Filomat 33:14 (2019), 4613–4611
https://doi.org/10.2298/FIL1914613S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Partial S-Metric Spaces and Coincidence Points

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Abstract. In this paper, the concept partial S-metric space is introduced as a generalization of S-metric space. We prove certain coincidence point theorems in partial S-metric spaces. The results we obtain generalize many known results in fixed point theory. Also, some examples show the effectiveness of this approach.

1. Introduction and preliminaries

Metrical fixed point theory became one of the most interesting area of research in the last fifty years. A lot of fixed and coincidence point results have been obtained by several authors in various types of spaces, such as metric spaces, fuzzy metric spaces, uniform spaces and others. One of the most interesting are partial metric spaces, which were defined by Matthews in [17] on the following way.

Definition 1.1. *[17] A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that, for all $x, y, z \in X$:*

(p_1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

In this case, the pair (X, p) is called a partial metric space.

Many fixed point results in partial metric spaces have been proved, see [2–11, 13–15, 23]. On the other hand, S-metric space were initiated by Sedghi, Shobe and Aliouche in [21] (see also [12] and references cited therein).

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25
Keywords. Fixed point, coincidence point, partial metric space, S-metric space, partially weakly increasing mappings
Received: 15 July 2018; Accepted: 20 January 2019
Communicated by Vladimir Rakočević
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Definition 1.2. *[21] An S-metric on a nonempty set X is a function $S : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:*

(s_1) $S(x, y, z) = 0 \iff x = y = z$,

(s_2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

In this case, the pair (X, S) is called an S-metric space.

It is easy to see that in an S-metric space (X, S) we always have $S(x, x, y) = S(y, y, x)$, $x, y \in X$.

In this paper, combining these two concepts, we introduce the notion of partial S-metric space and prove a common fixed point theorem for weakly increasing mappings in ordered spaces of this kind.

We recall some notions and properties in S-metric spaces.

Definition 1.3. *[19] Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in X.*

(a) The sequence $\{x_n\}$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. In this case, we write $lim_{n\to\infty}x_n = x$.

(b) $\{x_n\}$ is said to be a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

(c) The space (X, S) is said to be complete if every Cauchy sequence in it converges.

Lemma 1.4. *[19] Let (X, S) be an S-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.*

2. Partial S-metric spaces

In this section, we introduce partial S-metric spaces and investigate some of their properties.

Definition 2.1. *A partial S-metric on a nonempty set X is a function $S^* : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$:*

(s_{p1}) $x = y = z$ *if and only if* $S^*(x, y, z) = S^*(x, x, x) = S^*(y, y, y) = S^*(z, z, z)$,

(s_{p2}) $S^*(x, x, x) \leq S^*(x, y, z)$,

(s_{p3}) $S^*(x, y, z) \leq S^*(x, x, a) + S^*(y, y, a) + S^*(z, z, a) - 2S^*(a, a, a)$.

The pair (X, S^*) is then called a partial S-metric space.

Each S-metric space is also a partial S-metric space. The converse is not true, as shown by the following example.

Example 2.2. Let $X = [0, +\infty)$ and let $S^* : X \times X \times X \to [0, +\infty)$ be defined by $S^*(x, y, z) = \max\{x, y, z\}$. Then, it is easy to check that (X, S^*) is a partial S-metric space. Obviously, (X, S^*) is not an S-metric space.

Lemma 2.3. *For a partial S-metric S^* on X, we have, for all $x, y \in X$:*

(a) $S^*(x, x, y) = S^*(y, y, x)$,

(b) if $S^*(x, x, y) = 0$ *then* $x = y$.

Proof. (a) By the condition (s_{p3}), we have

(i) $S^*(x,x,y) \leq S^*(x,x,x) + S^*(x,x,x) + S^*(y,y,x) - 2S^*(x,x,x) = S^*(y,y,x)$,

and similarly

(ii) $S^*(y,y,x) \leq S^*(y,y,y) + S^*(y,y,y) + S^*(x,x,y) - 2S^*(y,y,y) = S^*(x,x,y)$.

By (i) and (ii), we get $S^*(x,x,y) = S^*(y,y,x)$.

(b) By the condition (s_{p2}), we have:

(iii) $S^*(x,x,x) \leq S^*(x,x,y) = 0$,

and similarly by relation (a), we also have:

(iv) $S^*(y,y,y) \leq S^*(y,y,x) = S^*(x,x,y) = 0$.

By (iii), (iv), we get $S^*(x,x,y) = S^*(x,x,x) = S^*(y,y,y) = 0$, which, by the condition (s_{p1}) implies that $x = y$. □

Remark 2.4. Dung, Hieu and Radojević noted in [13, Examples 2.1 and 2.2] that the class of S-metric spaces is incomparable with the the class of G-metric spaces, in the sense of Mustafa and Sims [18]. The same examples show that the class of partial S-metric spaces is incomparable with the class of GP-metric spaces, in the sense of Zand and Nezhad [23].

Definition 2.5. *Let (X, S^*) be a partial S-metric space and $\{x_n\}$ be a sequence in X.*

(a) The sequence $\{x_n\}$ converges to $x \in X$ (denoted as $x_n \to x$ as $n \to \infty$) if

$$\lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n\to\infty} S^*(x_n, x_n, x_n) = S^*(x,x,x).$$

(b) The sequence $\{x_n\}$ is said to be a Cauchy sequence if there exists (finite) $\lim_{n,m\to\infty} S^*(x_n, x_n, x_m)$.

(c) The space (X, S^*) is complete if each Cauchy sequence in X converges.

Note that if $x_n \to x$ as $n \to \infty$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n, x_n, x) - S^*(x,x,x)| < \epsilon \tag{1}$$

and

$$|S^*(x_n, x_n, x_n) - S^*(x,x,x)| < \epsilon, \tag{2}$$

for all $n \geq n_0$. Hence, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n, x_n, x_n) - S^*(x_n, x_n, x)| < \epsilon, \tag{3}$$

for all $n \geq n_0$.

Lemma 2.6. *Let (X, S^*) be a partial S-metric space. If a sequence $\{x_n\}$ in X converges to $x \in X$, then x is unique.*

Proof. Let $\{x_n\}$ converges to x and y. Then we have

$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(x_n, x_n, x) = S^*(x,x,x), \tag{4}$$

and

$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(x_n, x_n, y) = S^*(y,y,y).$$

Then, by the condition (s_{p3}), relation (4) and Lemma 2.3, we have

$$\begin{aligned} S^*(x,x,y) &\leq 2S^*(x,x,x_n) + S^*(y,y,x_n) - 2S^*(x_n,x_n,x_n) \\ &= 2(S^*(x_n,x_n,x) - S^*(x_n,x_n,x_n)) + S^*(x_n,x_n,y) - S^*(y,y,y) + S^*(y,y,y). \end{aligned}$$

By taking the limit as $n \to \infty$, we get $S^*(x,x,y) \leq S^*(y,y,y)$.

Also, by the condition (s_{p2}), we have $S^*(y,y,y) \leq S^*(y,y,x) = S^*(x,x,y)$. Hence, we get $S^*(x,x,y) = S^*(y,y,y)$. Similarly, we have $S^*(x,x,y) = S^*(x,x,x)$. Hence, by the condition, (s_{p1}) it follows that $x = y$. □

Lemma 2.7. *Let (X, S^*) be a partial S-metric space. Then each convergent sequence $\{x_n\}$ in X is a Cauchy sequence.*

Proof. Let $\{x_n\}$ converges to x, that is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that inequalities (1), (2) and (3) hold for all $n \geq n_0$. Then, by the condition (s_{p3}) and these inequalities, we have, for all $m, n \geq n_0$,

$$\begin{aligned} S^*(x_n, x_n, x_m) &\leq S^*(x_n, x_n, x) + S^*(x_n, x_n, x) + S^*(x_m, x_m, x) - 2S^*(x, x, x) \\ &\leq 2(S^*(x_n, x_n, x) - S^*(x, x, x)) + S^*(x_m, x_m, x) - S^*(x, x, x) + S^*(x, x, x) \\ &< 2\epsilon + \epsilon + S^*(x, x, x). \end{aligned} \tag{5}$$

Similarly, by the condition (s_{p3}) and Lemma 2.6,

$$\begin{aligned} S^*(x, x, x) &\leq S^*(x, x, x_n) + S^*(x, x, x_n) + S^*(x, x, x_n) - 2S^*(x_n, x_n, x_n) \\ &= 2(S^*(x_n, x_n, x) - S^*(x_n, x_n, x_n)) + S^*(x, x, x_n) \\ &\leq 2(S^*(x_n, x_n, x) - S^*(x_n, x_n, x_n)) + 2S^*(x, x, x_m) \\ &\quad + S^*(x_n, x_n, x_m) - 2S^*(x_m, x_m, x_m). \\ &< 2\epsilon + 2\epsilon + S^*(x_n, x_n, x_m). \end{aligned} \tag{6}$$

Hence, by (5) and (6), we have

$$|S^*(x_n, x_n, x_m) - S^*(x, x, x)| < 4\epsilon$$

for all $m, n \geq n_0$. Thus, $\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = S^*(x, x, x)$, and the sequence $\{x_n\}$ is a Cauchy. □

The notion of S_b-metric spaces was introduced independently in [20] and [22].

Definition 2.8. *Let X be a nonempty set and $b \geq 1$ a given real number. An S_b-metric on X, with parameter b, is a function $S_b : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:*

(s_{b1}) $S_b(x, y, z) = 0 \iff x = y = z$,

(s_{b2}) $S_b(x, x, y) = S_b(y, y, x)$,

(s_{b3}) $S_b(x, y, z) \leq b[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$.

In this case, the pair (X, S_b) is called an S_b-metric space.

A connection between partial S-metric and S_b-metric spaces (for $b = 2$) is given by the following lemma.

Lemma 2.9. *If (X, S^*) is a partial S-metric space, then $S^s : X \times X \times X \to [0, +\infty)$, given by*

$$S^s(x, y, z) = S^*(x, x, y) + S^*(y, y, z) + S^*(z, z, x) - S^*(x, x, x) - S^*(y, y, y) - S^*(z, z, z),$$

is an S_b-metric on X, with parameter $b = 2$.

Proof. First of all, by the condition (s_{p2}) and the definition of S^s, we have $S^s(x, y, z) \geq 0$. Further, we check that the conditions of Definition 2.8 are fulfilled.

(s_{b1}) If $S^s(x, y, z) = 0$ then it follows that $S^*(x, y, z) = S^*(x, x, x) = S^*(y, y, y) = S^*(z, z, z)$. That is, $x = y = z$. Conversely, if $x = y = z$, then we have $S^s(x, y, z) = 0$.

(s_{b2}) By the definition of S^s and Lemma 2.3, we have

$$\begin{aligned} S^s(x, x, y) &= S^*(x, x, x) + S^*(x, x, y) + S^*(y, y, x) - S^*(x, x, x) - S^*(x, x, x) - S^*(y, y, y) \\ &= S^*(x, x, x) + S^*(x, x, y) + S^*(x, x, y) - S^*(x, x, x) - S^*(x, x, x) - S^*(y, y, y) \\ &= 2S^*(x, x, y) - S^*(x, x, x) - S^*(y, y, y). \end{aligned}$$

Similarly, we can show that

$$S^s(y,y,x) = 2S^*(x,x,y) - S^*(x,x,x) - S^*(y,y,y).$$

Therefore, $S^s(x,x,y) = S^s(y,y,x)$. Also, we have always that $S^*(x,x,y) - S^*(x,x,x) \leq S^s(x,x,y)$.

(s_{b3}) By the condition (s_{p3}) and Lemma 2.3, we have

$$\begin{aligned}
S^s(x,y,z) &= S^*(x,x,y) + S^*(y,y,z) + S^*(z,z,x) - S^*(x,x,x) - S^*(y,y,y) - S^*(z,z,z)\\
&\leq 2S^*(x,x,a) - 2S^*(a,a,a) + S^*(y,y,a) + 2S^*(y,y,a) - 2S^*(a,a,a)\\
&\quad + S^*(z,z,a) + 2S^*(z,z,a) - 2S^*(a,a,a) + S^*(x,x,a) - S^*(x,x,x)\\
&\quad - S^*(y,y,y) - S^*(z,z,z)\\
&\leq 3S^*(a,a,x) - 2S^*(a,a,a) - S^*(x,x,x) + S^*(a,a,x) - S^*(x,x,x)\\
&\quad + 3S^*(a,a,y) - 2S^*(a,a,a) - S^*(y,y,y) + S^*(a,a,y) - S^*(y,y,y)\\
&\quad + 3S^*(a,a,z) - 2S^*(a,a,a) - S^*(z,z,z) + S^*(a,a,z) - S^*(z,z,z)\\
&= 2[S^s(x,x,a) + S^s(y,y,a) + S^s(z,z,a)].
\end{aligned}$$

□

Remark 2.10. *An open question remains whether there is a connection between partial S-metric and S_b-metric spaces for the parameter $b \neq 2$.*

Lemma 2.11. *Let (X,S^*) be a partial S-metric space and S^s the respective S_b-metric introduced in Lemma 2.9. Then:*

(a) A sequence $\{x_n\}$ in X is a Cauchy sequence in (X,S^*) if and only if it is a Cauchy sequence in (X,S^s).

(b) The space (X,S^*) is complete if and only if the space (X,S^s) is complete. Furthermore, $\lim_{n\to\infty} S^s(x_n,x_n,x) = 0$ if and only if

$$S^*(x,x,x) = \lim_{n\to\infty} S^*(x_n,x_n,x) = \lim_{n,m\to\infty} S^*(x_n,x_n,x_m).$$

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X,S^*). Then there exists (finite)

$$\lim_{n,m\to\infty} S^*(x_n,x_n,x_m) = \lim_{n\to\infty} S^*(x_n,x_n,x_n).$$

Since

$$S^s(x_n,x_n,x_m) = 2S^*(x_n,x_n,x_m) - S^*(x_n,x_n,x_n) - S^*(x_m,x_m,x_m),$$

we have

$$\lim_{n,m\to\infty} S^s(x_n,x_n,x_m) = 2\lim_{n,m\to\infty} S^*(x_n,x_n,x_m) - \lim_{n\to\infty} S^*(x_n,x_n,x_n) - \lim_{m\to\infty} S^*(x_m,x_m,x_m) = 0.$$

We conclude that $\{x_n\}$ is a Cauchy sequence in (X,S^s).

Next we prove that completeness of (X,S^s) implies completeness of (X,S^*). Indeed, if $\{x_n\}$ is a Cauchy sequence in (X,S^*) then it is also a Cauchy sequence in (X,S^s). Since the space (X,S^s) is complete, we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} S^s(x_n,x_n,y) = 0$, since $S^s(x_n,x_n,y) = 2S^*(x_n,x_n,y) - S^*(y,y,y) - S^*(x_n,x_n,x_n)$. Also, we know that

$$0 \leq S^*(x_n,x_n,y) - S^*(y,y,y) < S^s(x_n,x_n,y),$$

and

$$0 \leq S^*(x_n,x_n,y) - S^*(x_n,x_n,x_n) < S^s(x_n,x_n,y).$$

Therefore, we have

$$\lim_{n\to\infty} S^*(x_n, x_n, y) = \lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(y, y, y).$$

Hence, we deduce that $\{x_n\}$ is a convergent sequence in (X, S^*).

Now we prove that every Cauchy sequence $\{x_n\}$ in (X, S^s) is a Cauchy sequence in (X, S^*). Let $\epsilon = \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that $S^s(x_n, x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_0$. Since

$$\begin{aligned}
&S^*(x_n, x_n, x_n) \\
&\quad \leq 4S^*(x_{n_0}, x_{n_0}, x_n) - 3S^*(x_n, x_n, x_n) - S^*(x_{n_0}, x_{n_0}, x_{n_0}) + S^*(x_n, x_n, x_n) \\
&\quad \leq 2S^s(x_n, x_n, x_{n_0}) + S^*(x_{n_0}, x_{n_0}, x_{n_0}),
\end{aligned}$$

we have

$$\begin{aligned}
S^*(x_n, x_n, x_n) &\leq 2S^s(x_n, x_n, x_{n_0}) + S^*(x_{n_0}, x_{n_0}, x_{n_0}) \\
&\leq 1 + S^*(x_{n_0}, x_{n_0}, x_{n_0}).
\end{aligned}$$

Consequently, the sequence $\{S^*(x_n, x_n, x_n)\}$ is bounded in $\mathbb{R}$, and so there exists an $\alpha \in \mathbb{R}$ such that a subsequence $\{S^*(x_{n_k}, x_{n_k}, x_{n_k})\}$ is convergent to α, i.e., $\lim_{k\to\infty} S^*(x_{n_k}, x_{n_k}, x_{n_k}) = \alpha$.

It remains to prove that $\{S^*(x_n, x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a Cauchy sequence in (X, S^s), for given $\epsilon > 0$, there exists n_ϵ such that $S^s(x_n, x_n, x_m) < \frac{\epsilon}{2}$ for all $n, m \geq n_\epsilon$. Thus, for all $n, m \geq n_\epsilon$,

$$\begin{aligned}
&|S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m)| \\
&\qquad \leq 4S^*(x_n, x_n, x_m) - 3S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m) + S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m) \\
&\qquad \leq 2S^s(x_n, x_n, x_m) < \epsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|S^*(x_n, x_n, x_n) - \alpha| &\leq |S^*(x_n, x_n, x_n) - S^*(x_{n_k}, x_{n_k}, x_{n_k})| + |S^*(x_{n_k}, x_{n_k}, x_{n_k}) - \alpha| \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

for all $n, n_k \geq n_\epsilon$. Hence $\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \alpha$. Now,

$$\begin{aligned}
|2S^*(x_n, x_n, x_m) - 2\alpha| &= |S^s(x_n, x_n, x_m) + S^*(x_n, x_n, x_n) - \alpha + S^*(x_m, x_m, x_m) - \alpha| \\
&\leq S^s(x_m, x_m, x_m) + |S^*(x_n, x_n, x_n) - \alpha| + |S^*(x_m, x_m, x_m) - \alpha| \\
&< \frac{\epsilon}{2} + 2\epsilon + 2\epsilon = \frac{9}{2}\epsilon.
\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in (X, S^*).

In order to complete the proof, we have to prove that (X, S^s) is complete if such is (X, S^*). Let $\{x_n\}$ be a Cauchy sequence in (X, S^s). Then $\{x_n\}$ is a Cauchy sequence in (X, S^*), and so it is convergent to a point $y \in X$ with

$$\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = \lim_{n\to\infty} S^*(y, y, x_n) = S^*(y, y, y).$$

Thus, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$|S^*(y, y, x_n) - S^*(y, y, y)| < \frac{\epsilon}{2} \quad \text{and} \quad |S^*(y, y, y) - S^*(x_n, x_n, x_n)| < \frac{\epsilon}{2},$$

whenever $n \geq n_\epsilon$. Hence, we have

$$\begin{aligned}
S^s(y, y, x_n) &= 2S^*(y, y, x_n) - S^*(x_n, x_n, x_n) - S^*(y, y, y) \\
&\leq |S^*(y, y, x_n) - S^*(y, y, y)| + |S^*(y, y, x_n) - S^*(x_n, x_n, x_n)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

whenever $n \geq n_\epsilon$. Therefore (X, S^s) is complete.

Finally, it is a simple matter to check that $\lim_{n\to\infty} S^s(a, a, x_n) = 0$ if and only if

$$S^*(a, a, a) = \lim_{n\to\infty} S^*(a, a, x_n) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m).$$

□

Lemma 2.12. *Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences to $x \in X$ and $y \in X$, respectively, in a partial S-metric space (X, S^*). Then*

$$\lim_{n\to\infty} S^*(x_n, x_n, y_n) = S^*(x, x, y).$$

In particular, $\lim_{n\to\infty} S^*(x_n, x_n, y) = S^*(x, x, y)$ *for every* $y \in X$.

Proof. By the assumptions, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n, x_n, x) - S^*(x, x, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y) - S^*(y, y, y)| < \frac{\epsilon}{4},$$
$$|S^*(x_n, x_n, x_n) - S^*(x, x, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y_n) - S^*(y, y, y)| < \frac{\epsilon}{4},$$
$$|S^*(x_n, x_n, x_n) - S^*(x_n, x_n, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y_n) - S^*(y_n, y_n, y)| < \frac{\epsilon}{4},$$

hold for all $n \geq n_0$. By the condition (s_{p3}), for $n \geq n_0$ we have

$$\begin{aligned} S^*(x_n, x_n, y_n) &\leq S^*(x_n, x_n, x) + S^*(x_n, x_n, x) + S^*(y_n, y_n, x) - 2S^*(x, x, x) \\ &\leq S^*(x_n, x_n, x) + S^*(x_n, x_n, x) + S^*(y_n, y_n, y) + S^*(y_n, y_n, y) \\ &\quad + S^*(x, x, y) - 2S^*(y, y, y) - 2S^*(x, x, x) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + S^*(x, x, y), \end{aligned}$$

and so we obtain

$$S^*(x_n, x_n, y_n) - S^*(x, x, y) < \epsilon.$$

Also,

$$\begin{aligned} S^*(x, x, y) &\leq S^*(x, x, x_n) + S^*(x, x, x_n) + S^*(y, y, x_n) - 2S^*(x_n, x_n, x_n) \\ &\leq S^*(x, x, x_n) + S^*(x, x, x_n) + S^*(y, y, y_n) + S^*(y, y, y_n) \\ &\quad + S^*(x_n, x_n, y_n) - 2S^*(y_n, y_n, y_n) - 2S^*(x_n, x_n, x_n) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + S^*(x_n, x_n, y_n). \end{aligned}$$

Thus,

$$S^*(x, x, y) - S^*(x_n, x_n, y_n) < \epsilon.$$

Hence for all $n \geq n_0$, we have $|S^*(x_n, x_n, y_n) - S^*(x, x, y)| < \epsilon$ and the result follows. □

3. A common fixed point result in ordered partial S-metric spaces

Let f and g be two selfmaps on X. A point $x \in X$ is called

1. A fixed point of f if $f(x) = x$ (fixed point equation);
2. Coincidence point of a pair (f, g) if $fx = gx$ (coincidence point equation).

Solving fixed point equation and coincidence point equations in certain cases is equivalent to solving complementarity and implicit complementarity problems respectively [16].

Let $\Phi = \{\phi : \mathbb{R}^5{}_+ \longrightarrow \mathbb{R}_+ : \phi$ is continuous and increasing in each coordinate such that $\phi(t,t,t,t,5t) \leq t$ for every $t \geq 0\}$.

We present examples showing that the class of mappings Φ is nonempty.

Example 3.1. *(i)* $\phi_1(a,b,c,d,t) = max\{a,b,c,d,\frac{t}{5}\}$;
(ii) $\phi_2(a,b,c,d,t) = \frac{a+b+c+d+t}{9}$;
(iii) $\phi_3(a,b,c,d,t) = a$, *note that* $\phi_i \in \Phi$, *where* $i \in \{1,2,3\}$.

Definition 3.2. *Let* (X,S^*) *be a partial* $S-$ *metric space. A mapping* $F : X \longrightarrow X$ *is said to be continuos at* $x_0 \in X$ *with respect to* $g : X \longrightarrow X$, *if for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $F[g^{-1}(B_{S^*}(gx_0,\delta))] \subseteq B_{S^*}(Fx_0,\epsilon)$. *We shall say that F is* $g-$ *continuous at* x_0.

In particular, if set $g = I$ *(an identity map) in above definition then we have a classical definition of continuity of F at* $x_0 \in X$.

Lemma 3.3. *Let* $F, g : X \longrightarrow X$ *be two mappings and F is continuous at* x_0 *with respect to g. If a map g is continuous at* x_0 *then F is continuous at* $x_0 \in X$.

Proof. Since F is $g-$ continuos at $x_0 \in X$. Therefore for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$F[g^{-1}(B_{S^*}(gx_0,\delta))] \subseteq B_{S^*}(Fx_0,\epsilon).$$

Given that g is continuous at $x_0 \in X$, so for $\delta > 0$ there exists $\delta' > 0$ such that $g(B_{S^*}(x_0,\delta')) \subseteq B_{S^*}(gx_0,\delta)$. This implies that $B_{S^*}(x_0,\delta') \subseteq g^{-1}(B_{S^*}(gx_0,\delta))$. Hence,

$$F(B_{S^*}(x_0,\delta')) \subseteq F[g^{-1}(B_{S^*}(gx_0,\delta))] \subseteq B_{S^*}(Fx_0,\epsilon).$$

Following is an example of a discontinues map which is $g-$continuous. □

Example 3.4. *Let* $X = \mathbb{R}$ *and* $S^*(x,y,z) =\mid x-y \mid + \mid y-z \mid + \mid x-z \mid$. *Define selfmaps F and g on X as follows:*

$$gx = \begin{cases} 2 \;, & x \geq 0, \\ 3x \;, & x < 0 \end{cases} \qquad Fx = \begin{cases} 1 \;, & x \geq 0, \\ x \;, & x < 0 \end{cases} .$$

Obviously F is not continuous at $x = 0$. *Note that, for* $0 < \delta < 1$ *we have*

$$B_{S^*}(g(0),\delta) = \{y : \; 2|2-y| < \delta\} = (2-\frac{\delta}{2}, 2+\frac{\delta}{2}),$$

also $g^{-1}(B_{S^*}(g(0),\delta)) = [0,\infty)$ *and*

$$F[g^{-1}(B_{S^*}(g(0),\delta))] = \{1\} \subseteq B_{S^*}(F(0),\epsilon) = (1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2}).$$

which shows that F is $g-$*continuous at* $x = 0$.

Definition 3.5. *[8] Suppose* $(X,\preceq)$ *is a partially ordered set and* $F, g : X \longrightarrow X$ *are mappings of X into itself. We say F is* $g-$*nondecreasing if for* $x, y \in X$,

$$gx \preceq gy \quad implies \quad Fx \preceq Fy.$$

Obviously if $g = I$ *(an identity map on X), then definition of* $g-$*nondecreasing map coincides with the classical definition of nondecreasing map.*

Following example shows that $g-$nondecreasing mappings need not be nondecreasing in the classical sense.

Example 3.6. *Consider $(\mathbb{R}, \leq)$ with the usual order in $\mathbb{R}$ and $F, g : \mathbb{R} \longrightarrow \mathbb{R}$ are given by*

$$F(x) = x^2 + 1 \quad \text{and} \quad g(x) = x^2.$$

Obviously, F is a $g-$nondecreasing map and it is not nondecreasing.

The following is the main result of this section.

Theorem 3.7. *Let $(X, \preceq)$ is a partially ordered set and suppose that there is a partial $S-$metric S^* on X such that (X, S^*) is a complete partial $S-$metric space. Suppose that $F : X \longrightarrow X$ is a $g-$ continuous and $g-$nondecreasing mapping, with $g(X) = X$. Also,*

$$S^*(Fx, Fy, Fz) \leq q\phi \begin{pmatrix} S^*(gx, gy, gz), S^*(Fx, Fx, gx), \\ S^*(Fy, Fy, gy), S^*(Fz, Fz, gz), \\ S^*(Fy, Fy, gx) + S^*(Fz, Fz, gx) + \\ S^*(Fx, Fx, gy) + S^*(Fx, Fx, gz) \end{pmatrix} \tag{7}$$

for all $x, y, z \in X$ with $gz \preceq gy \preceq gx$, where $\phi \in \Phi$ and $0 \leq q < \frac{1}{2}$. If there exists an $x_0 \in X$ with $gx_0 \preceq Fx_0$, then coincidence point equation $gx = Fx$ has solution in X. Moreover, $S^*(gx, gx, gx) = 0$.

Proof. If $Fx_0 = gx_0$, then result follows, so suppose that $gx_0 \neq Fx_0$. Now let $gx_n = Fx_{n-1}$ for $n = 1, 2, \ldots$. If $gx_{n_0} = gx_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then it is clear that $Fx_{n_0} = gx_{n_0}$. Thus assume $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Since, $gx_0 \preceq Fx_0$ and F is $g-$nondecreasing, we have

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq \ldots \preceq gx_n \preceq gx_{n+1} \preceq \ldots.$$

Since $gx_{n-1} \preceq gx_n$, so inequality (3.1) implies that

$$\begin{aligned} S^*(gx_{n+1}, gx_{n+1}, gx_n) &= S^*(Fx_n, Fx_n, Fx_{n-1}) \\ &\leq q\phi \begin{pmatrix} S^*(gx_n, gx_n, gx_{n-1}), S^*(Fx_n, Fx_n, gx_n), \\ S^*(Fx_n, Fx_n, gx_n), S^*(Fx_{n-1}, Fx_{n-1}, gx_{n-1}), \\ S^*(Fx_n, Fx_n, gx_n) + S^*(Fx_{n-1}, Fx_{n-1}, gx_n) + \\ S^*(Fx_n, Fx_n, gx_n) + S^*(Fx_n, Fx_n, gx_{n-1}) \end{pmatrix} \\ &= q\phi \begin{pmatrix} S^*(gx_n, gx_n, gx_{n-1}), S^*(gx_{n+1}, gx_{n+1}, gx_n), \\ S^*(gx_{n+1}, gx_{n+1}, gx_n), S^*(gx_n, gx_n, gx_{n-1}), \\ S^*(gx_{n+1}, gx_{n+1}, gx_n) + S^*(gx_n, gx_n, gx_n) + \\ S^*(gx_{n+1}, gx_{n+1}, gx_n) + S^*(gx_{n+1}, gx_{n+1}, gx_{n-1}) \end{pmatrix} \\ &= q\phi \begin{pmatrix} S^*(gx_n, gx_n, gx_{n-1}), S^*(gx_{n+1}, gx_{n+1}, gx_n), \\ S^*(gx_{n+1}, gx_{n+1}, gx_n), S^*(gx_n, gx_n, gx_{n-1}), \\ 2S^*(gx_{n+1}, gx_{n+1}, gx_n) + S^*(gx_n, gx_n, gx_n) \\ +S^*(gx_{n+1}, gx_{n+1}, gx_{n-1}) \end{pmatrix} \end{aligned} \tag{8}$$

Since

$$\begin{aligned} & S^*(gx_{n+1}, gx_{n+1}, gx_{n-1}) \\ \leq\ & 2S^*(gx_{n+1}, gx_{n+1}, gx_n) + S^*(gx_{n-1}, gx_{n-1}, gx_n) - 2S^*(gx_n, gx_n, gx_n) \end{aligned}$$

and ϕ is nondecreasing we get from (3.2)

$$S^*(gx_{n+1}, gx_{n+1}, gx_n) \leq q\phi \begin{pmatrix} S^*(gx_n, gx_n, gx_{n-1}), S^*(gx_{n+1}, gx_{n+1}, gx_n), \\ S^*(gx_{n+1}, gx_{n+1}, gx_n), S^*(gx_n, gx_n, gx_{n-1}), \\ 4S^*(gx_{n+1}, gx_{n+1}, gx_n) + S^*(gx_{n-1}, gx_{n-1}, gx_n) \end{pmatrix}$$

Now if

$$S^*(gx_{n-1}, gx_{n-1}, gx_n) \leq S^*(gx_{n+1}, gx_{n+1}, gx_n)$$

for some n, then we have

$$\begin{aligned} S^*(gx_{n+1}, gx_{n+1}, gx_n) &\leq q\phi\begin{pmatrix} S^*(gx_{n+1}, gx_{n+1}, gx_n), S^*(gx_{n+1}, gx_{n+1}, gx_n), \\ S^*(gx_{n+1}, gx_{n+1}, gx_n), S^*(gx_{n+1}, gx_{n+1}, gx_n), \\ 5S^*(gx_{n+1}, gx_{n+1}, gx_n) \end{pmatrix} \\ &\leq qS^*(gx_{n+1}, gx_{n+1}, gx_n) \\ &< S^*(gx_{n+1}, gx_{n+1}, gx_n), \end{aligned}$$

a contradiction as $S^*(gx_{n+1}, gx_{n+1}, gx_n) > 0$. Thus

$$S^*(gx_{n+1}, gx_{n+1}, gx_n) \leq S^*(gx_{n-1}, gx_{n-1}, gx_n)$$

for all n. Therefore we have

$$S^*(gx_{n+1}, gx_{n+1}, gx_n) \leq qS^*(gx_{n-1}, gx_{n-1}, gx_n)$$

and so

$$S^*(gx_{n+1}, gx_{n+1}, gx_n) \leq q^n S^*(gx_0, gx_0, gx_1). \tag{9}$$

Therefore

$$\begin{aligned} & S^s(gx_n, gx_n, gx_{n+1}) \\ = \; & 2S^*(gx_{n+1}, gx_{n+1}, gx_n) - S^*(gx_n, gx_n, gx_n) - S^*(gx_{n+1}, gx_{n+1}, gx_{n+1}) \\ \leq \; & 2S^*(gx_{n+1}, gx_{n+1}, gx_n) \\ \leq \; & 2q^n S^*(gx_0, gx_0, gx_1) \end{aligned}$$

shows that $\lim_{n\longrightarrow\infty} S^s(gx_n, gx_n, gx_{n+1}) = 0$.

By the triangle inequality in S_b- metric space, for $m > n$ we have

$$\begin{aligned} S^s(gx_n, gx_n, gx_m) &\leq 2.2S^s(gx_n, gx_n, gx_{n+1}) + 2.2^2 S^s(gx_{n+1}, gx_{n+1}, gx_{n+2}) \\ &+ \ldots + 2.2^{m-n} S^s(gx_{m-1}, gx_{m-1}, gx_m) \\ &\leq 2^3 q^n S^*(gx_0, gx_0, gx_1) + 2^4 q^{n+1} S^*(gx_0, gx_0, gx_1) \\ &+ \ldots + 2^{m-n+2} q^{m-1} S^*(gx_0, gx_0, gx_1) \\ &\leq 2^3 q^n [1 + 2q + 2^2 q^2 + ...] S^*(gx_0, gx_0, gx_1) \\ &\leq \frac{2^3 q^n}{1-2q} S^*(gx_0, gx_0, gy_1) \longrightarrow 0. \end{aligned}$$

Therefore $\{gx_n\}$ is a Cauchy sequence in the S_b- metric space (X, S^s). Since (X, S^*) is complete then from Lemma 2.10, the sequence $\{gx_n\}$ converges in the S_b- metric space (X, S^s). Hence $\lim_{n\longrightarrow\infty} S^s(gx_n, gx_n, gx) = 0$ for some x in X. Again from Lemma 2.10, we have

$$S^*(gx, gx, gx) = \lim_{n\longrightarrow\infty} S^*(gx_n, gx_n, gx) = \lim_{n,m\longrightarrow\infty} S^*(gx_n, gx_n, gx_m). \tag{10}$$

Moreover since $\{gx_n\}$ is a Cauchy sequence in the S_b- metric space (X, S^s), $\lim_{n,m\longrightarrow\infty} S^s(gx_n, gx_n, gx_m) = 0$ and from (4) we have $\lim_{n\longrightarrow\infty} S^*(gx_n, gx_n, gx_n) = 0$. Using definition of S^s we get

$$\lim_{n,m\longrightarrow\infty} S^*(gx_n, gx_n, gx_m) = 0.$$

Therefore from (3.4), we obtain

$$S^*(gx, gx, gx) = \lim_{n\longrightarrow\infty} S^*(gx_n, gx_n, gx) = \lim_{n,m\longrightarrow\infty} S^*(gx_n, gx_n, gx_m) = 0$$

Now we claim that $Fx = gx$. Suppose $S^*(gx, gx, Fx) > 0$. Since F is $g-$ continuous, given $\epsilon > 0$, there exists $\delta > 0$ such that $F(g^{-1}(B_{S^*}(gx, \delta))) \subseteq B_{S^*}(Fx, \epsilon)$. Since $S^*(gx, gx, gx) = \lim_{n\longrightarrow\infty} S^*(gx_n, gx_n, gx) = 0$, there exists $k \in \mathbb{N}$ such that $S^*(gx_n, gx_n, gx) < S^*(gx, gx, gx) + \delta$ for all $n \geq k$. therefore, we have $gx_n \in B_{S^*}(gx, \delta)$ and $x_n \in g^{-1}(B_{S^*}(gx, \delta))$ for all $n \geq k$. Thus $F(x_n) \in F(g^{-1}(B_{S^*}(gx, \delta))) \subseteq B_{S^*}(Fx, \epsilon))$ and so $S^*(Fx_n, Fx_n, Fx) < S^*(Fx, Fx, Fx) + \epsilon$ for all $n \geq k$. This shows that $S^*(Fx, Fx, Fx) = \lim_{n\longrightarrow\infty} S^*(gx_{n+1}, gx_{n+1}, Fx)$. Using inequality (3.1) we have

$$\begin{aligned} S^*(Fx, Fx, Fx) &\leq q\phi\begin{pmatrix} S^*(gx, gx, gx), S^*(Fx, Fx, gx), \\ S^*(Fx, Fx, gx), S^*(Fx, Fx, gx), \\ S^*(Fx, Fx, gx) + S^*(Fx, Fx, gx)+ \\ S^*(Fx, Fx, gx) + S^*(Fx, Fx, gx) \end{pmatrix} \\ &\leq qS^*(Fx, Fx, gx). \end{aligned}$$

Therefore, obtain

$$\begin{aligned} & S^*(gx, gx, Fx) \\ \leq\ & S^*(gx, gx, gx_{n+1}) + S^*(gx, gx, gx_{n+1}) + S^*(Fx, Fx, gx_{n+1}) \\ -\ & 2S^*(gx_{n+1}, gx_{n+1}, gx_{n+1}) \\ \leq\ & S^*(gx, gx, gx_{n+1}) + S^*(gx, gx, gx_{n+1}) + S^*(Fx, Fx, gx_{n+1}), \end{aligned}$$

on taking limit as $n \longrightarrow \infty$, we arrive at the following inequality

$$\begin{aligned} & S^*(gx, gx, Fx) \\ \leq\ & \lim_{n\longrightarrow\infty} S^*(gx, gx, gx_{n+1}) + \lim_{n\longrightarrow\infty} S^*(gx, gx, gx_{n+1}) + \lim_{n\longrightarrow\infty} S^*(Fx, Fx, gx_{n+1}) \\ =\ & S^*(Fx, Fx, Fx) \\ \leq\ & qS^*(Fx, Fx, gx) \\ <\ & S^*(Fx, Fx, gx) = S^*(gx, gx, Fx), \end{aligned}$$

a contradiction. Thus $S^*(gx, gx, Fx) = 0$ and so $gx = Fx$. □

Taking $\phi(a, b, c, d, t) = a$ in Theorem 3.7, one obtains the following

Corollary 3.8. *Let $(X, \leq)$ is a partially ordered set and suppose that there is a partial $S-$metric S^* on X such that (X, S^*) is a complete partial $S-$metric space. Suppose that $F : X \longrightarrow X$ is a $g-$ continuous and $g-$nondecreasing mapping, with $g(X) = X$. Also,*

$$S^*(Fx, Fy, Fz) \leq qS^*(gx, gy, gz), \tag{11}$$

for all $x, y, z \in X$ with $gz \leq gy \leq gx$, where $0 \leq q < \frac{1}{2}$. If there exists an $x_0 \in X$ with $gx_0 \leq Fx_0$, then coincidence point equation $gx = Fx$ has solution in X. Moreover, $S^*(gx, gx, gx) = 0$.

Proof. It is enough in above Theorem, set $\phi(a, b, c, d, t) = a$. □

Now, we present an example which supports Corollary 3.8.

Example 3.9. *Let $X = \mathbb{R}^+$, and $S^* : X \times X \times X \longrightarrow \mathbb{R}^+$ be defined as*

$$S^*(x, y, z) = max\{|x - y|, |y - z|, |z - x|\}, \forall x, y, z \in X.$$

It is easy to see that (X, S^*) is a complete partial $S-$ metric space. Suppose $(X, \leq)$ with the usual order in X and $F, g : X \longrightarrow X$ are given by

$$F(x) = 2x + 1 \quad and \quad g(x) = 8x.$$

We will check that the conditions of Corollary 3.8 are fulfilled.

$$
\begin{aligned}
S^*(Fx,Fy,Fz) &= \max\{|Fx-Fy|,|Fx-Fz|,|Fy-Fz|\}\\
&= 2\max\{|x-y|,|x-z|,|y-z|\},
\end{aligned}
$$

and

$$
\begin{aligned}
S^*(gx,gy,gz) &= \max\{|gx-gy|,|gx-gz|,|gy-gz|\}\\
&= 8\max\{|x-y|,|x-z|,|y-z|\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
S^*(Fx,Fy,Fz) &= 2\max\{|x-y|,|x-z|,|y-z|\}\\
&\leq 8\max\{|x-y|,|x-z|,|y-z|\} = S^*(gx,gy,gz)\\
&\leq qS^*(gx,gy,gz),
\end{aligned}
$$

the condition (3.5) which is fulfilled for $\frac{1}{4} \leq q < \frac{1}{2}$. By Corollary 3.8, coincidence point equation $gx = Fx$ has solution $x = \frac{1}{6} \in X$.

If set $g = I$ identity map in Theorem (3.7) leads to the following corollary.

Corollary 3.10. *Let $(X, \preceq)$ a partially ordered set and suppose that there is a partial $S-$metric S^* on X such that (X, S^*) is a complete partial $S-$metric space. Suppose $F : X \longrightarrow X$ is a continuous and nondecreasing mapping such that*

$$
S^*(Fx,Fy,Fz) \leq q\phi\left(\begin{array}{l}
S^*(x,y,z), S^*(Fx,Fx,x),\\
S^*(Fy,Fy,y), S^*(Fz,Fz,z),\\
S^*(Fy,Fy,x) + S^*(Fz,Fz,x)+\\
S^*(Fx,Fx,y) + S^*(Fx,Fx,z)
\end{array}\right)
$$

for all $x, y, z \in X$ with $z \preceq y \preceq x$, where $\phi \in \Phi$ and $0 \leq q < \frac{1}{2}$. If there exists an $x_0 \in X$ with $x_0 \preceq Fx_0$, then fixed point equation $x = Fx$ has a solution in X. Moreover, $S^*(x, x, x) = 0$.

In the following theorem we drop the continuity of F and impose a condition on increasing convergent sequence in X.

Theorem 3.11. *Let $(X, \preceq)$ is a partially ordered set and suppose that there is a partial $S-$metric S^* on X such that (X, S^*) is a complete partial $S-$metric space. Suppose that $F : X \longrightarrow X$ is a $g-$nondecreasing mapping, with $g(X) = X$ such that*

$$
S^*(Fx,Fy,Fz) \leq q\phi\left(\begin{array}{l}
S^*(gx,gy,gz), S^*(Fx,Fx,gx),\\
S^*(Fy,Fy,gy), S^*(Fz,Fz,gz),\\
S^*(Fy,Fy,gx) + S^*(Fz,Fz,gx)+\\
S^*(Fx,Fx,gy) + S^*(Fx,Fx,gz)
\end{array}\right) \tag{12}
$$

for all $x, y, z \in X$ with $gz \preceq gy \prec gx$ (that is, $gz \preceq gy \preceq gx$ and $gy \neq gx$), where $\phi \in \Phi$ and $0 \leq q < \frac{1}{2}$. Also, the condition

$$
\left\{\begin{array}{c}
\text{If } \{gx_n\} \subset X \text{is a increasing sequence}\\
\text{with } gx_n \longrightarrow gx \text{ in X, then } gx_n \prec gx \text{ for all } n
\end{array}\right. \tag{13}
$$

holds. If there exists an $x_0 \in X$ with $gx_0 \preceq Fx_0$,then coincidence point equation $gx = Fx$ has a solution in X. Moreover, $S^*(gx, gx, gx) = 0$.

Proof. As given in the proof of Theorem 3.7, we construct a sequence $\{gx_n\}$ in X by $gx_n = Fx_{n-1}$ for $n = 1, 2, \cdots$. Also we can assume that the consecutive terms of $\{gx_n\}$ are distinct. Otherwise we are finished.

Therefore we have

$$gx_0 \prec gx_1 \prec gx_2 \prec \cdots \prec gx_n \prec gx_{n+1} \prec \cdots .$$

Following arguments similar to those given in the proof of Theorem 3.7, $\{gx_n\}$ is a Cauchy sequence in the S_b-metric space (X, S^s) and therefore there exists $x \in X$ such that

$$S^*(gx, gx, gx) = \lim_{n\longrightarrow\infty} S^*(gx_n, gx_n, gx) = \lim_{n,m\longrightarrow\infty} S^*(gx_n, gx_n, gx_m) = 0.$$

Now we claim that $Fx = gx$. Suppose $S^*(gx, gx, Fx) > 0$. In view of (3.7), we use (3.6) for $z = x_n$ and $x = y$ to obtain

$$\begin{aligned} & S^*(Fx, Fx, Fx_n) \\ \leq \quad & q\phi \begin{pmatrix} S^*(gx, gx, gx_n), S^*(Fx, Fx, gx), \\ S^*(Fx, Fx, gx), S^*(Fx_n, Fx_n, gx_n), \\ S^*(Fx, Fx, gx) + S^*(Fx_n, Fx_n, gx)+ \\ S^*(Fx, Fx, gx) + S^*(Fx, Fx, gx_n) \end{pmatrix}, \end{aligned}$$

which on taking limit as $n \longrightarrow \infty$ and using the continuity of ϕ implies that

$$\lim_{n\longrightarrow\infty} S^*(Fx, Fx, Fx_n) \leq qS^*(Fx, Fx, gx).$$

Therefore, we obtain

$$\begin{aligned} S^*(gx, gx, Fx) &\leq \lim_{n\longrightarrow\infty} S^*(gx, gx, gx_{n+1}) + \lim_{n\longrightarrow\infty} S^*(gx, gx, gx_{n+1}) + \lim_{n\longrightarrow\infty} S^*(Fx, Fx, gx_{n+1}) \\ &= \lim_{n\longrightarrow\infty} S^*(gx, gx, gx_{n+1}) + \lim_{n\longrightarrow\infty} S^*(gx, gx, gx_{n+1}) + \lim_{n\longrightarrow\infty} S^*(Fx, Fx, Fx_n) \\ &\leq qS^*(Fx, Fx, gx) \\ &< S^*(Fx, Fx, gx) = S^*(gx, gx, Fx), \end{aligned}$$

a contradiction. Thus $S^*(gx, gx, Fx) = 0$ and so $gx = Fx$. □

The above theorem leads to the following corollary.

Corollary 3.12. *Let $(X, \preceq)$ a partially ordered set and suppose that there is a partial $S-$ metric S^* on X such that (X, S^*) is a complete partial $S-$metric space. Suppose $F : X \longrightarrow X$ is a $g-$ nondecreasing mapping with $g(X) = X$ such that*

$$S^*(Fx, Fy, Fz) \leq q\phi \begin{pmatrix} S^*(gx, gy, gz), S^*(Fx, Fx, gx), \\ S^*(Fy, Fy, gy), S^*(Fz, Fz, gz), \\ S^*(Fy, Fy, gx) + S^*(Fz, Fz, gx) + \\ S^*(Fx, Fx, gx) + S^*(Fx, Fx, gz) \end{pmatrix}$$

for all $x, y, z \in X$ with $gz \preceq gy \preceq gx$ and $gy \neq gx$), where $\phi \in \Phi$ and $0 \leq q < \frac{1}{2}$. Also, the condition

$$\begin{cases} \text{If } \{gx_n\} \subset X \text{ is a increasing sequence} \\ \text{with } gx_n \longrightarrow gx \text{ in X, then } gx_n \prec gx \text{ for all } n \end{cases} \tag{14}$$

holds. If there exists an $x_0 \in X$ with $gx_0 \preceq Fx_0$,then there exists $x \in X$ such that $gx = Fx$. Moreover, $S^*(gx, gx, gx) = 0$.

Remark 3.13. *It should be noted that from the results obtained at a common fixed point in the partial S-metric spaces, a series of results on the fixed points is followed.*

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