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Infinitely Many Solutions for Mixed Dirichlet-Neumann Problems Driven by the (p, q)-Laplace Operator

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Abstract. We study a nonlinear problem with mixed Dirichlet-Neumann boundary conditions involving the p-Laplace operator and the q-Laplace operator ((p, q)-Laplace operator). Using variational tools and appropriate hypotheses on the behavior either at infinity or at zero of the reaction term, we prove that such a problem has infinitely many solutions.

1. Introduction

Let $N \geq 3$ and Ω be a nonempty open bounded subset of the Euclidean space $(\mathbb{R}^N, |\cdot|)$ with C^1-boundary $\partial\Omega$. Let Ω_1 and Ω_2 be two smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ satisfying the following conditions:

$$\Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega}_1 \cup \overline{\Omega}_2 = \partial\Omega, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 = \Omega_3,$$

where Ω_3 is a smooth $(N-2)$-dimensional submanifold of $\partial\Omega$.

In this paper, we study the following problem involving the (p, q)-Laplace operator (which is the sum of a p-Laplace operator and of a q-Laplace operator):

$$\begin{cases} -\triangle_p u - \triangle_q u + a(z)|u|^{p-2}u + b(z)|u|^{q-2}u = \lambda f(z, u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega_1, \\ \dfrac{\partial u}{\partial n_{pq}} = 0 & \text{on } \Omega_2. \end{cases} \tag{$P_\lambda(p,q)$}$$

Here, $\lambda > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ (reaction term) is a Carathéodory function and $a, b \in L^\infty(\Omega)$ are two functions such that $\operatorname{ess\,inf}_\Omega a > 0$ and $\operatorname{ess\,inf}_\Omega b > 0$. We recall that a lower order term of the type $a(z)|u|^{p-2}u$ has a regularizing effect on the solutions (see Porzio [12]). Furthermore, the p-Laplace and q-Laplace operators are defined by $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\triangle_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$, respectively. Finally, we have $p > q > N$ and

$$\frac{\partial u}{\partial n_{pq}} = (|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u, n)_{\mathbb{R}^N},$$

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with $n(\cdot)$ being the outward unit normal on Ω_2.

The study of nonlinear elliptic equations involving the p-Laplace and (p, q)-Laplace operators received the attention of many researchers (see [14] and the references therein). The motivation of this interest is in the usefulness of such equations to model various real phenomena arising in the physical processes (elasticity theory, quantum physics, reaction-diffusion systems). Some recent results in this direction were obtained by Cherfils-Il'yasov [4], Tanaka [13] (spectral theory, eigenvalue problems), Papageorgiou-Vetro [11] and Vetro [15] (variable version of the (p, q)-Laplace and of the p-Laplace-like operators, respectively), Nastasi-Vetro-Vetro [10] (discrete version of the (p, q)-Laplace operator). Komiya and Kajikiya [7] give a sufficient condition on the nonlinear term for the existence of a sequence of solutions converging to zero or diverging to infinity for the (p, q)-Laplace equation in a bounded domain under the Dirichlet boundary condition. Precisely, they investigated the problem:

$$-\triangle_p u - \triangle_q u = f(z, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $f(z, u)$ is subcritical but may change its sign and it is not necessarily a power nonlinearity.

For mixed boundary value problems there are the recent works of Bonanno-D'Aguì-Papageorgiou [2], Colorado-Peral [5] (p-Laplace operator), Dávila [6] (Laplace operator), Vetro-Vetro [16] (variable version of the p-Laplace operator). Of course analogous results occur whenever we use Neumann boundary condition. However, we note that there are deep differences in study of such (p, q)-Laplace problems under the three boundary type mentioned above. Clearly, an immediate difference is related to the choice of the Sobolev space (see also [2, Remark 3.7]), that is $W_0^{1,p}(\Omega)$ for the Dirichlet problem and $W^{1,p}(\Omega)$ for the Neumann problem.

The structure of problem $(P_\lambda(p, q))$ is variational. So, we discuss this problem via critical point theorems. Precisely, we give two main results producing the existence of infinitely many solutions of $(P_\lambda(p, q))$ under appropriate hypotheses on the behavior either at infinity or at zero of the reaction term. Also, we derive some corollaries as consequences. Our work is closely related to [2], where the authors study an analogous problem involving just the p-Laplace operator. So, they establish the existence of infinitely many solutions under suitable hypotheses (see [2, Theorems 3.1 and 3.2]). For another contribution establishing the existence of infinitely many solutions see Cao-Peng-Yan [3].

2. Mathematical background

We collect some basic notions and fix notation. Let (X, X^*) be a Banach topological pair with X reflexive. A function $\Gamma : X \to \mathbb{R}$ is called Gâteaux differentiable at $u \in X$ if there is $\Gamma'(u) \in X^*$ such that

$$\lim_{t \to 0^+} \frac{\Gamma(u + tv) - \Gamma(u)}{t} = \Gamma'(u)(v), \quad \text{for each } v \in X.$$

Also, the function Γ is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the function $u \to \Gamma'(u)$ is a continuous map from X to its dual X^*.

Let us denote by $\Upsilon_1, \Upsilon_2 : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions satisfying the following conditions:

- Υ_1 is strongly continuous, coercive and sequentially weakly lower semicontinuous,
- Υ_2 is sequentially weakly upper semicontinuous.

In the sequel, for all $s > \inf_X \Upsilon_1$, we set

$$\gamma(s) := \inf_{\Upsilon_1(v) < s} \frac{\left(\sup_{\Upsilon_1(u) < s} \Upsilon_2(u)\right) - \Upsilon_2(v)}{s - \Upsilon_1(v)}, \tag{1}$$

$$\eta := \liminf_{s \to +\infty} \gamma(s) \quad \text{and} \quad \delta := \liminf_{s \to (\inf_X \Upsilon_1)^+} \gamma(s). \tag{2}$$

The main spaces that we will use in the analysis of problem $(P_\lambda(p,q))$ are the Sobolev spaces $W^{1,p}(\Omega)$, $W^{1,q}(\Omega)$. Indeed, we use

$$\begin{aligned}
X_a &= W^{1,p}_{0,\Omega_1}(\Omega) = \{u \in W^{1,p}(\Omega) : \ u_{|\Omega_1} = 0\},\\
\|u\|_a &= \left(\int_\Omega (|\nabla u(z)|^p + a(z)|u(z)|^p dz\right)^{\frac{1}{p}},\\
X_b &= W^{1,q}_{0,\Omega_1}(\Omega) = \{u \in W^{1,q}(\Omega) : \ u_{|\Omega_1} = 0\},\\
\|u\|_b &= \left(\int_\Omega (|\nabla u(z)|^q + b(z)|u(z)|^q dz\right)^{\frac{1}{q}},\\
X &= X_a \cap X_b,\\
\|u\| &= \|u\|_a + \|u\|_b \quad \text{for all } u \in X.
\end{aligned}$$

Also we consider the space $C_{0,\Omega_1}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : \ u_{|\Omega_1} = 0\}$. We notice that $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ are embedded in $C(\overline{\Omega})$ (see [9], Theorem 1.49 we recall that $p > q > N$). Consequently, $W^{1,p}_{0,\Omega_1}(\Omega)$ and $W^{1,q}_{0,\Omega_1}(\Omega)$ are embedded in $C_{0,\Omega_1}(\overline{\Omega})$. So, we have

$$\|u\|_\infty \le k\|u\|_b \quad \text{for all } u \in X, \tag{3}$$

where

$$k = \sup_{u \in X\setminus\{0\}} \frac{\sup_{z\in\Omega}|u(z)|}{\|u\|_b}. \tag{4}$$

We mention that a weak solution of problem $(P_\lambda(p,q))$ is any $u \in X$ such that

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u(z)|^{p-2}\nabla u(z)\nabla v(z) + |\nabla u(z)|^{q-2}\nabla u(z)\nabla v(z)\\
&\quad + a(z)|u(z)|^{p-2}u(z)v(z) + b(z)|u(z)|^{q-2}u(z)v(z)\Big)dz\\
&= \lambda \int_\Omega f(z,u(z))v(z)dz \quad \text{for each } v \in X.
\end{aligned}$$

Let $\varpi := \sup_{z\in\Omega} \operatorname{dist}(z, \partial\Omega)$. It is easy to see that there exists $z_0 \in \Omega$ such that $B(z_0,\varpi) \subseteq \Omega$ and, for $\nu \in]0,1[$, we also have $B(z_0,\nu\varpi) \subset B(z_0,\varpi)$.

From now on, we put

$$w(z) = \begin{cases} 0 & \text{if } z \in \Omega \setminus B(z_0,\varpi),\\ \frac{1}{\varpi(1-\nu)}(\varpi - |z - z_0|) & \text{if } z \in B(z_0,\varpi) \setminus B(z_0,\nu\varpi),\\ 1 & \text{if } z \in B(z_0,\nu\varpi) \end{cases} \tag{5}$$

for all $z \in \Omega$ and

$$F(z,t) = \int_0^t f(z,\xi)d\xi \quad \text{for all } (z,t) \in \Omega \times \mathbb{R}.$$

To obtain our multiplicity results for $(P_\lambda(p,q))$, we use the following key-theorem of Bonanno-Molica Bisci [1], see also Marano-Motreanu [8].

Theorem 1 ([1], Theorem 2.1). *Let X, Υ_1, Υ_2 given as above and $\Gamma_\lambda = \Upsilon_1 - \lambda\Upsilon_2$ with $\lambda > 0$, then we have:*

(i) *If $\eta < +\infty$ (see (2)) then, for each $\lambda \in \left]0, \frac{1}{\eta}\right[$, one of the following assertions is true:*

(i_1) *Γ_λ possesses a global minimum,*

(i_2) *one can find a sequence* $\{u_n\}$ *of critical points (local minima) of* Γ_λ *with* $\Upsilon_1(u_n) \to +\infty$ *as* $n \to +\infty$.

(ii) *If* $\delta < +\infty$ (*see* (2)) *then, for each* $\lambda \in \left]0, \frac{1}{\delta}\right[$, *one of the following assertions is true:*

(ii_1) Υ_1 *possesses a global minimum which is a local minimum of* Γ_λ,

(ii_2) *one can find a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* Γ_λ, *with* $\Upsilon_1(u_n) \to \inf_X \Upsilon_1$ *as* $n \to +\infty$, *which weakly converges to a global minimum of* Υ_1.

Remark 1. *We say that* $u \in X$ *is a critical point of* Γ_λ *when* $\Gamma'_\lambda(u) = 0_{X^*}$, *that is,* $\Gamma'_\lambda(u)(v) = 0$ *for each* $v \in X$.

3. Main Results

Let us denote by $\Upsilon_1, \Upsilon_2, \Gamma_\lambda : X \to \mathbb{R}$ the functions defined by:

$$\Upsilon_1(u) := \frac{1}{p}\|u\|_a^p + \frac{1}{q}\|u\|_b^q, \quad \Upsilon_2(u) := \int_\Omega F(z, u(z))dz \quad \text{and} \quad \Gamma_\lambda = \Upsilon_1(u) - \lambda\Upsilon_2(u)$$

for all $u \in X$, where λ is a positive real parameter. We notice that Υ_1 and Υ_2 satisfy the regularity hypotheses of Theorem 1. Furthermore, if u is a critical point of Γ_λ, then u is a weak solution of $(P_\lambda(p, q))$.

We consider the following hypothesis (which means an appropriate oscillating behavior of F at infinity):

(H_1) For each $\xi \geq 0$, we have $\int_{\Omega\setminus B(z_0,\nu\varpi)} F(z,\xi)dz \geq 0$ and $\mathcal{M} > Ck^q\mathcal{L}$, where $\mathcal{M} = \limsup_{\xi\to+\infty}\left(\xi^{-p}\int_{B(z_0,\nu\varpi)} F(z,\xi)dz\right)$, $C = \|w\|_a^p + \|w\|_b^q$, k is given by (4) and $\mathcal{L} = \liminf_{\xi\to+\infty}\left(\xi^{-q}\int_\Omega \max_{|t|\leq\xi} F(z,t)dz\right)$.

Now we are ready for our multiplicity theorem producing infinitely many solutions.

Theorem 2. *If* (H_1) *holds, then problem* $(P_\lambda(p, q))$ *admits an unbounded sequence of weak solutions in* X *for each* $\lambda \in \left]\frac{C}{q\mathcal{M}}, \frac{1}{qk^q\mathcal{L}}\right[$.

Proof. Let $\lambda \in \left]\frac{C}{q\mathcal{M}}, \frac{1}{qk^q\mathcal{L}}\right[$. We show that η in (2) is finite. Let $\{\alpha_n\} \subset]0, +\infty[$ with $\alpha_n \to +\infty$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \max_{|t|\leq\alpha_n} F(z,t)dz}{\alpha_n^q} = \mathcal{L}.$$

Set $s_n := \frac{1}{q}\left(\frac{\alpha_n}{k}\right)^q$ for each $n \in \mathbb{N}$. If $u \in X$ is such that $\Upsilon_1(u) < s_n$, then

$$\Upsilon_1(u) = \frac{1}{p}\|u\|_a^p + \frac{1}{q}\|u\|_b^q < s_n = \frac{1}{q}\left(\frac{\alpha_n}{k}\right)^q,$$
$$\Rightarrow \quad \|u\|_b^q < q\, s_n.$$

Since $\|u\|_\infty \leq k\|u\|_b$, we also get $|u(z)| < \alpha_n$ for all $z \in \Omega$. As $\Upsilon_1(0) = \Upsilon_2(0) = 0 \in \{u \in X : \Upsilon_1(u) < s_n\}$, we have

$$\begin{aligned}\gamma(s_n) &= \inf_{\Upsilon_1(v)<s_n} \frac{\left(\sup_{\Upsilon_1(u)<s_n} \Upsilon_2(u)\right) - \Upsilon_2(v)}{s_n - \Upsilon_1(v)} \quad \text{(recall (1))}, \\ &\leq \frac{\sup_{\Upsilon_1(u)<s_n} \Upsilon_2(u)}{s_n}\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\int_\Omega \max_{|t|\leq\alpha_n} F(z,t)dz}{s_n} \quad \text{(since } |u(z)| < \alpha_n\text{)} \\
&= q\,k^q \frac{\int_\Omega \max_{|t|\leq\alpha_n} F(z,t)dz}{\alpha_n^q} \quad \text{for each } n \in \mathbb{N}, \\
\Rightarrow \quad & \eta = \liminf_{s\to+\infty} \gamma(s) \leq \liminf_{n\to+\infty} \gamma(s_n) \leq q\,k^q \mathcal{L} < +\infty. \qquad (6)
\end{aligned}$$

Next, we prove that Γ_λ does not possess a global minimum. Let $\{\beta_n\} \subset]0,+\infty[$ with $\beta_n \to +\infty$ as $n \to +\infty$, $\beta_n \geq 1$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to+\infty} \frac{\int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz}{\beta_n^p} = \mathcal{M}. \qquad (7)$$

Let us denote by $w_n : \mathbb{R} \to \mathbb{R}$ the function defined, for each $n \in \mathbb{N}$, by

$$w_n(z) = \beta_n\, w(z) \quad \text{(where } w \text{ is given by (5))}.$$

So, we have

$$\begin{aligned}
\Upsilon_1(w_n) &= \frac{1}{p}\|w_n\|_a^p + \frac{1}{q}\|w_n\|_b^q \\
&= \frac{1}{p}\beta_n^p\|w\|_a^p + \frac{1}{q}\beta_n^q\|w\|_b^q \\
&\leq \frac{1}{q}\beta_n^p(\|w\|_a^p + \|w\|_b^q) \quad \text{(since } \beta_n \geq 1\text{)} \\
&= \frac{1}{q}\beta_n^p C, \quad \text{where } C = \|w\|_a^p + \|w\|_b^q
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon_2(w_n) &= \int_\Omega F(z, w_n(z))dx \\
&\geq \int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz \quad \text{for each } n \in \mathbb{N}.
\end{aligned}$$

Hence, it follows that

$$\Gamma_\lambda(w_n) \leq \frac{1}{q}\beta_n^p C - \lambda \int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz.$$

By choosing $\epsilon \in \left]0, \mathcal{M} - \frac{C}{q\lambda}\right[$ (for $\mathcal{M} < +\infty$) and taking into account that by (7) we have

$$\lim_{n\to+\infty} \int_{B(z_0,\nu\varpi)} \frac{F(z,\beta_n)}{\beta_n^p} dz = \mathcal{M},$$

one can find n_ϵ satisfying

$$\int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz \geq (\mathcal{M} - \epsilon)\beta_n^p \quad \text{for each } n \geq n_\epsilon.$$

This assures that

$$\Gamma_\lambda(w_n) \leq \frac{1}{q}\beta_n^p C - \lambda(\mathcal{M} - \epsilon)\beta_n^p$$

$$= \frac{\beta_n^p}{q}\Big[C - q\lambda(\mathcal{M} - \epsilon)\Big] \quad \text{for each } n \geq n_\epsilon.$$

Since $C - q\lambda(\mathcal{M} - \epsilon) < 0$, we get $\Gamma_\lambda(w_n) \to -\infty$ as $n \to +\infty$.
Analogously, if $\mathcal{M} = +\infty$ we fix $V > \dfrac{C}{\lambda q}$ and thus, by (7), one can find n_V such that

$$\int_{B(z_0,\nu\varpi)} F(z, \beta_n)dz \geq V\beta_n^p \quad \text{for all } n \geq n_V.$$

So, we get

$$\Gamma_\lambda(w_n) \leq \frac{\beta_n^p}{q}\Big[C - q\lambda V\Big] \quad \text{for all } n \geq n_V.$$

Now, taking into account that $C - q\lambda V < 0$, we again have $\Gamma_\lambda(w_n) \to -\infty$ as $n \to +\infty$.
This assures that Γ_λ does not possess a global minimum. Finally, we point out that

$$\Big]\frac{C}{q\mathcal{M}}, \frac{1}{qk^q\mathcal{L}}\Big[\subset \Big]0, \frac{1}{\eta}\Big[\qquad \text{(see (6))},$$

$\Rightarrow$ Γ_λ has an unbounded sequence of critical points (by Theorem 1(i)),

$\Rightarrow$ $(P_\lambda(p,q))$ has an unbounded sequence of weak solutions in X.

□

As immediate consequence of Theorem 2 we get the following result.

Corollary 1. *If* (H_1) *holds with* $\mathcal{M} = +\infty$ *and* $\mathcal{L} = 0$*, then problem* $(P_\lambda(p,q))$ *admits an unbounded sequence of weak solutions in* X *for each* $\lambda > 0$.

Now, we consider the following problem, where we drop the z-dependence on the reaction term:

$$\begin{cases} -\triangle_p u - \triangle_q u + |u|^{p-2}u + |u|^{q-2}u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega_1, \\ \dfrac{\partial u}{\partial n_{pq}} = 0 & \text{on } \Omega_2. \end{cases} \tag{8}$$

Here $\lambda > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative function and $N < q < p < +\infty$. Also, we put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$ and consider the following hypothesis:

(h_1) $\limsup\limits_{\xi\to+\infty} \xi^{1-p} f(\xi) = +\infty$ and $\liminf\limits_{\xi\to+\infty} \xi^{1-q} f(\xi) = 0$.

Corollary 2. *If* (h_1) *holds, then problem* (8) *admits an unbounded sequence of weak solutions in* X *for each* $\lambda > 0$.

Proof. By (h_1) we get

$$\limsup_{\xi\to+\infty} \xi^{-p} F(\xi) = +\infty \quad \text{and} \quad \liminf_{\xi\to+\infty} \xi^{-q} F(\xi) = 0.$$

Also, from $F(\xi) \geq 0$ for all $\xi \geq 0$, we get $\int_{\Omega\setminus B(z_0,\nu\varpi)} F(\xi)dz \geq 0$. Consequently, we can say that (H_1) holds with $\mathcal{M} = +\infty$ and $\mathcal{L} = 0$. So, by Corollary 1, the problem (8) admits an unbounded sequence of weak solutions in X for all $\lambda > 0$. □

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(\xi) = \begin{cases} \xi^2 + (1 - \cos\xi)\xi^4 & \xi \geq 0, \\ 0 & \xi < 0, \end{cases}$$

and $N = 3 < q < p < 5$. We have

$$\limsup_{\xi \to +\infty} \frac{f(\xi)}{\xi^{p-1}} = +\infty \quad \text{and} \quad \liminf_{\xi \to +\infty} \frac{f(\xi)}{\xi^{q-1}} = 0.$$

So, by Corollary 2, the problem

$$\begin{cases} -\triangle_p u - \triangle_q u + |u|^{p-2}u + |u|^{q-2}u = \lambda[u^2 + (1 - \cos u)u^4] & \text{in } \Omega, \\ u = 0 & \text{on } \Omega_1, \\ \dfrac{\partial u}{\partial n_{pq}} = 0 & \text{on } \Omega_2, \end{cases}$$

admits an unbounded sequence of weak solutions in X for all $\lambda > 0$.

To get our second result, we change hypothesis (H_1) as follows:

(H_1') For each $\xi \geq 0$, we have $\int_{\Omega \setminus B(z_0,\nu\varpi)} F(z,\xi)dz \geq 0$ and $\mathcal{M}_1 > Ck^q\mathcal{L}_1$, where $\mathcal{M}_1 = \limsup_{\xi \to 0^+} \Big(\xi^{-q} \int_{B(z_0,\nu\varpi)} F(z,\xi)dz\Big)$, $C = \|w\|_a^p + \|w\|_b^q$, k is given by (4) and $\mathcal{L}_1 = \liminf_{\xi \to 0^+} \Big(\xi^{-q} \int_\Omega \max_{|t| \leq \xi} F(z,t)dz\Big)$.

This time, we assume an oscillating behavior of F at zero. So, we are ready for our multiplicity theorem producing a sequence of pairwise distinct weak solutions with some regularities.

Theorem 3. *If (H_1') holds, then problem $(P_\lambda(p,q))$ admits a sequence of pairwise distinct weak solutions which uniformly converges to zero in X for each $\lambda \in \Big]\dfrac{C}{q\mathcal{M}_1}, \dfrac{1}{qk^q\mathcal{L}_1}\Big[$.*

Proof. We prove that the unique global minimum of Υ_1 is not a local minimum for Γ_λ.

Fix $\lambda \in \Big]\dfrac{C}{q\mathcal{M}_1}, \dfrac{1}{qk^q\mathcal{L}_1}\Big[$ and denote by $\{\alpha_n\} \subset]0, +\infty[$ a sequence converging to zero with

$$\lim_{n \to +\infty} \frac{\int_\Omega \max_{|t| \leq \alpha_n} F(z,t)dz}{\alpha_n^q} = \mathcal{L}_1.$$

Let $s_n := \dfrac{1}{q}\Big(\dfrac{\alpha_n}{k}\Big)^q$ for each $n \in \mathbb{N}$. If $u \in X$ is such that $\Upsilon_1(u) < s_n$, then

$$\Upsilon_1(u) = \frac{1}{p}\|u\|_a^p + \frac{1}{q}\|u\|_b^q < s_n = \frac{1}{q}\left(\frac{\alpha_n}{k}\right)^q,$$
$$\Rightarrow \quad \|u\|_b^q < q\, s_n.$$

As $\|u\|_\infty \leq k\|u\|_b$, we have $|u(z)| < \alpha_n$ for all $z \in \Omega$. Since $\Upsilon_1(0) = \Upsilon_2(0) = 0 \in \{u \in X : \Upsilon_1(u) < s_n\}$, proceeding as in the proof of Theorem 2, we get

$$\gamma(s_n) \leq q\, k^q \frac{\int_\Omega \max_{|t| \leq \alpha_n} F(z,t)dz}{\alpha_n^q} \qquad \text{for each } n \in \mathbb{N},$$

$$\Rightarrow \quad \delta = \liminf_{s\to 0^+} \gamma(s) \leq \liminf_{n\to+\infty} \gamma(s_n) \leq q\, k^q \mathcal{L}_1 < +\infty. \tag{9}$$

Now, we prove that the unique global minimum of Υ_1, say $u = 0$, is not a local minimum of Γ_λ. Let $\{\beta_n\} \subset]0,1]$ with

$$\lim_{n\to+\infty} \frac{\int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz}{\beta_n^q} = \mathcal{M}_1. \tag{10}$$

Let us denote by $w_n : \mathbb{R} \to \mathbb{R}$ the function defined, for each $n \in \mathbb{N}$, by

$$w_n(z) = \beta_n\, w(z) \quad \text{(where } w \text{ is given by (5))}.$$

Again, proceeding as in the proof of Theorem 2, we get

$$\Upsilon_1(w_n) \leq \frac{1}{q}\beta_n^q C$$

and

$$\begin{aligned}\Upsilon_2(w_n) &:= \int_\Omega F(z, w_n(z))dz \\ &\geq \int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz \quad \text{for each } n \in \mathbb{N}.\end{aligned}$$

Hence, it follows that

$$\Gamma_\lambda(w_n) \leq \frac{1}{q}\beta_n^q C - \lambda \int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz.$$

By choosing $\epsilon \in \left]0, \mathcal{M}_1 - \frac{C}{q\lambda}\right[$ (for $\mathcal{M}_1 < +\infty$) and taking into account that by (10) we have

$$\lim_{n\to+\infty} \int_{B(z_0,\nu\varpi)} \frac{F(z,\beta_n)}{\beta_n^q} dz = \mathcal{M}_1,$$

one can find n_ϵ satisfying

$$\int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz \geq (\mathcal{M}_1 - \epsilon)\beta_n^q \quad \text{for each } n \geq n_\epsilon.$$

This leads to

$$\begin{aligned}\Gamma_\lambda(w_n) &\leq \frac{1}{q}\beta_n^q C - \lambda(\mathcal{M}_1 - \epsilon)\beta_n^q \\ &= \frac{\beta_n^q}{q}\left[C - q\lambda(\mathcal{M}_1 - \epsilon)\right] \quad \text{for each } n \geq n_\epsilon.\end{aligned}$$

Since $C - q\lambda(\mathcal{M}_1 - \epsilon) < 0$, we get

$$\Gamma_\lambda(w_n) < 0 \quad \text{for all } n \geq n_\epsilon.$$

Analogously, if $\mathcal{M}_1 = +\infty$ we fix $V > \frac{C}{\lambda q}$ and thus, by (10), one can find n_V such that

$$\int_{B(z_0,\nu\varpi)} F(z,\beta_n)dz \geq V\beta_n^q \quad \text{for all } n \geq n_V.$$

Hence, we get

$$\Gamma_\lambda(w_n) \leq \frac{\beta_n^q}{q}\Big[C - q\lambda V\Big] \quad \text{for all } n \geq n_V.$$

Now, taking into account that $C - q\lambda V < 0$, we again have $\Gamma_\lambda(w_n) < 0$ for all $n \geq n_V$.
We point out that $\Gamma_\lambda(0) = 0$ and so 0 is not a local minimum of Γ_λ. So, we get

$$\Big]\frac{C}{q\mathcal{M}_1}, \frac{1}{qk^q\mathcal{L}_1}\Big[\subset \Big]0, \frac{1}{\delta}\Big[\quad \text{(see (9))},$$

$\Rightarrow$ Γ_λ has a sequence $\{u_n\} \subset X$ of pairwise distinct critical points such that $\lim_{n\to+\infty} \Upsilon_1(u_n) = \inf_X \Upsilon_1 = 0$ (by Theorem 1(*ii*)),

$\Rightarrow$ $(P_\lambda(p,q))$ has a sequence of pairwise distinct weak solutions which uniformly converges to zero in X.

□

From Theorem 3 we deduce the following result.

Corollary 3. *If (H_1') holds with $\mathcal{M}_1 = +\infty$ and $\mathcal{L}_1 = 0$, then problem $(P_\lambda(p,q))$ admits a sequence of pairwise distinct weak solutions which uniformly converges to zero in X for each $\lambda > 0$.*

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