Filomat 33:14 (2019), 4585–4591
https://doi.org/10.2298/FIL1914585G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Transcendental Continued β-Fraction with Quadratic Pisot Basis over $\mathbb{F}_q((x^{-1}))$

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Abstract. Let $\mathbb{F}_q$ be a finite field and $\mathbb{F}_q((x^{-1}))$ is the field of formal power series with coefficients in $\mathbb{F}_q$. Let $\beta \in \mathbb{F}_q((x^{-1}))$ be a quadratic Pisot series with $deg(\beta) = 2$. We establish a transcendence criterion depending on the continued β-fraction of one element of $\mathbb{F}_q((x^{-1}))$.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of characteristic $q \geq 0$, $\mathbb{F}_q[x]$ is the ring of polynomials with coefficients in $\mathbb{F}_q$ and $\mathbb{F}_q((x^{-1}))$ is the field of formal power series of the form:

$$f = \sum_{k \geq l} f_k x^{-k}, \quad f_k \in \mathbb{F}_q, l \in \mathbb{Z},$$

where $l = deg(f)$ and by convention $deg(0) = -\infty$. We define the absolute value $|f| = q^{deg(f)}$ if $f \neq 0$, $|f| = 0$ otherwise. This absolute value is not archimedean over $\mathbb{F}_q((x^{-1}))$.
Let $\beta \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$. The continued β-fraction is a generalization of classic continued fraction with formal power series basis. Similar to the real case, the β-expansion of a formal power series f is a unique representation $f = \sum\limits_{i \geq n} d_i \beta^{-i}$ where $n \in \mathbb{Z}$ and $(d_i)_{i \geq n}$ is a polynomial sequence such that $deg(d_i) < deg(\beta)$ for all $i \geq n$ (see [4]). Thus, f has a unique decomposition as follows$f = \sum\limits_{i=0}^{N} d_i \beta^i + \sum\limits_{i \in \mathbb{N}^*} d_{-i} \beta^{-i}$. Note that $\sum\limits_{i=0}^{N} d_i \beta^i = [f]_\beta$ is called β-polynomial part of f and $\sum\limits_{i \in \mathbb{N}^*} d_{-i} \beta^{-i} = \{f\}_\beta$ is called β-fractional part of f. If $\{f\}_\beta = 0$, then f is β-polynomial. The β-polynomial's set is denoted $\mathbb{F}_q[x]_\beta$.
Let $f \in \mathbb{F}_q((x^{-1}))$. Consider the transformation T_β defined over $\mathbb{F}_q((x^{-1}))$ by $T_\beta(f) = \frac{1}{f} - [\frac{1}{f}]_\beta$. The continued

2010 *Mathematics Subject Classification*. Primary 11R06; Secondary 37B50
Keywords. Pisot series, β-polynomial, finite field, continued β-fraction
Received: 03 April 2019; Accepted: 14 June 2019
Communicated by Dragan S. Djordjević
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β-fraction of f has the following form: Set $f = f_0$,

$$\begin{aligned} f &= [f_0]_\beta + T_\beta(\frac{1}{f_0}) \\ &= [f_0]_\beta + \frac{1}{f_1} \\ &= [f_0]_\beta + \frac{1}{[f_1]_\beta + T_\beta(\frac{1}{f_1})} \\ &= [f_0]_\beta + \cfrac{1}{[f_1]_\beta + \cfrac{1}{\ddots \cfrac{1}{[f_i]_\beta + T_\beta(\frac{1}{f_i})}}} \end{aligned}$$

where for all $i \geq 0$ the equality $\frac{1}{f_{i+1}} = T_\beta(\frac{1}{f_i})$ is satisfied if $f_i \notin \mathbb{F}_q[x]_\beta$. Otherwise, the algorithm ends, and the sequence $(f_i)_i$ is finite.

In this paper, we give a transcendence criterion of continued β-fraction with quadratic Pisot series basis β over $\mathbb{F}_q((x^{-1}))$. This result has been developed on several researchers works [1, 7, 8], which contributes to prove Khintchine conjecture [5].

In [1] Baker proved that if $[a_0, a_1, a_2, ...]$ is classic continued fraction of real number x such that $a_n = a_{n+1} = ... = a_{n+\alpha(n)-1}$, where $\alpha(n)$ is a sequence of integers satisfying certain increasing properties, then x is transcendental. In 2004, Mkaouar [7] built an other transcendence criterion of the classic continued fraction over $\mathbb{F}_q((x^{-1}))$.

This paper is organized as follows. In section 2, basic arithmetics properties of continued β-fraction with Pisot series basis are introduced in $\mathbb{F}_q((x^{-1}))$. In section 3, the main results are proved.

2. Continued β-fraction with Pisot series basis over $\mathbb{F}_q((x^{-1}))$

Let $\beta \in \mathbb{F}_q((x^{-1}))$ with $|\beta| > 1$ and $f \in \mathbb{F}_q((x^{-1}))$, we have $f = [f]_\beta + \{f\}_\beta$.
If $\lambda_0 = [f]_\beta$. Then, we get

$$f = \lambda_0 + \cfrac{1}{\lambda_1 + \cfrac{1}{\lambda_2 + \cfrac{1}{\ddots}}},$$

where $\lambda_i = \left[\frac{1}{T_\beta^i(\frac{1}{f})}\right]_\beta \in \mathbb{F}_q[x]_\beta$ for all $i \geq 1$ and the map $T_\beta : f \to \{\frac{1}{f}\}_\beta$. The previous continued β-fraction of f is denoted by $[\lambda_0, \lambda_1, \lambda_2, ...]_\beta$. The sequence $(\lambda_i)_{i\geq 0}$ is called the partial β-quotients of f and the expansion $[\lambda_n, \lambda_{n+1}, ...]_\beta$ is called the n^{th}-complete β-quotients of f, denoted by f_n.

Similarly, let $f = [\lambda_0, \lambda_1, ..., \lambda_n, ...]_\beta$, we define two sequences $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0} \in \mathbb{F}_q[x, \beta^{-1}]$ as follows:

$$\begin{cases} P_0 = \lambda_0, & P_1 = \lambda_0\lambda_1 + 1 \\ Q_0 = 1, & Q_1 = \lambda_1 \end{cases} \text{ and } \begin{cases} P_n = \lambda_n P_{n-1} + P_{n-2} \\ Q_n = \lambda_n Q_{n-1} + Q_{n-2} \end{cases} \forall n \geq 2.$$

$\frac{P_n}{Q_n} = [\lambda_0, \lambda_1, ..., \lambda_n]_\beta$ is called the n^{th} β-convergent of f.

Recall that $\mathbb{F}_q[x]_\beta$ is not stable under usual multiplication. Let

$$L_\otimes = \{n \in \mathbb{N} / \forall P_1, P_2 \in \mathbb{F}_q[x]_\beta ; d_\beta(P_1P_2) \text{ is finite } \Rightarrow \beta^n P_1 P_2 \in \mathbb{F}_q[x]_\beta\}.$$

In [3], the value of $L_\otimes$ is already calculated for some Pisot series over $\mathbb{F}_q((x^{-1}))$.

Corollary 2.1. *[3] Let β be a quadratic Pisot series with $deg(\beta) \geq 2$. Then, $L_\otimes = 1$.*

3. Results

Let β be a quadratic Pisot series with $deg(\beta) = 2$. For $P = a_s\beta^s + ... + a_0 \in \mathbb{F}_q[x]_\beta$. Define $\gamma(P)$ as the β-degree of P as follows:

$$deg(\beta)deg_\beta(P) + deg(a_s) = 2s + deg(a_s).$$

In this case, we have $|P| = q^{\gamma(P)}$.

Theorem 3.1. *Let β be a quadratic Pisot series with $deg(\beta) = 2$ and f is a formal power series such that its continued β-fraction is $[\lambda_1, \lambda_2, ..., \lambda_n, ...]_\beta$. If*

$$\limsup_{n\to+\infty} \frac{\log(\sum_{i=1}^{n} \gamma(\lambda_i))}{n} = +\infty,$$

then f is transcendental.

First, we use the following proposition and lemma.

Proposition 3.2. *Let β be a quadratic Pisot series with $deg(\beta) = 2$. If f is a formal power series such that its n^{th} β-convergent is $\frac{P_n}{Q_n}$, then*

$$(\beta^{\frac{n}{2}} P_n) \in \mathbb{F}_q[x]_\beta, \quad (\beta^{\frac{n}{2}} Q_n) \in \mathbb{F}_q[x]_\beta$$

and

$$\beta^{\frac{2n+1}{2}} (P_{n+1}Q_n - P_nQ_{n+1}) \in \mathbb{F}_q[x]_\beta.$$

Proof. According to Corollary 2.1, if $A_1, ..., A_n \in \mathbb{F}_q[x]_\beta$, then $\beta^{\frac{n}{2}}(A_1A_2...A_n) \in \mathbb{F}_q[x]_\beta$. □

Lemma 3.3. *Let β be a quadratic Pisot series with $deg(\beta) = 2$ and f is an algebraic series of algebraic degree d. Then, there exists $c = c(f) > 0$ such that, for all $P, Q \in \mathbb{F}_q[x]_\beta$,*

$$|\frac{P}{Q} - f| > \frac{c}{|\beta|^{\frac{d+1}{2}} |Q|^d}.$$

Proof. Similar to Liouville's inequality [6]. Let $K(y) = A_dy^d + ... + A_0 \in \mathbb{F}_q[x][y]$ be an irreducible polynomial of f such that $K(f) = 0$. According to Proposition 3.2, for all $P, Q \in \mathbb{F}_q[x]_\beta$ we get

$$\beta^{\frac{d+1}{2}} Q^d K(\frac{P}{Q}) \in \mathbb{F}_q[x]_\beta.$$

So

$$|K(\frac{P}{Q})| \geq \frac{1}{|\beta|^{\frac{d+1}{2}} |Q|^d}.$$

As $K(f) = 0$, then

$$|K(\frac{P}{Q})| = |K(\frac{P}{Q}) - K(f)| \leq |\frac{P}{Q} - f| \max_{1\leq i\leq d} |A_if^{i-1}|.$$

Let $c_1 = \max_{1\leq i\leq d} |A_if^{i-1}|$ and $c = \min(1, \frac{1}{2c_1})$. We obtain

$$|\frac{P}{Q} - f| \geq \frac{1}{c_1|\beta|^{\frac{d+1}{2}} |Q|^d} > \frac{c}{|\beta|^{\frac{d+1}{2}} |Q|^d}.$$

□

Proof. of Theorem 3.1
Let $(\frac{P_n}{Q_n})_n$ be a β-convergent sequence of f. Similar to the classical case,

$$|f - \frac{P_n}{Q_n}| = |\frac{P_{n+1}Q_n - P_nQ_{n+1}}{Q_{n+1}Q_n}|.$$

By Proposition 3.2, we obtain

$$|\beta^{\frac{2n+1}{2}}(f - \frac{P_n}{Q_n})| \leq \frac{|\beta|^{\frac{2n+1}{2}}}{|Q_{n+1}||Q_n|}.$$

If f is an algebraic series of algebraic degree $d > 1$. Then, according to Lemma 3.3, there exists $c > 0$ such that

$$\frac{c}{|\beta|^{\frac{nd+1}{2}}|Q_n|^d} < |f - \frac{P_n}{Q_n}| < \frac{|\beta|^{\frac{2n+1}{2}}}{|Q_n||Q_{n+1}|}.$$

So,

$$|Q_{n+1}| < \frac{1}{c}|\beta|^{\frac{n(d+2)+2}{2}}|Q_n|^{d-1}.$$

Let $c_1 = \frac{1}{c}$. Then,

$$|Q_{n+1}| < c_1^{(d-1)^n}|\beta|^{\frac{(n(d+2)+2)(d-1)^n}{2}}|Q_1|^{(d-1)^n}.$$

This implies,

$$|Q_n| < c_1^{(d-1)^n}|\beta|^{\frac{(n(d+2)+2)(d-1)^n}{2}}|Q_1|^{(d-1)^n}.$$

Moreover,

$$\sum_{i=1}^{n}\gamma(\lambda_i) < (d-1)^n[\log(c_1) + \log(|\beta|^{\frac{n(d+2)+2}{2}}) + \log(|Q_1|)].$$

Hence,

$$\log(\sum_{i=1}^{n}\gamma(\lambda_i)) < n\log(d-1) + \log[\log(c_1) + (\frac{n(d+2)+2}{2})\log(|\beta|) + \log(|Q_1|)],$$

which is a contradiction with the fact that

$$\limsup_{n\to+\infty}\frac{\log(\sum_{i=1}^{n}\gamma(\lambda_i))}{n} = +\infty.$$

□

Theorem 3.4. *Let β be a quadratic Pisot series with deg$(\beta) = 2$ and f is a formal power series such that its continued β-fraction $[B_1, B_2, B_3, ...]_\beta$ is not ultimately periodic, where B_i are finite blocks beginning with the repetition n_i-times of same partial β-quotient λ and $\gamma(\lambda) > 1$. We denote by d_i the sum of β-degrees of B_i's terms. If*

$$\liminf_{i\to+\infty}\frac{\sum_{j=1}^{i-1}d_j}{n_i} = 0,$$

then f is transcendental.

The proof of Theorem 3.4 requires the following lemmas indeed.
Let f be an algebraic formal power series of minimal polynomial $P(Y) = A_mY^m + ... + A_0$ where $A_i \in \mathbb{F}_q[x]$. Set $H(f) = \max_{0\leq i\leq m}|A_i|$ and $\sigma(f) = A_m$.

Lemma 3.5. *Let β be a quadratic Pisot series with $deg(\beta) = 2$. If f is an algebraic series of algebraic degree d such that*

$$f = \lambda_1 + \cfrac{1}{\lambda_2 + \cfrac{1}{\ddots + \cfrac{1}{\lambda_t + \cfrac{1}{h}}}},$$

where $\lambda_i \in \mathbb{F}_q[x]_\beta$, $h \in \mathbb{F}_q((x^{-1}))$ and $|h| > 1$. Then h is an algebraic series of algebraic degree d and

$$H(h) \leq H(f)|\beta|^{\frac{td}{2}}|\prod_{i=1}^{t}\lambda_i|^d.$$

Proof. Assume that f is an algebraic series of algebraic degree d. Then,

$$A_d f^d + A_{d-1}f^{d-1} + ... + A_0 = 0,$$

where $A_i \in \mathbb{F}_q[x]$.
If $f = \lambda_1 + \frac{1}{h}$ such that $\lambda_1 \in \mathbb{F}_q[x]_\beta$, $h \in \mathbb{F}_q((x^{-1}))$ and $|h| > 1$. Thus, according to Proposition 3.2, we obtain

$$\beta^{\frac{d}{2}} h^d (A_d(\lambda_1 + \frac{1}{h})^d + A_{d-1}(\lambda_1 + \frac{1}{h})^{d-1} + ... + A_0) = 0.$$

This implies,

$$B_d h^d + B_{d-1}h^{d-1} + ... + B_0 = 0 \qquad (*)$$

where

$$B_{d-k} = \beta^{\frac{d}{2}} \sum_{j=k}^{d} \binom{j}{k} A_j \lambda_1^{j-k} \in \mathbb{F}_q[x]_\beta.$$

Let $P = a_s\beta^s + ... + a_1\beta + a_0 \in \mathbb{F}_q[x]_\beta$, we denote by $TC_\beta(P) = a_0$. Then, by $(*)$ we have

$$TC_\beta(B_d)h^d + TC_\beta(B_{d-1})h^{d-1} + ... + TC_\beta(B_0) = 0.$$

Thus, we get

$$\begin{aligned}
|TC_\beta(B_d)| &= |TC_\beta(\beta^{\frac{d}{2}} \sum_{j=0}^{d-1} \binom{j}{0} A_j \lambda_1^j + \binom{d}{0} A_d \beta^{\frac{d}{2}} \lambda_1^d)| \\
&= |TC_\beta(\binom{d}{0} A_d \beta^{\frac{d}{2}} \lambda_1^d))| \\
&\leq |A_d||\beta^{\frac{d}{2}} \lambda_1^d| \\
&\leq H(f)|\beta|^{\frac{d}{2}}|\lambda_1|^d
\end{aligned}$$

and for $k \geq 1$,

$$|TC_\beta(B_{d-k})| \leq |TC_\beta(B_d)| \leq H(f)|\beta|^{\frac{d}{2}}|\lambda_1|^d.$$

Hence,

$$H(h) \leq \sup_{0\leq k\leq d} |TC_\beta(B_k)| \leq H(f)|\beta|^{\frac{d}{2}}|\lambda_1|^d.$$

Consequently, if $f = [\lambda_1, \lambda_2, ..., \lambda_t, h]_\beta$ where $\lambda_i \in \mathbb{F}_q[x]_\beta$, $h \in \mathbb{F}_q((x^{-1}))$ and $|h| > 1$, and set $f_i = [\lambda_i, \lambda_{i+1}, ..., \lambda_t, h]_\beta$. Then, we get by iterating the last case $H(h) \leq H(f)|\beta|^{\frac{td}{2}}|\prod_{i=1}^{t}\lambda_i|^d$. □

Lemma 3.6. *Let f be an algebraic series of algebraic degree d and g is a formal power series such that its continued β-fraction is purely periodic with its period is $Per(g) = 1$. We denote by $\sigma(g) = Q_1$ such that $\frac{P_1}{Q_1}$ is the first β-convergent of g. If $f \neq g$, then*

$$|f - g| \geq \frac{1}{|g|^{d-2}|\beta|^2|\sigma(g)|^{(2d-1)}H(f)^2}.$$

Proof. Combine Lemma (2) in [7] and Proposition 3.2, we get $|f - g| \geq \frac{1}{|g|^{d-2}|\beta|^2|\sigma(g)|^{(2d-1)}H(f)^2}$. □

Proof. of Theorem3.4
Assume that f is an algebraic series of algebraic degree d such that its continued β-fraction $[B_1, B_2, B_3, ...]_\beta$ is not ultimately periodic where, for all $i \geq 1$, B_i is finite block which begins with n_i-times of $\lambda \in \mathbb{F}_q[x]_\beta$ and $\gamma(\lambda) > 1$. We denote by $g = [\lambda, \lambda, \lambda, ...]_\beta$ and we set $f_i = [B_i, B_{i+1}, B_{i+2}, ...]_\beta$ such that its n_i^{th} β-convergent is $\frac{P_{n_i}}{Q_{n_i}}$. As f_i and g have same first n_i-terms of their continued β-fraction, then according to Proposition 3.2, we have

$$|\beta|^{\frac{2n_i+1}{2}}|f_i - g| \leq \sup\{|\beta|^{\frac{2n_i+1}{2}}|f_i - \frac{P_{n_i}}{Q_{n_i}}|, |\beta|^{\frac{2n_i+1}{2}}|g - \frac{P_{n_i}}{Q_{n_i}}|\}$$
$$\leq \frac{|\beta|^{\frac{2n_i+1}{2}}}{|Q_{n_i}|^2}.$$

By Lemma 3.6, we obtain

$$\frac{1}{|g|^{d-2}|\beta|^2|\sigma(g)|^{(2d-1)}H(f_i)^2} \leq |\beta|^{\frac{2n_i+1}{2}}|f_i - g| \leq \frac{|\beta|^{\frac{2n_i+1}{2}}}{|Q_{n_i}|^2}.$$

Thus,

$$|Q_{n_i}|^2 \leq |\beta|^{\frac{2n_i+5}{2}}|g|^{d-2}|\sigma(g)|^{(2d-1)}H(f_i)^2.$$

As $deg(Q_{n_i}) = n_i\gamma(\lambda) - deg(g)$, then

$$2n_i\gamma(\lambda) \leq (\frac{2n_i + 5}{2})deg(\beta) + 2deg(H(f_i)) + d\ deg(g) + (2d - 1)\ deg(\sigma(g)).$$

Let $\alpha = \sum_{j=1}^{i-1} \alpha_j$, with α_j is the number of B_j's terms. Then, by Lemma 3.5, we get

$$2n_i\gamma(\lambda) \leq 2deg(H(f)) + (\alpha d + \frac{2n_i + 5}{2})deg(\beta) + 2d\sum_{j=1}^{i-1} d_j + d\ deg(g) + (2d - 1)\ deg(\sigma(g)).$$

Hence

$$n_i(\gamma(\lambda) - 1) \leq deg(H(f)) + (\alpha d + \frac{5}{2}) + d\sum_{j=1}^{i-1} d_j + d\ deg(g) + d\ deg(\sigma(g)).$$

Therefore, we get

$$\liminf_{i\to+\infty} \frac{\sum_{j=1}^{i-1} d_j}{n_i} > 0.$$

□

Example 3.7. *Let β be a quadratic Pisot series with $deg(\beta) = 2$. Consider $f = [B_1, B_2, B_3, ...]_\beta$ where*

$$B_i = [\underbrace{h\beta, \ldots, h\beta}_{i^i-terms}, (h+1)\beta]_\beta,$$

with $h \in \mathbb{F}_q[X]$ and $deg(h) = 1$. We get:

$$\lim_{i\to+\infty} \frac{\sum_{j=1}^{i-1} j^j + 1}{i^i} = 0.$$

Then, by Theorem 3.4, f is transcendental.

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