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Landau-Bloch Theorems for Bounded Biharmonic Mappings

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Abstract. We determine coefficient bounds for bounded planar biharmonic mappings and generalize the Landau-Bloch univalency theorems for such bounded biharmonic functions. The univalence radii presented here improve many related results published to date, including the most recent one [Complex Var. Elliptic Equ. 58(12) (2013), 1667-1676] and are sharp in some given cases.

1. Introduction

A function $f(z) = u(z) + iv(z)$ defined on a domain $\Omega \in \mathbb{C}$ is a harmonic mapping if and only if f is twice continuously differentiable and $\Delta f = 4f_{z\bar{z}} = 0$. In a remarkable paper, Clunie and Sheil-Small [7] explored the class of harmonic functions and showed that if Ω is simply connected, then f can be written as $f = h + \overline{g}$, where h and g are holomorphic in Ω. Harmonic mappings can be regarded as generalizations of holomorphic functions while biharmonic mappings are generalizations of harmonic mappings. A four times continuously differentiable complex-valued function $F(z) = U(z) + iV(z)$ is said to be biharmonic in a domain $\Omega \in \mathbb{C}$ if and only if ΔF is harmonic in Ω, that is, if and only if F satisfies the biharmonic equation $\Delta^2 F = \Delta(\Delta F) = 0$ in Ω.

For a continuously differentiable function f in Ω we define

$$\Lambda_f(z) = \max_{0 \le \theta \le 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0 \le \theta \le 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||.$$

Lewy [15] showed that a harmonic function f is locally univalent in Ω if its Jacobian $J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |h'|^2 - |g'|^2$ does not vanish anywhere in Ω. We note that local univalence of f does not imply global univalence in a given domain Ω and also note that $|J_f| = \Lambda_f \lambda_f$.

It is known (e.g. see [2], [3]) that a mapping F is biharmonic in a simply connected domain Ω if and only if F has the representation

$$F = |z|^2 G + K, \qquad z \in \Omega \tag{1}$$

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where G and K are harmonic in Ω.

Without loss of generality, for functions $F = |z|^2 G + K$ biharmonic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ we may express G and K by

$$G(z) = g_1(z) + \overline{g_2(z)} = \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$$
$$K(z) = k_1(z) + \overline{k_2(z)} = \sum_{n=1}^{\infty} c_n z^n + \overline{\sum_{n=1}^{\infty} d_n z^n} \tag{2}$$

where g_1, g_2, k_1, and k_2 are analytic in $\mathbb{D}$.

The classical Landau Theorem for bounded holomorphic functions states that if f is a holomorphic function in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{D}$, then f is univalent (schlicht) in the disk $|z| < \rho_0 = (M + \sqrt{M^2 - 1})^{-1}$ and $f(|z| < \rho_0)$ contains the disk $|z| < M\rho_0^2$. It is known (e.g. see [14] or [4]) that these bounds are sharp. Moreover, for f as defined above with $f(0)$ not necessarily zero, there is the Bloch Theorem which asserts the existence of a positive constant b such that $f(\mathbb{D})$ contains a schlicht disk, that is, a disk of radius b which is the univalent image of some region in $\mathbb{D}$. The Bloch constant is defined as the supremum of all such b (e.g. see [8], [11] or [9]).

In the sequel, for $\zeta \in \mathbb{C}$ we let $\mathbb{D}_\rho(\zeta) := \{z \in \mathbb{C} : |z - \zeta| < \rho\}$, $\mathbb{D}_\rho = \mathbb{D}_\rho(0)$ and for $\rho = 1$ we simply use $\mathbb{D}_1 = \mathbb{D}$. The following theorem is proved by Zhu and Liu ([17], Theorem 3.2).

Theorem 1.1. *Suppose that $F(z) = |z|^2 G(z) + H(z)$ is a biharmonic mapping of the unit disk $\mathbb{D}$ such that $|G(z)| \le M_1$ and $|H(z)| \le M_2$ for $z \in \mathbb{D}$ with $\lambda_F(0) = 1$.*
(i) If $M_2 > 1$ or $M_2 = 1$ and $M_1 > 0$, then F is univalent in the disk $\mathbb{D}_{r_3}$, and $F(\mathbb{D}_{\sigma_2})$ contains a schlicht disk $\mathbb{D}_{\sigma_2}(F(0))$, where $r_3 = r_3(M_1, M_2)$ is the minimum positive root of the following equation

$$1 - 2M_1 r - \frac{4M_1 r^2}{\pi(1 - r^2)} - \sqrt{2(M_2^2 - 1)}.\frac{r\sqrt{4 - 3r^2 + r^4}}{(1 - r^2)^{3/2}} = 0 \tag{3}$$

and

$$\sigma_2 = r_3 - M_1 r_3^2 - \sqrt{2(M_2^2 - 1)}.\frac{r_3^2}{(1 - r_3^2)^{1/2}}. \tag{4}$$

(ii) If $M_2 = 1$ and $M_1 = 0$, then F is univalent in $\mathbb{D}$ and $F(\mathbb{D}) = \mathbb{D}$.

In this paper we give better results than those given in Theorem 1.1 (also see Remark 2.1 and Table 1). Moreover, we extend these results to Landau-Bloch theorems for the mappings $L(F)$ where the differential operator L is defined by

$$L = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}.$$

We observe that (e.g. see [1]) the operator L preserves both harmonicity and biharmonicity and is a complex linear operator that satisfies the usual product rule $L(af + bg) = aL(f) + bL(g)$ and $L(fg) = fL(g) + gL(f)$ where a and b are complex constants.

2. Main Results

First we state the following two lemmas, the first of which is a modification of a result due to Zhu and Liu [17] (also see Liu [16]).

Lemma 2.1. *Let* $f(z) = \overline{g(z)} + h(z) = \overline{\sum_{n=1}^{\infty} a_n z^n} + \sum_{n=1}^{\infty} b_n z^n$ *be a harmonic mapping in the unit disk* $\mathbb{D}$.
(i) If $|f(z)| < M$, *then*

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|)^2 \leq 2M^2.$$

(ii) If $\lambda_f(0) = 1$ *and* $|f(z)| < M$, *then*

$$\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 \leq 2M^2 - 2.$$

(iii) If $|J_f(0)| = 1$ *and* $|f(z)| < M$, *then*

$$\sqrt{\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2} \leq T_1(M) := \min\{\sqrt{2M^2 - 2}, \sqrt{M^4 - 1}.\lambda_f(0)\},$$

where

$$\lambda_f(0) \geq \lambda_f(M) = \frac{\sqrt{2}}{\sqrt{M^2 - 1} + \sqrt{M^2 + 1}}.$$

Lemma 2.2. *Let* $f(z) = \overline{g(z)} + h(z) = \overline{\sum_{n=1}^{\infty} a_n z^n} + \sum_{n=1}^{\infty} b_n z^n$ *be a harmonic mapping in the unit disk* $\mathbb{D}$ *with* $|g(z)| + |h(z)| < M$ *in* $\mathbb{D}$, $a_n \neq 0$ *and* $b_n \neq 0$; $n \geq 1$.
(i) If $\Lambda_f(0) = 1$ *and*

$$\arg \frac{a_n b_1}{b_n a_1} = 2k\pi n, \qquad k \in \{0, 1, 2, 3, ...\}, \tag{5}$$

then

$$\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 \leq M^2 - 1. \tag{6}$$

(ii) If $\lambda_f(0) = 1$ *and* (5) *then* (6).
(iii) If $J_f(0) = 1$ *and (5) then*

$$\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 \leq \frac{\lambda_f^2(0) M^2 - 1}{\lambda_f^2(0)}.$$

Proof. We shall provide a brief proof for part (*i*). The proofs for the other two parts are similar and we skip them. Set $F(z) = \sum_{n=1}^{\infty} (a_n + e^{i\phi} b_n) z^n$ where $\phi = \arg \frac{a_1}{b_1}$. Then by the hypothesis we have $|F(z)| < M$. So Parseval's identity yields

$$\sum_{n=1}^{\infty} |a_n + e^{i\phi} b_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |F(z)|^2 d\theta \leq M^2.$$

Hence in view of $\Lambda_f(0) = |a_1| + |b_1|$ and letting $r \to 1^-$, we obtain

$$\sum_{n=2}^{\infty} (|a_n| + |b_n|)^2 \leq (M^2 - 1).$$

□

Our first theorem provides a sufficient condition for univalency of bounded biharmonic functions.

Theorem 2.1. *Let $F = r^2G + K$ given by (1) be biharmonic in $\mathbb{D}$ so that G and K are given by (2). If*

$$\sum_{n=1}^{\infty}(n+2)(|a_n|+|b_n|)r^{n+1} + \sum_{n=2}^{\infty} n(|c_n|+|d_n|)r^{n-1} \leq \lambda_F(0) \tag{7}$$

then F is univalent in $\mathbb{D}$.

Proof. Suppose z_1 and z_2 are in $\mathbb{D}_r$ so that $z_1 \neq z_2$ and $0 < r < 1$. Since $\mathbb{D}_r$ is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2 \in \mathbb{D}_r$, where $0 \leq t \leq 1$. Then (using a method first used in [13], Proof of Therem 1) we can write

$$\begin{aligned} F(z_2) - F(z_1) &= \int_0^1 (|z(t)|^2 G(z(t)) + K(z(t)))' dt \\ &= \int_0^1 \Big\{ [z'(t)\overline{z(t)} + z(t)\overline{z'(t)}][g_1(z(t)) + \overline{g_2(z(t))}] \\ &+ |z(t)|^2 [z'(t) g_1'(z(t)) + \overline{z'(t) g_2'(z(t))}] + [z'(t) k_1'(z(t)) + \overline{z'(t) k_2'(z(t))}] \Big\} dt. \end{aligned}$$

Dividing the above equation by $z_2 - z_1 \neq 0$ and letting $\omega = z(t)$, we obtain

$$\begin{aligned} \left| \frac{F(z_2) - F(z_1)}{z_2 - z_1} \right| &= \left| c_1 - \frac{\overline{z_2 - z_1}}{z_2 - z_1} \overline{d_1} + \int_0^1 (A(\omega) + B(\omega)) dt \right| \\ &\geq ||c_1| - |d_1|| - \int_0^1 |A(\omega) + B(\omega)| dt \\ &\geq \lambda_F(0) - \int_0^1 (|A(\omega)| + |B(\omega)|) dt, \end{aligned}$$

where

$$A(\omega) = \overline{\omega} \sum_{n=1}^{\infty} a_n \omega^n + \overline{\sum_{n=1}^{\infty} b_n \omega^{n+1}} + |\omega|^2 \sum_{n=1}^{\infty} n a_n \omega^{n-1} + \sum_{n=2}^{\infty} n c_n \omega^{n-1},$$

and

$$B(\omega) = \frac{\overline{z_2 - z_1}}{z_2 - z_1} \left(\sum_{n=1}^{\infty} a_n \omega^{n+1} + \omega \overline{\sum_{n=1}^{\infty} b_n \omega^n} + |\omega|^2 \overline{\sum_{n=1}^{\infty} n b_n \omega^{n-1}} + \overline{\sum_{n=2}^{\infty} n d_n \omega^{n-1}} \right).$$

Now for $|\omega| < r$, we have

$$|A(\omega)| + |B(\omega)| < \sum_{n=1}^{\infty} (n+2)(|a_n| + |b_n|) r^{n+1} + \sum_{n=2}^{\infty} n(|c_n| + |d_n|) r^{n-1}.$$

This in conjunction with (7) yield

$$\left| \frac{F(z_2) - F(z_1)}{z_2 - z_1} \right| > 0.$$

Therefore F is biharmonic univalent in the unit disc $\mathbb{D}$. □

Letting $r \mapsto 1$ in Theorem 2.1 yields a generalization of the well-known sufficient univalency condition for harmonic functions given in [13].

Corollary 2.1. *Let $F = r^2G + K$ given by (1) be biharmonic in $\mathbb{D}$ so that G and K are given by (2). If*

$$\sum_{n=1}^{\infty}(n+2)(|a_n| + |b_n|) + \sum_{n=2}^{\infty} n(|c_n| + |d_n|) \leq \lambda_F(0)$$

then F is univalent in $\mathbb{D}$.

For Theorem 2.1, we give the following example.

Example 2.1. *For $z \in \mathbb{D}$ consider the biharmonic function*

$$F(z) = a|z|^2(z + c\overline{z}) + b(dz + \overline{z}).$$

In view of Theorem 2.1, it is easy to see that if $||bd| - 1| \geq 3|a|(1 + |c|)$ then F is univalent in the unit disk $\mathbb{D}$ and if $||bd| - 1| < 3|a|(1 + |c|)$ then F is univalent in the disk $\mathbb{D}_\rho$ with $\rho = \sqrt{|b|||d| - 1|/3|a|(1 + |c|)}$.
The result is sharp in the second case when $\arg c = \pi + \arg b/a$, $\arg d = -\arg b/a$ and $|d| < 1$. In fact, in the second case, F is not univalent in the disk $\mathbb{D}_r$ for $r \in (\rho, 1]$ with $\rho > 1/3$. For a brief justification, set $r \in (\rho, 1]$, $\arg \frac{b}{a} = \theta_0$, $\varepsilon = \frac{r-\rho}{2} > 0$, $r_1 = \rho + \varepsilon$ and $r_2 = \rho - \delta$ with

$$\delta = \frac{3\rho + \varepsilon - \sqrt{3(\rho - \varepsilon)(3\rho + \varepsilon)}}{2} \in (0, 2\varepsilon).$$

Now for $z_1 = r_1e^{i(\pi+\theta_0)/2}$ and $z_2 = r_2e^{i(\pi+\theta_0)/2}$ in $\mathbb{D}_r$ we obtain

$$\begin{aligned}
F(z_1) &= a|z_1|^2(z_1 + c\overline{z_1}) + b(dz_1 + \overline{z_1}) \\
&= a\Big(r_1^2(r_1e^{i(\pi+\theta_0)/2} + |c|e^{i(\pi+\theta_0)}r_1e^{-i(\pi+\theta_0)/2}) \\
&+ |\frac{b}{a}|e^{i\theta_0}(|d|e^{-i\theta_0}r_1e^{i(\pi+\theta_0)/2} + r_1e^{-i(\pi+\theta_0)/2})\Big) \\
&= ae^{i(\pi+\theta_0)/2)}\left(r_1^3(1 + |c|) - |\frac{b}{a}|r_1(1 - |d|)\right) \\
&= ae^{i(\pi+\theta_0)/2)}\left(r_2^3(1 + |c|) - |\frac{b}{a}|r_2(1 - |d|)\right) = F(z_2).
\end{aligned}$$

Hence F is not univalent in the disc $\mathbb{D}_r$.

The Landau-Bloch Theorem for the bounded biharmonic functions $F = r^2G + K$ is given in the following theorem.

Theorem 2.2. *Let $F = r^2G + K$ given by (1) be biharmonic in $\mathbb{D}$ so that G and K are given by (2). Also let $|G| < M_1$, $|K| < M_2$ and $\lambda_F(0) = 1$.*
(i) If either $M_2 > 1$ or $M_2 = 1$ and $M_1 > 0$, then there exists a constant ρ ($0 < \rho < 1$) so that F is univalent in $\mathbb{D}_\rho$ where ρ is the smallest positive root of the equation

$$\sqrt{\frac{2M_1^2(9\rho^4 - 11\rho^6 + 4\rho^8)}{(1-\rho^2)^3}} + \sqrt{\frac{2(M_2^2 - 1)(4\rho^2 - 3\rho^4 + \rho^6)}{(1-\rho^2)^3}} - 1 = 0.$$

Moreover, $F(\mathbb{D}_\rho)$ covers the schlicht disk $\mathbb{D}_{r_1}$ where

$$r_1 = \rho - \frac{\sqrt{2}M_1\rho^3}{\sqrt{1-\rho^2}} - \frac{\sqrt{2(M_2^2 - 1)}\,\rho^2}{\sqrt{1-\rho^2}}.$$

(ii) If $M_1 = 0$ and $M_2 = 1$, then F is univalent in $\mathbb{D}$ and $F(\mathbb{D})$ contains $\mathbb{D}$.

Proof. According to Lemma 2.1, we have

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)^2 \leq 2M_1^2 \quad \textit{and} \quad \sum_{n=2}^{\infty}(|c_n|+|d_n|)^2 \leq 2M_2^2-2.$$

(i) For $z \in \mathbb{D}$ set $F_\rho(z) = F(\rho z)$. So in view of Theorem 2.1, it suffices to show that the inequality (7) holds for $|z| < \rho$. This is the case since

$$\begin{aligned}
&\sum_{n=1}^{\infty}(n+2)(|a_n|+|b_n|)\rho^{n+1} + \sum_{n=2}^{\infty} n(|c_n|+|d_n|)\rho^{n-1}\\
&\leq (\sum_{n=1}^{\infty}(|a_n|+|b_n|)^2)^{1/2}(\sum_{n=1}^{\infty}(n+2)^2\rho^{2n+2})^{1/2}\\
&+ (\sum_{n=2}^{\infty}(|c_n|+|d_n|)^2)^{1/2}(\sum_{n=2}^{\infty} n^2\rho^{2n-2})^{1/2}\\
&\leq M_1\rho^2\sqrt{\frac{18-22\rho^2+8\rho^4}{(1-\rho^2)^3}} + \sqrt{2M_2^2-2}\sqrt{\frac{4\rho^2-3\rho^4+\rho^6}{(1-\rho^2)^3}} = 1.
\end{aligned}$$

To show that $F(\mathbb{D}_\rho)$ contains the disk $\mathbb{D}_{r_1}$, let $z \in \partial\mathbb{D}_\rho$. Then

$$\begin{aligned}
|F(z)-F(0)| &= ||z|^2G(z)+K(z)| = |\rho^2\sum_{n=1}^{\infty}(a_nz^n+\overline{b_nz^n}) + \sum_{n=1}^{\infty}(c_nz^n+\overline{d_nz^n})|\\
&\geq |c_1z+\overline{d_1}\overline{z}| - \rho^2(\sum_{n=1}^{\infty}(|a_n|+|b_n|)^2)^{\frac{1}{2}}(\sum_{n=1}^{\infty}\rho^{2n})^{\frac{1}{2}} - (\sum_{n=2}^{\infty}(|c_n|+|d_n|)^2)^{\frac{1}{2}}(\sum_{n=2}^{\infty}\rho^{2n})^{\frac{1}{2}}\\
&\geq \rho\lambda_F(0) - \frac{\sqrt{2}M_1\rho^3}{\sqrt{1-\rho^2}} - \frac{\rho^2\sqrt{2M_2^2-2}}{\sqrt{1-\rho^2}}\\
&= \rho - \frac{\sqrt{2}M_1\rho^3}{\sqrt{1-\rho^2}} - \frac{\rho^2\sqrt{2M_2^2-2}}{\sqrt{1-\rho^2}} = r_1.
\end{aligned}$$

(ii) If $M_1 = 0$ and $M_2 = 1$ then by Lemma 2.1, $a_n = b_n = 0; n \geq 1$ and $c_n = d_n = 0; n \geq 2$ and so $F(z) = c_1z + \overline{d_1z}$. Now for $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$ we have

$$\begin{aligned}
&|F(z_1)-F(z_2)| = |c_1(z_1-z_2) - \overline{d_1(z_1-z_2)}|\\
&\geq ||c_1|-|d_1|||z_1-z_2| = \lambda_F(0)|z_1-z_2| = |z_1-z_2|.
\end{aligned}$$

Hence F is univalent in the disc $\mathbb{D}$.
The covering result is also immediate since for any $z \in \partial\mathbb{D}$ we have

$$|F(z)-F(0)| = |c_1z - \overline{d_1z}| \geq \lambda_F(0)|z| = 1.$$

□

Remark 2.1. *It is claimed in [17] that Theorem 1.1 for certain values of M_1 and M_2 improves the results given in ([1], [5], [6], [9], [10], [12], [16]). Our Theorem 2.2 is an improvement to all those results published prior to [17] including that given by Zhu and Liu ([17], Theorem 3.2). The following table of values demonstrates examples of cases that Theorem 2.2 provides better results than those given by Zhu and Liu [17].*

Table 1: Values of ρ are for Theorem 2.2 and the values of r are for Theorem 1.1.

M_1	M_2	ρ	r	M_1	M_2	ρ	r
1	2	0.1889	0.1391	3/4	2	0.1904	0.1498
1	3	0.1211	0.0979	3/4	3	0.1215	0.1031
2	1	0.7071	0.2181	1/3	2	0.1931	0.1721
2	2	0.1832	0.1081	1/3	3	0.1222	0.1132
2	3	0.1194	0.0815	1/2	2	0.1920	0.1624
3	1	0.5773	0.1516	1/2	3	0.1219	0.1089
3	2	0.1780	0.0886	0.56	2	0.1916	0.1592
3	3	0.1179	0.0698	0.56	3	0.1218	0.1074

Next we extend Theorem 2.2 to the case where the coefficients of the biharmonic function $F = r^2G + K$ satisfy certain varying argument conditions.

Theorem 2.3. *Let $F = r^2G + K$ given by (1) be biharmonic in $\mathbb{D}$ so that G and K are given by (2) with $\Lambda_G(0) = \lambda_F(0) = 1$, $|g_1| + |g_2| \le M_1$ and $|k_1| + |k_2| \le M_2$. Also for $b_na_1 \neq 0$ and $d_nc_1 \neq 0$ let the coefficients a_n, b_n, c_n and d_n satisfy the following varying argument conditions*

$$\arg\frac{a_nb_1}{b_na_1} = 2k\pi n, \quad \arg\frac{c_nd_1}{d_nc_1} = 2m\pi n \qquad k, m \in \{0, 1, 2, 3, ...\}.$$

(*i*) *If $M_1 > 1$ and $M_2 > 1$, then there exists a constant ρ $(0 < \rho < 1)$ so that F is univalent in $\mathbb{D}_\rho$ where ρ is the smallest positive root of the equation*

$$3\rho^2 + \rho^3\sqrt{\frac{(M_1^2-1)(16-23\rho^2+9\rho^4)}{(1-\rho^2)^3}} + \sqrt{\frac{(M_2^2-1)(4\rho^2-3\rho^4+\rho^6)}{(1-\rho^2)^3}} - 1 = 0.$$

Moreover, $F(\mathbb{D}_\rho)$ covers the schlicht disk $\mathbb{D}_{\rho_1}$, where

$$\rho_1 = \rho - \frac{\rho^2\sqrt{M_1^2-1}}{\sqrt{1-\rho^2}} - \rho^3 - \frac{\rho^4\sqrt{M_2^2-1}}{\sqrt{1-\rho^2}}.$$

(*ii*) *If $M_1 = 1$ and $M_2 = 1$, then F is univalent in the disk $\mathbb{D}_{\sqrt{1/3}}$.*
Moreover, $F(\mathbb{D}_{\sqrt{1/3}})$ covers the disc $\mathbb{D}_{\frac{2\sqrt{3}}{9}}$ and the result is sharp.

Proof. It follows from Lemma 2.2 that

$$\sum_{n=2}^{\infty}(|a_n| + |b_n|)^2 \le M_1^2 - 1 \quad \text{and} \quad \sum_{n=2}^{\infty}(|c_n| + |d_n|)^2 \le M_2^2 - 1.$$

(i) Letting $F_\rho(z) = F(\rho z)$ and using Theorem 2.1, the proof for this part is similar to that of Theorem 2.2.(i). In brief, for the inequality (7), we get

$$\sum_{n=1}^{\infty}(n+2)(|a_n| + |b_n|)\rho^{n+1} + \sum_{n=2}^{\infty} n(|c_n| + |d_n|)\rho^{n-1}$$
$$\le 3\rho^2 + \rho^3\sqrt{M_1^2-1}\sqrt{\frac{16-23\rho^2+9\rho^4}{(1-\rho^2)^3}} + \sqrt{M_2^2-1}\sqrt{\frac{4\rho^2-3\rho^4+\rho^6}{(1-\rho^2)^3}} = 1.$$

Similarly, to show that $F(\mathbb{D}_\rho)$ contains the disk $\mathbb{D}_{r_1}$ we observe that

$$|F(z)-F(0)| = ||z|^2G(z)+K(z)| = |\rho^2\sum_{n=1}^{\infty}(a_nz^n+\overline{b_nz^n})+\sum_{n=1}^{\infty}(c_nz^n+\overline{d_nz^n})|$$
$$\geq \rho-\rho^3-\frac{\rho^4\sqrt{M_1^2-1}}{\sqrt{1-\rho^2}}-\frac{\rho^2\sqrt{M_2^2-1}}{\sqrt{1-\rho^2}}=r_1.$$

(ii) For the second part, since $M_1 = 1$ and $M_2 = 1$, we get $a_n = b_n = 0$ and $c_n = d_n = 0$ for $n \geq 2$. This yields $F(z) = |z|^2(a_1z+\overline{b_1z})+c_1z+\overline{d_1z}$. By comparing this F with that in Example 2.1 we find that $a = a_1, \overline{ac} = b_1, bd = c_1$ and $\overline{b} = d_1$. Once again, Example 2.1 for $1 = \Lambda_G(0) = |a_1|+|b_1| = |a|(1+|c|)$ and $1 = \lambda_F(0) = ||c_1|-|d_1|| = |c|||d|-1|$ yields the sharp bound $\rho = \frac{\sqrt{3}}{3}$.
On the other hand, if $z \in \partial\mathbb{D}_{\frac{\sqrt{3}}{3}}$ then

$$|F(z)-F(0)|\geq|z|||c_1|-|d_1||-|z|^3(|a_1|+|b_1|) = |z|-|z|^3 = \frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{9}=\frac{2\sqrt{3}}{9}.$$

Finally, for $\arg d = -\arg(b/a) := \theta_0$, $\arg c = \pi+\arg(b/a)$ and $z = \frac{\sqrt{3}}{3}e^{i(\pi+\theta_0)/2}$ we obtain $|F(z)-F(0)| = \frac{2\sqrt{3}}{9}$. □

The next two theorems are extensions of Theorem 2.2 and Theorem 2.3 to Landau-Bloch theorems for the mapping $L(F)$.

Theorem 2.4. *Let $F = r^2G+K$ given by (1) be biharmonic in $\mathbb{D}$ so that G and K are given by (2). Also let $|G| < M_1$, $|K| < M_2$ and $\lambda_F(0) = \lambda_{L(F)}(0) = 1$.*
(i) If either $M_2 > 1$ or $M_2 = 1$ and $M_1 > 0$ then there is a constant ρ $(0 < \rho < 1)$ so that $L(F)$ is univalent in $\mathbb{D}_\rho$ where ρ is the smallest positive root of the equation

$$M_1\rho^2\sqrt{\frac{18+38\rho^2-10\rho^4+2\rho^6}{(1-\rho^2)^5}}+\rho\sqrt{\frac{2(M_2^2-1)(16+\rho^2+11\rho^4-5\rho^6+\rho^8)}{(1-\rho^2)^5}}-1=0.$$

Moreover, $L(F(\mathbb{D}_\rho))$ contains a schlicht disk $\mathbb{D}_{r_1}$ where

$$r_1 = \rho-M_1\rho^3\sqrt{\frac{2+2\rho^2}{(1-\rho^2)^3}}-\rho\sqrt{2M_2^2-2}\sqrt{\frac{4-3\rho^2+\rho^4}{(1-\rho^2)^3}}.$$

(ii) If $M_1 = 0$ and $M_2 = 1$, then $L(F)$ is univalent in $\mathbb{D}$ and $L(F(\mathbb{D}))$ contains $\mathbb{D}$.

Proof. Of course, by Lemma 2.1 the inequality (7) holds for the coefficients of G and K. The rest of the proof will be similar to that given for Theorem 2.2, only if we note that for part (i) we have

$$\sum_{n=1}^{\infty}n(n+2)(|a_n|+|b_n|)\rho^{n+1}+\sum_{n=2}^{\infty}n^2(|c_n|+|d_n|)\rho^{n-1}$$
$$\leq M_1\rho^2\sqrt{\frac{18+38\rho^2-10\rho^4+2\rho^6}{(1-\rho^2)^5}}+\sqrt{2M_2^2-2}\rho\sqrt{\frac{16+\rho^2+11\rho^4-5\rho^6+\rho^8}{(1-\rho^2)^5}}=1$$

and that $L(F(\mathbb{D}_\rho))$ contains the disk $\mathbb{D}_{r_1}$ since

$$|L(F(z))-L(F(0))| = ||z|^2L(G(z))+L(K(z))|$$
$$\geq\rho-M_1\rho^3\sqrt{\frac{2+2\rho^2}{(1-\rho^2)^3}}-\rho\sqrt{2M_2^2-2}\sqrt{\frac{4-3\rho^2+\rho^4}{(1-\rho^2)^3}}=r_1.$$

For part (ii) we get $L(F(z)) = c_1 z - \overline{d_1 z}$ which is univalent in the disc $\mathbb{D}$ and

$$|L(F(z)) - L(F(0))| = |c_1 z - \overline{d_1 z}| \geq \lambda_F(0)|z| = 1.$$

□

The extension to $L(F)$ for $F = r^2 G + K$ with varying arguments is given next.

Theorem 2.5. *Let $F = r^2 G + K$ given by (1) be biharmonic in $\mathbb{D}$ so that $\Lambda_G(0) = \lambda_F(0) = 1, |g_1| + |g_2| \leq M_1$ and $|k_1| + |k_2| \leq M_2$ for $z \in \mathbb{D}$. Also for $b_n a_1 \neq 0$ and $d_n c_1 \neq 0$ let the coefficients a_n, b_b, c_n and d_n satisfy*

$$\arg \frac{a_n b_1}{b_n a_1} = 2k\pi n, \quad \arg \frac{c_n d_1}{d_n c_1} = 2m\pi n \qquad k, m \in \{0, 1, 2, 3, ...\}.$$

(i) If $M_1 > 1$ and $M_2 > 1$ then there is a constant ρ $(0 < \rho < 1)$ such that $L(F)$ is univalent in $\mathbb{D}_\rho$, where ρ is the minimum positive root of the equation

$$3\rho^2 + \rho^3 \sqrt{\frac{(M_1^2 - 1)(64 - 95\rho^2 + 91\rho^4 - 45\rho^6 + 9\rho^8)}{(1 - \rho^2)^5}}$$
$$+ \rho \sqrt{\frac{(M_2^2 - 1)(16 + \rho^2 + 11\rho^4 - 5\rho^6 + \rho^8)}{(1 - \rho^2)^5}} - 1 = 0.$$

Moreover, $L(F(\mathbb{D}_\rho))$ covers a schlicht disc $\mathbb{D}_{r_1}$, where

$$r_1 = \rho - \rho^3 - (\rho^2 \sqrt{M_1^2 - 1} + \rho^2 \sqrt{M_2^2 - 1}) \sqrt{\frac{4 - 3\rho^2 + \rho^6}{(1 - \rho^2)^3}}.$$

(ii) If $M_1 = 1$ and $M_2 = 1$, then $L(F)$ is univalent in $\mathbb{D}_{\sqrt{1/3}}$.
Moreover,$L(F(\mathbb{D}_{\sqrt{1/3}}))$ contains the disk $\mathbb{D}_{\frac{2\sqrt{3}}{9}}$ and the result is sharp.

Proof. The proof will be similar to that given for Theorems 2.2 and 2.4 only if we note that $L(F(z)) = |z|^2(a_1 z - \overline{b_1 z}) + c_1 z - \overline{d_1 z}$ for part (ii). □

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