v\} \leq (u+v)$, $\frac{1}{2}P(\mathcal{A},\mathcal{B}) \leq \mathcal{H}(\mathcal{A},\mathcal{B}) \leq P(\mathcal{A},\mathcal{B})$.

The metrical fixed point theorem for a set valued map appeared in 1968 with the papers of Markin [9], Nadler [12] and with the paper in 1969 by Covits and Nadler [4]. Consequently, 1968 marks the appearance of Pompeiu-Hausdorff metric in metrical fixed point theory, under the name "Hausdorff metric" or "Hausdorff distance", which is still being used in the majority of cases. Although it is better to use the word Pompeiu-Hausdorff metric as Pompeiu intiated the work to measure distance between two closed and bounded sets.

Our aim is to prove a relation-theoretic variant of the main result of Ayadi et al. [2] in partial metric spaces equipped with an arbitrary binary relation for a set-valued map. Examples are given, to demonstrate the theoretical findings. Consequently, we unify and generalize numerous fixed point results for single-valued maps to the analogous set-valued maps. Also, we solve an integral inclusion of Fredholm type as an application of our main result. In the sequal we demonstrate the significance of partial Pompeiu-Hausdorff metric under Relation-Theoretic Contractions and point out that the list of research areas that uses partial Pompeiu-Hausdorff metric is very inspiring.

2. Preliminaries

Throughout the paper, a map $\mathcal{M} : \mathcal{Z} \to 2^{\mathcal{Z}}$ of a non empty set $\mathcal{Z}$ is a set-valued map, $(\mathcal{Z}, p)$ is a partial metric space [11] and $\mathcal{CB}_p(\mathcal{Z})$ denote the set of all nonempty, bounded and closed subsets of $\mathcal{Z}$.
An element $z^* \in \mathcal{Z}$ is called a fixed point of $\mathcal{M}$ if $z^* \in \mathcal{M}z^*$. If τ_p is topology induced by p, closedness is taken from $(\mathcal{Z}, \tau_p)$. $\mathcal{U}$ is a bounded subset in $(\mathcal{Z}, p)$ if there exists $z_0 \in \mathcal{Z}$ and $M > 0$ such that $p(z_0, z) < p(z, z) + M$, i.e., $z \in \mathcal{B}_p(z_0, M)$. For $\mathcal{U}, \mathcal{V} \in \mathcal{Z}$, define

- $p(z,\mathcal{U}) = \inf\{p(z,u) : u \in \mathcal{U}\}$,
- $\delta_p(\mathcal{U},\mathcal{V}) = \sup\{p(u,\mathcal{V}) : u \in \mathcal{U}\}$,
- $\delta_p(\mathcal{V},\mathcal{U}) = \sup\{p(v,\mathcal{U}) : v \in \mathcal{V}\}$.

Definition 2.1. *[2] Let* $\mathcal{U},\mathcal{V},\mathcal{W} \in \mathcal{CB}_p(\mathcal{Z})$ *and* $\mathcal{H}_p(\mathcal{U},\mathcal{V}) = \max\left\{\delta_p(\mathcal{U},\mathcal{V}), \delta_p(\mathcal{V},\mathcal{U})\right\}$. *We have*

(*i*) $\mathcal{H}_p(\mathcal{U},\mathcal{U}) \leq \mathcal{H}_p(\mathcal{U},\mathcal{V})$;

(*ii*) $\mathcal{H}_p(\mathcal{U},\mathcal{V}) = \mathcal{H}_p(\mathcal{V},\mathcal{U})$;

(*iii*) $\mathcal{H}_p(\mathcal{U},\mathcal{V}) \leq \mathcal{H}_p(\mathcal{U},\mathcal{W}) + \mathcal{H}_p(\mathcal{W},\mathcal{V}) - \inf_{w\in\mathcal{W}} p(w,w)$.

$\mathcal{H}_p : \mathcal{CB}_p(\mathcal{Z}) \times \mathcal{CB}_p(\mathcal{Z}) \to [0,\infty)$ *is known as partial Pompeiu-Hausdorff metric induced by p.*

Example 2.2. *Let* $\mathcal{Z} = \{-1, 0, 1\}$ *and* $p(z, w) = \max\{|z|, |w|\}, z, w \in \mathcal{Z}$ *defines a partial metric p on* $\mathcal{Z}$ *then* $\mathcal{H}_p(\mathcal{U},\mathcal{V}) = 0$ *or* 1, *where* $\mathcal{U}$ *and* $\mathcal{V}$ *are subsets of* $\mathcal{Z}$.

Remark 2.3. *Every Pompeiu-Hausdorff metric is a partial Pompeiu-Hausdorff metric but reverse is not applicable.*

Lemma 2.4. *[2] Let* $\mathcal{U}$ *and* $\mathcal{V}$ *are nonempty bounded and closed subsets of a partial metric space* $(\mathcal{Z}, p)$ *and* $\xi > 1$. *Then* $u \in \mathcal{U}$ *there exists* $v \in \mathcal{V}$ *such that* $p(u, v) \leq \xi\mathcal{H}_p(\mathcal{U},\mathcal{V})$.

Definition 2.5. *[8] A binary relation denoted by* $\mathcal{R}$, *is a subset of* $\mathcal{Z} \times \mathcal{Z}$. *If* $(s, v) \in \mathcal{R}$, *then s is related to v.*

Definition 2.6. *[10]* $\mathcal{R}$ *is complete if* $s, v \in \mathcal{R}, [s, v] \in \mathcal{R}$, *(i.e., either* $(s, v) \in \mathcal{R}$ *or* $(v, s) \in \mathcal{R}$*).*

Definition 2.7. *[8] The symmetric closure $\mathcal{R}^s$ is the smallest symmetric relation containg $\mathcal{R}$, i.e., $\mathcal{R}^s = \mathcal{R} \bigcup \mathcal{R}^{-1}$.*

Definition 2.8. *[7] $\mathcal{R}$ is $\mathcal{M}-$closed if $(s, v) \in \mathcal{R} \Longrightarrow (\mathcal{M}s, \mathcal{M}v) \in \mathcal{R}$, $s, v \in \mathcal{U}$.*

Definition 2.9. *[1] A sequence $\{s_n\}$ in $\mathcal{Z}$ is $\mathcal{R}-$preserving if $(s_n, s_{n+1}) \in \mathcal{R}$, $n \in \mathbb{N}_0$.*

Definition 2.10. *[6] For any $m \in \mathbb{N}$, $w \in \mathcal{Z}$ is $m-$connected to z if there exists a path of length m from z to w, i.e., there exists $\{u_i\} \subseteq \mathcal{Z}$, $i = 0, 1, \ldots, m$ such that $u_0 = z$, $u_k = w$ and $(u_i, u_{i+1}) \in \mathcal{R}$. For some $z \in \mathcal{Z}$,*

$$\mathcal{P}(z, m) = \{w \in \mathcal{Z} : there\ exists\ a\ path\ of\ length\ m\ from\ z\ to\ w\}.$$

A sequence $\{z_n\} \subseteq \mathcal{Z}$ is called a trajectory of the map $\mathcal{M}$, starting at z_1, if $z_{n+1} \in \mathcal{M}z_n$, $n \in \mathbb{N}$.

A graph of a map $\mathcal{M}$ is defined as $\mathcal{G}(\mathcal{M}) = \{(z, w) : z \in \mathcal{Z}, w \in \mathcal{M}z\}$. The map $\mathcal{M}$ is closed if $\mathcal{G}(\mathcal{M})$ is closed subset of $\mathcal{Z} \times \mathcal{Z}$.

Definition 2.11. *[19] Let $\boldsymbol{\psi}$ be the class of functions $\psi : [0, \infty) \longrightarrow [0, \infty)$, such that*

(i) ψ is Lebesgue integrable, summable on compact subset of $[0, \infty)$;

(ii) $\int_0^{\epsilon} \psi(t)dt > 0$ for each $\epsilon > 0$.

3. Main Results

We introduce $\mathcal{R}-$completeness and $\mathcal{M}_p-$closedness in a partial Pompieu-Hausdorff metric space $(\mathcal{CB}_p(\mathcal{Z}), \mathcal{H}_p)$ to establish a set-valued relation theoretic variant of the main result of Aydi *et al*. [2].

Definition 3.1. *$(\mathcal{CB}_p(\mathcal{Z}), \mathcal{H}_p)$ is $\mathcal{R}-$complete if every $\mathcal{R}-$preserving Cauchy sequence in $\mathcal{Z}$ converges to a point in $\mathcal{Z}$.*

Every complete partial Pompeiu-Hausdorff metric space is $\mathcal{R}-$complete, however reverse is not essentially true.

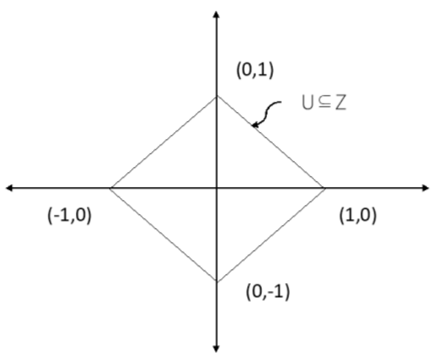
Definition 3.2. *Let $\mathcal{R}$ be a binary relation on $\mathcal{Z}$ and $\mathcal{M} : \mathcal{Z} \to \mathcal{CB}_p(\mathcal{Z})$ be a set-valued map. Then $\mathcal{R}$ is called $\mathcal{M}_p-$closed if*

$$(w, z) \in \mathcal{R},\ u \in \mathcal{M}w,\ v \in \mathcal{M}z,\ p(u, v) \leq p(w, z)\ implies\ that\ (u, v) \in \mathcal{R}.$$

Remark 3.3. *For a single valued map $\mathcal{M} : \mathcal{Z} \to \mathcal{Z}$, $\mathcal{R}$ is $\mathcal{M}_p-$closed if*

$$(w, z) \in \mathcal{R},\ p(\mathcal{M}w, \mathcal{M}z) \leq p(w, z)\ implies\ that\ (\mathcal{M}w, \mathcal{M}z) \in \mathcal{R}.$$

Example 3.4. *Let $\mathcal{Z} = \mathbb{R}^2$ and a partial metric $p : \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ be defined by $p((z, w), (u, v)) = \max\left\{\sqrt{z^2 + w^2}, \sqrt{u^2 + v^2}\right\}$, $(z, w), (u, v) \in \mathcal{Z}$. Let a relation $\mathcal{R}$ and a self map $\mathcal{M}$ on $\mathcal{Z}$ be defined as $\mathcal{R} = \left\{((z, w), (u, v)) : z^2 + w^2 \leq 1,\ u^2 + v^2 \leq 1\right\}$ and $\mathcal{M}(z, w) = \begin{cases} \frac{1}{4}(z, w), & (z, w) \in \mathcal{U} \\ 4(z, w), & otherwise. \end{cases}$ Let $\mathcal{U}$ be open subset of $\mathcal{Z}$ as shown in following figure.*



Noticeably, $\mathcal{R}$ is $\mathcal{M}_p$−closed but not $\mathcal{M}$−closed. Since $((1,0),(0,1)) \in \mathcal{R}$ but $(\mathcal{M}(1,0),\mathcal{M}(0,1)) = ((4,0),(0,4)) \notin \mathcal{R}$.

Remark 3.5. *In the above example if $p : \mathcal{Z}\times\mathcal{Z} \to [0,\infty)$ be defined as:*

$$p((z,w),(u,v)) = \begin{cases} \frac{1}{2}, & (z,w) \neq (u,v) \\ \frac{1}{4}, & (z,w) = (u,v). \end{cases}$$

Then $\mathcal{R}$ is not $\mathcal{M}_p$−closed. Here $p(\mathcal{M}(1,0),\mathcal{M}(0,1)) = p((4,0),(0,4)) = \frac{1}{2} = p((1,0),(0,1))$ and $p(\mathcal{M}(1,0),\mathcal{M}(1,0)) = p((4,0),(4,0)) = \frac{1}{4} = p((1,0),(1,0))$. However $(M(1,0),\mathcal{M}(0,1)),(M(1,0),\mathcal{M}(1,0)) \notin \mathcal{R}$.

It is clear that $\mathcal{M}_p$−closedness depends on both, a set-valued map $\mathcal{M}$ as well as a partial metric p and is weaker than $\mathcal{M}$−closedness.

Theorem 3.6. *Let $\mathcal{M}$ be a set-valued self map of a complete partial Pompeiu-Hausdorff metric space $(\mathcal{CB}_p(\mathcal{Z}),\mathcal{H}_p)$ equipped with a binary relation $\mathcal{R}$, satisfying :*

(i) there exist $z_0 \in \mathcal{Z}$ such that $\mathcal{P}(z_0,k)\bigcap \mathcal{M}z_0$ is non-empty, $k \in \mathbb{N}$.

(ii) $\mathcal{R}$ is $\mathcal{M}_p$-closed.

(iii) $\mathcal{M} : \mathcal{Z} \to \mathcal{CB}_p(\mathcal{Z})$ is a set-valued map such that

$$\mathcal{H}_p(\mathcal{M}z,\mathcal{M}w) \leq \eta p(z,w) \text{ for all } (z,w) \in \mathcal{R}, \text{ where } \eta \in (0,1). \tag{1}$$

(iv) either $\mathcal{M}$ is a closed map or if for any trajectory $\{z_n\} \subseteq \mathcal{Z}$ of $\mathcal{M}$, if $\{z_n\} \to z^*$ and $z_{n+1} \in P(z_n,k)$, $n \in \mathbb{N}$, then there exists a subsequence $\left\{z_{n_j}\right\}$ of $\{z_n\}$ such that $[z_{n_j},z^*] \in \mathcal{R}$, $j \in \mathbb{N}$.

Then there exists a trajectory $\{z_n\} \subseteq \mathcal{Z}$ of $\mathcal{M}$, converging to a fixed point of $\mathcal{M}$.

Proof. Because $\mathcal{P}(z_0,m)\bigcap\mathcal{M}z_0 \neq \phi$, so let $z_1 \in \mathcal{P}(z_0,m)\bigcap\mathcal{M}z_0$. Therefore $z_1 \in \mathcal{M}z_0$ implies if $z_0 = z_1 \in \mathcal{M}z_0$, z_0 is a fixed point of $\mathcal{M}$ and the constant sequence $\{z_0\}$ is a trajectory of $\mathcal{M}$ converging to a fixed point of $\mathcal{M}$ and hence the proof is completed.

So we consider a path of minimum length, i.e., $z_0 \neq z_1$. Now $z_1 \in \mathcal{P}(z_0,m)$ implies that there exists $\{w_i\} \subseteq \mathcal{Z}$, $i = 0,1,\ldots,m$ such that $w_0 = z_0, w_m = z_1$ and $(w_i,w_{i+1}) \in \mathcal{R}, i = 0,1,\ldots,m-1$. If $w_i = w_{i+1}$ for some $i = 0,1,\ldots,m-1$, we redefine m such that $w_i \neq w_{i+1}, i = 0,1,\ldots,m-1$.
Since $z_1 \in \mathcal{M}z_0 = \mathcal{M}w_0, p(w_0,w_1) > 0$. From Lemma 2.4 there exists $\xi = \frac{1}{\sqrt{\eta}}$ then there exists $w_1^1 \in \mathcal{M}w_1$ such that

$$p(z_1,w_1^1) \leq \frac{1}{\sqrt{\eta}}\mathcal{H}_p(\mathcal{M}w_0,\mathcal{M}w_1). \tag{2}$$

Choice of w_1^1 depends on the choice of η, since w_0 and w_1 are fixed.
$\mathcal{M}$ is set-valued relation theoretic contraction and $(w_0,w_1) \in \mathcal{R}$, so inequality (1) follows that

$$p(z_1,w_1^1) \leq \sqrt{\eta}\, p(w_0,w_1), \tag{3}$$

$$\text{i.e., } p(z_1,w_1^1) < p(w_0,w_1).$$

By the $\mathcal{M}_p$−closedness of $\mathcal{R}$, $(z_1,w_1^1) \in \mathcal{R}$.
Following the similiar steps, for $i = 0,1,2,\ldots$ there exists $w_i^1 \in \mathcal{M}w_i$ such that $(w_{i-1}^1,w_i^1) \in \mathcal{R}$ and

$$p(w_{i-1}^1,w_i^1) \leq \sqrt{\eta}p(w_{i-1},w_i).$$

Taking $w_0^1 = z_1$ and $w_m^1 = z_2$, $\left\{w_i^1 : i = 0,1,\ldots,m\right\}$ forms a path of lenght m from z_1 to z_2. So $z_2 \in \mathcal{P}(z_1,m)\bigcap\mathcal{M}z_1$. There is no guarantee that all $\left\{w_i^1\right\} \subseteq \mathcal{Z}$ are different.
Following the same pattern, we get a trajectory $\{z_n\}$ of $\mathcal{M}$ such that $z_{n+1} \in \mathcal{P}(z_n,k)\bigcap\mathcal{M}z_n$ as there exists $\left\{w_i^n\right\} \subseteq \mathcal{Z}, i = 0,1,\ldots,m$, $w_0^n = z_n, w_m^n = z_{n+1}$. Although, all $\left\{w_i^n\right\} \subseteq \mathcal{Z}$ may not be different. Consider if:

(i) $p(w_{i-1}^{n-1}, w_i^{n-1}) > 0$, then $w_i^n \in \mathcal{M}w_i^{n-1}$ such that $(w_{i-1}^n, w_i^n) \in \mathcal{R}$ and

$$p(w_{i-1}^n, w_i^n) < \sqrt{\eta}p(w_{i-1}^{n-1}, w_i^{n-1}).$$

(ii) If $p(w_{i-1}^{n-1}, w_i^{n-1}) = 0$, i.e., $w_{i-1}^{n-1} = w_i^{n-1}$. Since $w_{i-1}^{n-1} \in \mathcal{M}w_{i-1}^{n-2}, w_i^{n-1} \in \mathcal{M}w_i^{n-2}$ and $p(w_{i-1}^{n-1}, w_i^{n-1}) = 0 < p(w_{i-1}^{n-1}, w_i^{n-1})$. So by $\mathcal{M}_p-$closedness of $\mathcal{R}$, we get $(w_{i-1}^{n-1}, w_i^{n-1}) \in \mathcal{R}$.

In both cases, $(w_{i-1}^{n-1}, w_i^{n-1}) \in \mathcal{R},\ i = 1, 2, \ldots, m$.
Next, for notational convenience, take $p_i^0 = p(w_{i-1}^0, w_i^0)$ and $p_i^n = p(w_{i-1}^n, w_i^n), n \in \mathbb{N}$. Using inequality (1) repeatedly, we get, for $n \in \mathbb{N}$ and $i = 1, 2, \ldots, m$,

$$\begin{aligned} p_i^n = p(w_{i-1}^n, w_i^n) &\leq \sqrt{\eta}p(w_{i-1}^{n-1}, w_i^{n-1}) \\ &\ \ \vdots \\ &< (\sqrt{\eta})^n p_i^0. \end{aligned}$$

$$i.e.,\ p_i^n \leq (\sqrt{\eta})^n p_i^0. \tag{4}$$

Now by using (p_4), $n \in \mathbb{N}$ and $i = 1, 2, \ldots, m$

$$\begin{aligned} p(z_n, z_{n+1}) = p(w_0^n, w_m^n) &\leq \Sigma_{i=1}^m p_i^n - \Sigma_{i=1}^m p(w_i^{n-1}, w_i^{n-1}) \\ &< \Sigma_{i=1}^m p_i^n \\ &< (\sqrt{\eta})^n \Sigma_{i=1}^m p_i^0. \end{aligned}$$

For $n, m \in \mathbb{N}$ and $n > m$, using (p_4)

$$\begin{aligned} p(z_n, z_m) &\leq \Sigma_{j=m}^{n-1} p(z_j, z_{j+1}) - \Sigma_{j=m+1}^{n-1} p(z_j, z_j) \\ &< \Sigma_{j=m}^{n-1} p(z_j, z_{j+1}) \\ &\leq \Sigma_{j=m}^{n-1} (\sqrt{\eta})^j \Sigma_{i=1}^m p_i^0 = \frac{(\sqrt{\eta})^m}{1-\sqrt{\eta}} \Sigma_{i=1}^m p_i^0 \longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned}$$

Thus $\{z_n\}$ is a Cauchy sequence in a complete partial Pompeiu-Hausdorff metric space $(\mathcal{CB}_p(\mathcal{Z}), \mathcal{H}_p)$. Therefore, there exists $z^* \in \mathcal{Z}$ such that $z_n \longrightarrow z^*$ as $n \longrightarrow \infty$. Since $\{z_n\}$ is a trajectory of $\mathcal{M}$ and $\mathcal{M}$ is a closed map, so $(z_n, z_{n+1}) \longrightarrow (z^*, z^*) \in \mathcal{G}(\mathcal{M})$, i.e., $z^* \in \mathcal{M}z^*$. Hence z^* is a fixed point of $\mathcal{M}$.
In case, if there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $[z_{n_j}, z^*] \in \mathcal{R},\ j \in \mathbb{N}$. Since $z_{n_{j+1}} \in \mathcal{M}z_{n_j}, [z_{n_j}, z^*] \in \mathcal{R}, j \in \mathbb{N}$, there exists a sequence $\{v_j\}$ with $v_j \in \mathcal{M}z^*$ such that
$p(z_{n_{j+1}}, v_j) \leq \mathcal{H}_p(\mathcal{M}z_{n_j}, \mathcal{M}z^*)$.
Now, using inequality (1) and (p_4)

$$\begin{aligned} &\leq \sqrt{\eta}p(z_{n_j}, z^*) \\ &\leq \sqrt{\eta}(p(z_{n_j}, z_{n_{j+1}}) + p(z_{n_{j+1}}, z^*) - p(z_{n_{j+1}}, z_{n_{j+1}})). \end{aligned}$$

Choose the sequences $\{v_j\}$ in such a way that as $j \to \infty$, then $p(v_j, z^*) \to p(z^*, z^*)$. Since $v_j \in \mathcal{M}z^*$ for all $j \in \mathbb{N}$ and $\mathcal{M}z^*$ is closed, so $z^* \in \mathcal{M}z^*$. Hence z^* is a fixed point of $\mathcal{M}$. □

Remark 3.7. *If $\mathcal{R} = \mathcal{Z} \times \mathcal{Z}$ in Theorem 3.6, we get the extension of Nadler's Theorem [12] to partial metric spaces.*

To demonstrate the significance of partial Pompeiu-Hausdorff metric and valadity of Theorem 3.6, we give the following examples.

Example 3.8. *Let $\mathcal{Z} = \mathbb{R}$. Let partial metric on $\mathcal{Z}$ and $\mathcal{M} : \mathcal{Z} \to \mathcal{CB}_p(\mathcal{Z})$ be defined as:*

$$p(z, w) = |z - w| + \max\{|z|, |w|\} \text{ and } \mathcal{M}z = \begin{cases} \left\{0, \frac{z}{4}\right\}, & z \in \mathbb{Q}^+ \\ \{0, 4z + 1\}, & z \notin \mathbb{Q}^-. \end{cases}$$

Let the binary relation $\mathcal{R}$ on $\mathcal{Z}$ be defined as $\mathcal{R} = \{(z, w) : z, w \in Q^+\}$.

(i) Consider a sequence $\{z_n\} = \frac{\alpha}{2^{2n}}, n \in \mathbb{N}, \alpha \in \mathbb{Q}$. Then $(z_n, z_{n+1}) \in \mathcal{R}$ and $z_{n+1} \in \mathcal{M}z_n$, $n \in \mathbb{N}$. Therefore $\mathcal{P}(z_n, 1) \cap \mathcal{M}z_n \neq \phi$, $n \in \mathbb{N}$. Also $z_1 \in \mathcal{P}(z_1, 1) \cap \mathcal{M}z_1$.

(ii) *Let* $(z,w) \in \mathcal{R}, z, w \in \mathbb{Q}$. *Also there exist* $u \in \mathcal{M}z$ *and* $v \in \mathcal{M}w$, *i.e.,* $(u,v) \in \mathcal{R}, u, v \in \mathbb{Q}$. *Hence* $\mathcal{R}$ *is* $\mathcal{M}_p-$*closed.*

(iii) *By definition*
$\mathcal{H}_p(\mathcal{M}z, \mathcal{M}w) = \mathcal{H}_p(\left\{0, \frac{z}{4}\right\}, \left\{0, \frac{w}{4}\right\})$
$= \max\left\{\sup\left\{0, \min(\left|\frac{z}{4}\right| + \left|\frac{z}{4}\right|, \left|\frac{z}{4} - \frac{w}{4}\right| + \max\left\{\left|\frac{z}{4}\right|, \left|\frac{w}{4}\right|\right\})\right\}, \sup\left\{0, \min(\left|\frac{w}{4}\right| + \left|\frac{w}{4}\right|, \left|\frac{z}{4} - \frac{w}{4}\right| + \max\left\{\left|\frac{z}{4}\right|, \left|\frac{w}{4}\right|\right\})\right\}\right\}$
$= \max\left\{\min(\left|\frac{z}{4}\right| + \left|\frac{z}{4}\right|, \left|\frac{z}{4} - \frac{w}{4}\right| + \max\left\{\left|\frac{z}{4}\right|, \left|\frac{w}{4}\right|\right\}), \min(\left|\frac{w}{4}\right| + \left|\frac{w}{4}\right|, \left|\frac{z}{4} - \frac{w}{4}\right| + \max\left\{\left|\frac{z}{4}\right|, \left|\frac{w}{4}\right|\right\})\right\}$
$\leq \frac{1}{4}[|z - w| + \max\{|z|, |w|\}]$.
So, $\mathcal{H}_p(\mathcal{M}z, \mathcal{M}w) \leq \frac{1}{4}p(z,w)$,
i.e., $\mathcal{M}$ *is a set-valued relation theoretic contraction with* $\frac{1}{4}$ *as a contractivity constant.*

(iv) *Let* $\{z_n\} \subseteq \mathcal{Z}$ *be a trajectory of* $\mathcal{M}$ *and* $\{z_n\} \to z$, *since* $z_{n+1} \in \mathcal{P}(z_n, 1)$. *Hence* $(z_n, z_{n+1}) \in \mathcal{R}$ *and* $z_n \in \mathbb{Q}, n \in \mathbb{N}$. *So either* $z_2 = 0$ *or* $z_2 = \frac{z_1}{4}$. *If* $z_2 = 0$ *then* $z_3 = 0$ *and so on. If* $z_2 = \frac{z_1}{4}$, *conitinuing similiarly, we get* $z_n = \frac{1}{4}z_{n-1} = \frac{1}{4^2}z_{n-2} = \cdots = \frac{1}{4^{n-1}}z_1$. *So* $z_n \to 0$ *as* $n \to \infty$. *Therefore,* $(z_n, 0) \in \mathcal{R}, n \in \mathbb{N}$.

Thus, all the hypotheses of Theorem 3.6 are verified and the set of fixed point of $\mathcal{M}, \mathcal{F}(\mathcal{M}) = \left\{0, -\frac{1}{3}\right\}$.
But for any $z \in \mathbb{Q}^-$ *:*
$\mathcal{H}_p(\mathcal{M}z, \mathcal{M}0) = \mathcal{H}_p(\{0, 4z+1\}, 0) = |4z+1| \geq 2|z| = p(z,0)$.
Therefore, $\mathcal{M}$ *is not a Nadlar contraction.*

Example 3.9. *Let* $\mathcal{Z} = \mathbb{R}$ *and* $p(z,w) = \{|z|, |w|\}$. *Let a binary relation* $\mathcal{R}$ *on* $\mathcal{Z}$ *and* $\mathcal{M} : \mathcal{Z} \longrightarrow C\mathcal{B}_p(\mathcal{Z})$ *be defined as* $\mathcal{R} = \{(z,w) : z, w \in \mathbb{R}^+\}$ *and* $\mathcal{M}z = \begin{cases} \left\{\frac{1}{2}, \frac{1}{2z}\right\}, & z > 1 \\ \left\{0, \frac{z}{2}\right\}, & 0 \leq z \leq 1 \\ \{-2, -8\}, & \textit{otherwise.} \end{cases}$

(i) *Consider a sequence* $\{z_n\} = \frac{2n}{2n+1}, n \in \mathbb{N}$. *Then* $(z_n, z_{n+1}) \in \mathcal{R}$ *and* $z_{n+1} \in \mathcal{M}z_n$, $n \in \mathbb{N}$. *Therefore* $\mathcal{P}(z_n, 1) \cap \mathcal{M}z_n \neq \phi, n \in \mathbb{N}$. *Also* $z_1 \in \mathcal{P}(z_1, 1) \cap \mathcal{M}z_1$.

(ii) *Let* $(z,w) \in \mathcal{R}$. *Also there exist* $u \in \mathcal{M}z$ *and* $v \in \mathcal{M}w$, *i.e.,* $(u,v) \in \mathcal{R}$. *Hence* $\mathcal{R}$ *is* $\mathcal{M}_p-$*closed.*

(iii) $\mathcal{H}_p(\mathcal{M}z, \mathcal{M}w) \leq \frac{1}{2}p(z,w), (z,w) \in \mathcal{R}$.

(iv) *Let* $\{z_n\} \subseteq \mathcal{Z}$ *be a trajectory of* $\mathcal{M}$ *and* $\{z_n\} \longrightarrow z$, *since* $z_{n+1} \in P(z_n, 1)$. *Hence* $(z_n, z_{n+1}) \in \mathcal{R}, n \in \mathbb{N}$. *So either* $z_2 = 0, z_2 = \frac{z_1}{2}$ *or* $z_2 = \frac{1}{2z_1}$. *If* $z_2 = 0$ *then* $z_3 = 0$ *and so on. Following the same pattern for* $z_2 = \frac{z_1}{2}$ *or* $z_2 = \frac{1}{2z_1}$, *we get either or* $z_{n+1} = \frac{z_n}{2^n}$ *if* $\{z_n\} \subseteq [0,1]$ *or* $z_{n+1} = \frac{1}{2^n z_n}$ *if* $\{z_n\} \subseteq (1, \infty)$. *So* $z_n \longrightarrow 0$ *as* $n \longrightarrow \infty$. *Therefore,* $(z_n, 0) \in \mathcal{R}$.

Thus all the hypotheses of Theorem 3.6 are verified and the set of fixed point of $\mathcal{M}, \mathcal{F}(\mathcal{M}) = \{0\}$.
But for $z < 0$, $\mathcal{H}_p(\mathcal{M}(-1), \mathcal{M}(-2)) = 8 > p((-1), (-2)) = 2$. *Therefore,* $\mathcal{M}$ *is not a Nadlar contraction.*

Remark 3.10. *It is worth mentioning here that each Nadler contraction is a set-valued relation-theoretic contraction with the same binary relation* $\mathcal{R}$ *as the universal relation* $\mathcal{Z} \times \mathcal{Z}$ *and the same contractivity constant. But, each set-valued relation-theoretic contraction need not be a Nadler contraction in the space under consideration. So Nadler's fixed point theorem can not be applied in Examples 3.8 and 3.9.*

The following result is a an immediate extension of Theorem 3.6 to an integral type contraction.

Corollary 3.11. *Theorem 3.6 is valid even if contraction condition (iii) is replaced by*

$$\int_0^{\mathcal{H}_p(\mathcal{M}z, \mathcal{M}w)} \psi(t)dt \leq \eta \int_0^{p(z,w)} \psi(t)dt, (z,w) \in \mathcal{R}, \ \psi \in \boldsymbol{\psi}.$$

Remark 3.12. *It is intersting to see that, since single-valued map is a particular case of multi-valued map* ($\mathcal{M}z = \{\mathcal{M}z\}, z \in \mathcal{Z}$). *Therefore, Theorem 3.6 is an extension of Alam and Imdad [1] to a set-valued case.*

Remark 3.13. *Noticeably $\mathcal{M}$ is not continuous in Examples 3.8 and 3.9, hence we may conclude that set-valued relation theoretic contraction does not force the map to be continuous at the fixed point. Further, we have not assumed any variant of continuity for establishing unique fixed point using set-valued relation theoretic contraction. For details on the variants of continuity one may refer to Tomar and Karapinar [22]. Thus, we provide one more novel answer to the open question posed by Rhoades [18]. It is worth to mention here that the question of existence of a contractive condition which generate a fixed point but does not force a map to be continuous at the fixed point was posed by Rhoades [18] as an open problem and was first settled in the affirmative by Pant [14]. The study of contractive conditions which admit discontinuity at the fixed point and applications of such results in neural networks with discontinuous activation functions is presently a very active area of research e.g. Bisht and Rakočević [3],Özgür and Taş [13], Pant et al. [15], Rashid et al. [17], Taş and Özgür [20], Taş et al. [21]. Recently, Pant et al. [15] demonstrated that the problem of continuity of contractive maps at the fixed point has an affirmative answer in Menger PM spaces also. Bisht and Rakočević [3] obtained an interesting theorem which not only provides a new answer to the problem of continuity at the fixed point but, as shown below, also characterizes completeness of the metric space under the assumption of k-continuity.*

If f is a single-valued self-map of a metric space (X, d), let us denote:

$$m(x, y) = \max\left\{d(x, y), a\,d(x, fx) + (1-a)\,d(y, fy), (1-a)\,d(x, fx) + a\,d(y, fy), \frac{b[d(x, fy) + d(y, fx)]}{2}\right\},\ 0 \le a, b < 1$$

Bisht and Rakočević [3] proved the following theorem:

Theorem 3.14 (Theorem 2.1 of [3]). *Let (X, d) be a complete metric space. Let f be a self-map on X such that for any $x, y \in X$:*

(i) for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < m(x, y) < \epsilon + \delta$ implies $d(fx, fy) \le \epsilon$;
(ii) $d(Tx, Ty) < m(x, y)$, whenever $m(x, y) > 0$.

Then f has a unique fixed point, say z, and $f^n x \to z$ for each x in X. Moreover, f is continuous at z iff $\lim_{x \to z} m(x, z) = 0$.

Definition 3.15. *[15] A self-map f of a metric space X is called k-continuous, where $k = 1, 2, 3, \ldots$, if $f^k x_n \to ft$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1} x_n \to t$.*

We now prove that under the assumption of k-continuity Theorem 3.14 (Theorem 2.1 of [3]) of Bisht and Rakočević [3] characterizes completeness of the metric space.

Theorem 3.16. *Let (X, d) be a metric space. If every k-continuous self-map of X satisfying conditions (i) and (ii) of Theorem 3.14 has a fixed point, then X is complete.*

Proof. Suppose that every k-continuous self-map of X satisfying conditions (i) and (ii) of Theorem 3.14 possesses a fixed point. We assert that X is complete. If possible, suppose X is not complete. Then there exists a Cauchy sequence in X, say $S = \{u_1, u_2, u_3, \ldots\}$, consisting of distinct points which does not converge. Let $x \in X$ be given. Then since x is not a limit point of the sequence $S, d(x, S - \{x\}) > 0$ and there exists a least positive integer $N(x)$ such that $x \neq u_{N(x)}$ and for each $m \ge N(x)$ we have

$$d(u_{N(x)}, u_m) < \frac{1}{3} d(x, u_{N(x)}). \tag{5}$$

Let us define a map $f : X \to X$ by $f(x) = u_{N(x)}$. Then, $f(x) \neq x$ for each x and, using (5), for any $x, y \in X$ we get

$$d(fx, fy) = d(u_{N(x)}, u_{N(y)}) < \frac{1}{3} d(x, u_{N(x)}) = \frac{1}{3} d(x, fx) \text{ if } N(x) \le N(y)$$

or

$$d(fx, fy) = d(u_{N(x)}, u_{N(y)}) < \frac{1}{3} d(y, u_{N(y)}) = \frac{1}{3} d(y, fy) \text{ if } N(x) > N(y).$$

This implies that

$$\begin{aligned} d(fx, fy) &< \frac{1}{3}\max\left\{d(x,y), a\,d(x,fx) + (1-a)\,d(y,fy), (1-a)\,d(x,fx) + a\,d(y,fy), \frac{b[d(x,fy)+d(y,fx)]}{2}\right\} \\ &= \frac{1}{3}m(x,y) \end{aligned} \tag{6}$$

In other words, given $\epsilon > 0$ we can select $\delta(\epsilon) = \epsilon$ such that

$$\epsilon < m(x,y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon. \tag{7}$$

It is clear from (6) and (7) that the map f satisfies conditions (i) and (ii) of Theorem 3.14. Moreover, f is a fixed point free map whose range is contained in the non-convergent Cauchy sequence $S = \{u_n\}$. Hence, there exists no sequence $\{x_n\}$ in X for which $\{fx_n\}$ converges; that is, there exists no sequence $\{x_n\}$ in X for which the condition $fx_n \to t \Rightarrow f^k x_n \to ft$ is violated. Therefore, f is a k-continuous map. Thus, we have a self-map f of X which satisfies all the conditions of Theorem 3.14, but does not possess a fixed point. This contradicts the hypotheses of Theorem 3.14. Hence, X is complete. □

4. Application

Now, we solve an integral inclusion as an application of Theorem 3.6 .

Theorem 4.1. *Consider an integral inclusion of Fredholm type*

$$\xi(t) \in \phi(t) + \int_0^1 K(t,s,\xi(s))ds,\ t \in [0,1], \tag{8}$$

where $K : \mathcal{I} \times \mathcal{I} \times \mathbb{R} \to 2^{\mathbb{R}}$ is lower semicontinuous $(t,u) \in \mathcal{I} \times \mathcal{I}$ and $\xi, \phi \in C(\mathcal{I}, \mathbb{R})$.
Also for each $t \in \mathcal{I}$, there exists $f(t,.) \in \mathcal{L}_1(I)$ such that $\sup_{t\in\mathcal{I}} \int_0^1 f(t,s)ds = \frac{u}{4}$ with $u \in [0,1]$ and if $\xi(t) \leq \phi(t)$ for all $t \in \mathcal{I}$ then $\mathcal{H}_p(K(t,s,\xi(s)) - \mathcal{K}(t,s,\phi(s)) \leq f(t,s)(\max_{s\in\mathcal{I}} |\xi(s)|, |\phi(s)|)$.
Then the integral inclusion has at least one solution.

Proof. Take $\mathcal{Z} = C(\mathcal{I}, \mathbb{R})$. Consider the set-valued operator $\mathcal{M} : \mathcal{Z} \to 2^{\mathcal{Z}}$ defined by
$\mathcal{M}z(t) = \left\{\xi(t) \in C(\mathcal{I}, \mathbb{R}) : \xi(t) \in \phi(t) + \int_0^1 K(t,s,\xi(s))ds, t \in \mathcal{I}\right\}$.
$\mathcal{M}$ is well defined, since by Michael selection theorem, for each $K(t,s,\xi(s)) : \mathcal{I} \times \mathcal{I} \times \mathbb{R} \to 2^{\mathbb{R}}$ there exists a continuous operator $k : \mathcal{I} \times \mathcal{I} \times \mathbb{R} \to 2^{\mathbb{R}}$ such that $k(t,s,\xi(s)) \in K(t,s,\xi(s))$, $t,s \in \mathcal{I}$.
$\mathcal{M}z(t) \neq \phi$, since $\phi(t) + \int_0^1 K(t,s,\xi(s))ds \in \mathcal{M}z(t)$. So, the solution of an integral inclusion is the fixed point of the operator $\mathcal{M}$.
Let partial metric $p : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^+$ and binary relation $\mathcal{Z}$ be defined as
$p(z(t), w(t)) = \max_{t\in\mathcal{I}} \{|z(t)|, |w(t)|\}$ and $\mathcal{R} = \{(z,w) : z \leq w \text{ iff } z(t) \leq w(t),\ t \in \mathcal{I}\}$.

(i) Cosider a $\mathcal{R}$–presereving sequence $\{z_n\}$ in $\mathcal{M}z$ such that $z_n \to z$ and $z_{n+1} \in \phi(t) + \int_0^1 K(t,s,z_n(s))$. Therefore $\mathcal{P}(z_n, 1) \cap \mathcal{M}z_n \neq \phi$, $n \in \mathbb{N}$.

(ii) Let $z, w \in \mathcal{Z}$ with $(z,w) \in \mathcal{R}$, so $z(t) \leq w(t)$ for all $t \in \mathcal{Z}$. Also by hypothesis
$\mathcal{H}_p(K(t,s,z(s)) - K(t,s,w(s)) \leq f(t,s) \max_{s\in\mathcal{I}} \{|z(s)|, |w(s)|\}$, $t,s \in \mathcal{I}$.
Let $g \in \mathcal{M}z$, $h \in \mathcal{M}w$ then there exists $k(t,s,z(s)) \in K(t,s,z(s))$,$k(t,s,w(s)) \in K(t,s,w(s))$, $t,s \in \mathcal{I}$ such that
$g(t) = \phi(t) + \int_0^1 k(t,s,z(s))ds$, $h(t) = \phi(t) + \int_0^1 k(t,s,w(s))ds$, $t \in \mathcal{I}$ and
$|k(t,s,z(s)) - k(t,s,w(s))| \leq f(t,s)[\max |z(s)|, |w(s)|]$, $t,s \in \mathcal{I}$.
Therefore,
$\delta_p(\mathcal{M}z(t), \mathcal{M}w(t)) = \sup_{g\in\mathcal{M}z} \inf_{h\in\mathcal{M}w}(p(g(t), h(t))$
$= \sup_{g\in\mathcal{M}z} \inf_{h\in\mathcal{M}w} \max_{t\in\mathcal{I}} \left\{\left|\phi(t) + \int_0^1 k(t,s,z(s))ds\right|, \left|\phi(t) + \int_0^1 k(t,s,w(s))ds\right|\right\}$

$= \sup_{g\in\mathcal{M}z} \inf_{h\in\mathcal{M}w} \max_{t\in\mathcal{I}} \left\{\left|\int_0^1 k(t,s,z(s))\right|, \left|\int_0^1 k(t,s,w(s))\right|\right\}$
$\leq \sup_{g\in\mathcal{M}z} \inf_{h\in\mathcal{M}w} \int_0^1 \max_{t\in\mathcal{I}} \{|k(t,s,z(s))|, |k(t,s,w(s))|\} ds$
$\leq \sup_{g\in\mathcal{M}z} \inf_{h\in\mathcal{M}w} \int_0^1 f(t,s) \max_{t\in\mathcal{I}} \{|z(s)|, |w(s)|\} ds$
$\leq \max_{t\in\mathcal{I}} \{|z(s)|, |w(s)|\} \sup_{t\in\mathcal{I}} \int_0^1 f(t,s)ds$
$= \frac{u}{4}p(z(t), w(t)),\ u \in [0,1].$

Similiarly $\delta_p(\mathcal{M}w(t), \mathcal{M}z(t) \leq \frac{u}{4}p(w(t), z(t)),\ u \in [0,1].$

Now, $\mathcal{H}_p(\mathcal{M}z(t), \mathcal{M}w(t)) = \max\left\{\delta_p(\mathcal{M}z(t), \mathcal{M}w(t)), \delta_p(\mathcal{M}w(t), \mathcal{M}z(t))\right\}$
$\leq \frac{u}{4}p(z(t), w(t)),$

i.e., $\mathcal{M}$ is a set-valued relation theoretic contraction.

(iii) Let $(z,w) \in \mathcal{R}$, i.e., $z(t) \leq w(t),\ t \in \mathcal{I}$ and $g(t) \in \mathcal{M}z(t), h(t) \in \mathcal{M}w(t)$. Then
$p(g(t), h(t)) = \max_{t\in\mathcal{I}} \left\{\left|g(t)\right|, |h(t)|\right\}$
$= \max_{t\in\mathcal{I}} \left\{\left|\phi(t) + \int_0^1 k(t,s,z(s))ds\right|, \left|\phi(t) + \int_0^1 k(t,s,w(s))ds\right|\right\}$
$= \max_{t\in\mathcal{I}} \left\{\left|\int_0^1 k(t,s,z(s))ds\right|, \left|\int_0^1 k(t,s,w(s))ds\right|\right\}$
$\leq \max_{t\in\mathcal{I}} \int_0^1 \{|k(t,s,z(s))|, |k(t,s,w(s))|\} ds$
$\leq \max_{t\in\mathcal{I}} \{z(s), w(s)\} \int_0^1 f(t,s)ds$
$\leq \frac{u}{4}p(z(t), w(t),$

i.e., $\mathcal{R}$ is $\mathcal{M}_p-$closed.

(iv) Since $\mathcal{M}z \neq \phi$. So it is easy to prove $\mathcal{M}z$ is closed.
Hence, all the hypotheses of Theorem 3.6 are verified and consequently, the integral inclusion has a solution in $\mathcal{Z}$.

□

5. Conclusion

We have used an arbitrary binary relation to establish a fixed point in partial Pompeiu-Hausdorff metric spaces for set-valued relation-theoretic map. However, different binary relations, e.g., preorder, partial order, transitive relation, strict order, tolerance symmetric closure and so on, are being used by numerous authors to establish fixed point in a metric space for single-valued map. Further we have provided illustrative examples and solved an integral inclusion to demonstrate the validity of our main result and significance of partial Pompeiu-Haudorff metric for relation theoretic contraction. In the sequel we provided one more answer to the open question of Rhoades [18] for a set-valued relation-theoretic nonlinear contraction in a partial Pompeiu-Hausdorff metric space equipped with a binary relation. We also demonstrated that the result of Bisht and Rakočević [3] also characterizes the completeness of metric space via $k-$continuity.

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