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A Note on the Common Spectral Properties for Bounded Linear Operators

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Abstract. Let X and Y be Banach spaces, $A : X \to Y$ and B, $C : Y \to X$ be bounded linear operators. We prove that if $A(BA)^2 = ABACA = ACABA = (AC)^2A$, then

$$\sigma_*(AC) \setminus \{0\} = \sigma_*(BA) \setminus \{0\}$$

where σ_* runs over a large of spectra originated by regularities.

1. Introduction

Throughout this paper $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators acting from a complex Banach space X into another one, Y, and $\mathcal{L}(X)$ is a short for $\mathcal{L}(X, X)$. Given two operators $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$, Jacobson's Lemma asserts that

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\} \tag{1}$$

where $\sigma(\cdot)$ denotes the ordinary spectrum.

Several works have been devoted to equality (1) by showing that $AB - I$ and $BA - I$ share many spectral properties. See [2, 3, 5, 6, 13, 15, 16, 18, 19] and the references therein. Barnes in [2] extended (1) to other part of the spectrum and showed that $AB - I$ and $BA - I$ share some spectral properties. In [3], Benhida and Zerouali investigated equation (1) for various Taylor joint spectra. For A and B satisfying $ABA = A^2$ and $BAB = B^2$, Schmoeger [15, 16] and Duggal [7] showed that A, B, AB and BA share spectral properties. Corach *et al.* [6] investigated common properties for $ac - 1$ and $ba - 1$ where a, b and c are elements in associative ring such that $aba = aca$. For bounded linear operators A, B and C, Zeng and Zhong [19] studied spectral properties for AC and BA under the condition $ABA = ACA$. If $C = I$ in the last condition, one can retrieve Schmoeger's result. For operators A, B, C and D satisfying $ACD = DBD$ and $BDA = ACA$, Yan and Fang [17] investigated spectral properties for AC and BD. Recently, [5] studied common properties for ac and ba for elements in a ring satisfying $a(ba)^2 = abaca = acaba = (ac)^2a$.

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The paper is a continuation of [5] and [20]. The aim of this paper is to extend recent results to bounded linear operators $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ satisfying

$$A(BA)^2 = ABACA = ACABA = (AC)^2A.$$

In section two we give basic definitions and notation which we need in the sequel. Section 3 is devoted to the main results of the paper. In Theorem 3.1 we prove that if $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ satisfy $A(BA)^2 = ABACA = ACABA = (AC)^2A$, then

$$\sigma_*(AC) \setminus \{0\} = \sigma_*(BA) \setminus \{0\}$$

where σ_* runs over a large of spectra originated by regularities.

2. Basic definitions and notations

For an operator $T \in \mathcal{L}(X)$, let $\mathcal{N}(T)$ and $\mathcal{R}(T)$ stand for the *kernel*, respectively the *range* of T. An operator $T \in \mathcal{L}(X)$ is said to be an *upper semi-Fredholm* operator if $\mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T) < \infty$, and T is said to be a *lower semi-Fredholm* operator if codim $\mathcal{N}(T) < \infty$. One says that T is a *Fredholm* operator if $\dim \mathcal{N}(T) < \infty$ and codim $\mathcal{N}(T) < \infty$. If T is either upper or lower semi-Fredholm then T is said *semi-Fredholm* operator. In this case the *index* of T is defined by $\mathrm{ind}(T) = \dim \mathcal{N}(T) - \dim \mathcal{R}(T)$.

The *ascent* of T, $asc(T)$, is the smallest nonnegative integer n for which $N(T^n) = N(T^{n+1})$, i.e.; $asc(T) = \inf\{n \in \mathbb{Z}_+ : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$. If no such integer exists, we shall say that T has infinite ascent. In a similar way, the *descent* of T, $dsc(T)$, is defined by $dsc(T) = \inf\{n \in \mathbb{Z}_+ : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$ and if no such integer exists, we shall say that T has infinite descent. We say that T is *left Drazin invertible* if $asc(T) < \infty$ and $\mathcal{R}(T^{asc(T)+1})$ is closed and T is *right Drazin invertible* if $dsc(T) < \infty$ and $\mathcal{R}(T^{dsc(T)})$ is closed. If T is both left and right Drazin invertible, then T is said to be *Drazin invertible* ; which is equivalent to $asc(T) = dsc(T) < \infty$ (see [1]). One says that T is *upper semi-Browder* if T is upper semi-Fredholm with finite ascent, and T is *lower semi-Browder* if T is lower semi-Fredholm with finite descent. If T is both upper and lower semi-Browder then T is said to be *Browder* operator (see [14]).

For each $n \in \mathbb{Z}_+$, let $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c'_n(T) = \dim N(T^{n+1})/N(T^n)$. It was proved in [8, Lemma 3.2] that for every n, we have

$$c_n(T) = \dim X/(R(T) + N(T^n)) \text{ and } c'_n(T) = \dim N(T) \cap R(T^n).$$

It is easy to see that $\{c_n(T)\}$ and $\{c'_n(T)\}$ are decreasing sequences and $dsc(T) = \inf\{n \in \mathbb{Z}_+ : c_n(T) = 0\}$, $asc(T) = \inf\{n \in \mathbb{Z}_+ : c'_n(T) = 0\}$.

Following [12], the *essential descent* $dsc_e(T)$ of T is defined by $dsc_e(T) = \inf\{n \in \mathbb{Z}_+ : c_n(T) < \infty\}$, and the *essential ascent* $asc_e(T)$ of T is defined by $asc_e(T) = \inf\{n \in \mathbb{Z}_+ : c'_n(T) < \infty\}$, where the infimum over the empty set is taken to be infinite.

Let $\mathcal{N}^\infty(T)$ and $\mathcal{R}^\infty(T)$ denote the *hyper-kernel* and the *hyper-range* of T defined by

$$\mathcal{N}^\infty(T) = \bigcup_{n=1}^{\infty} \mathcal{N}(T^n) \text{ and } \mathcal{R}^\infty(T) = \bigcap_{n=1}^{\infty} \mathcal{R}(T^n).$$

One says that T is *semi-regular* if $\mathcal{R}(T)$ is closed and $\mathcal{N}^\infty(T) \subseteq \mathcal{R}(T)$.

For each $n \in \mathbb{Z}_+$, $T \in \mathcal{L}(X)$ induces a linear maps Γ_n from the space $\mathcal{R}(T^n)/\mathcal{R}(T^{n+1})$ into $\mathcal{R}(T^{n+1})/\mathcal{R}(T^{n+2})$. The dimension of the null space of Γ_n will be denoted by $k_n(T)$, i.e., $k_n(T) = \dim \mathcal{N}(\Gamma_n)$. It follows from [8, Theorem 3.7] that for every n,

$$\begin{aligned} k_n(T) &= \dim((\mathcal{R}(T^n) \cap \mathcal{N}(T))/(\mathcal{R}(T^{n+1}) \cap \mathcal{N}(T))) \\ &= \dim(\mathcal{R}(T) + \mathcal{R}(T^{n+1}))/(\mathcal{R}(T) + \mathcal{N}(T^n)). \end{aligned}$$

Let

$$k(T) = \sum_{n=0}^{\infty} k_n(T).$$

Then it follows from [8, Theorem 3.7] that $k(T) = \dim \mathcal{N}(T)/(\mathcal{N}(T) \cap \mathcal{R}^\infty(T)) = \dim(\mathcal{R}(T) + \mathcal{N}^\infty(T))/\mathcal{R}(T)$. The *stable nullity* $c(T)$ and the *stable defect* $c'(T)$ of T are defined by

$$c(T) = \sum_{n=0}^{\infty} c_n(T) \text{ and } c'(T) = \sum_{n=0}^{\infty} c'_n(T).$$

Then we have $c(T) = \dim X/\mathcal{R}^\infty(T)$ and $c'(T) = \dim \mathcal{R}^\infty(T)$.

According to [11], the *degree of stable iteration* of $T \in \mathcal{L}(X)$ is defined by

$$dis(T) = \inf\{n \in \mathbb{Z}_+ : k_m(T) = 0 \text{ for all } m \geq n\},$$

and the *degree of essential stable iteration* of T ([18]) is defined is

$$dis_e(T) = \inf\{n \in \mathbb{Z}_+ : k_m(T) < \infty \text{ for all } m \geq n\}.$$

Definition 2.1. *Let R be a non-empty subset of $\mathcal{L}(X)$. R is called a regularity if it satisfies the following two conditions:*

i) if $n \in \mathbb{N}$, then $A \in R$ if and only if $A^n \in R$;

ii) if A, B, C and D are mutually commuting operators in $\mathcal{L}(X)$ such that $AC+BD = I$, then $AB \in R$ if and only if $A \in R$ and $B \in R$.

A regularity $R \subset \mathcal{L}(X)$ assigns to each $T \in \mathcal{L}(X)$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin R\}$$

and called the *spectrum of T corresponding to the regularity R*. We note that every regularity R contains all invertible operators, so that $\sigma_R(T) \subseteq \sigma(T)$. In general, $\sigma_R(T)$ is neither compact nor non-empty (see [10, 12, 14]).

The regularities R_i, where $1 \leq i \leq 15$, were introduced and studied in [10, 12, 14] but are in a different form. Regularity R_{18} was introduced by [4], while R_{16}, R_{17} and R_{19} were introduced by [18].

Definition 2.2.

$R_1 = \{T \in \mathcal{L}(X) : c(T) = 0\}$,
$R_2 = \{T \in \mathcal{L}(X) : c(T) < \infty\}$,
$R_3 = \{T \in \mathcal{L}(X) :$ *there exists* $d \in \mathbb{Z}_+$ *such that* $c_d(T) = 0$ *and* $\mathcal{R}(T^{d+1})$ *is closed*$\}$,
$R_4 = \{T \in \mathcal{L}(X) : c_n(T) < \infty, \forall n \in \mathbb{Z}_+\}$,
$R_5 = \{T \in \mathcal{L}(X) :$ *there exists* $d \in \mathbb{Z}_+$ *such that* $c_d(T) < \infty$ *and* $\mathcal{R}(T^{d+1})$ *is closed*$\}$,
$R_6 = \{T \in \mathcal{L}(X) : c'(T) = 0$ *and* $\mathcal{R}(T)$ *is closed*$\}$,
$R_7 = \{T \in \mathcal{L}(X) : c'(T) < \infty$ *and* $\mathcal{R}(T)$ *is closed*$\}$,
$R_8 = \{T \in \mathcal{L}(X) :$ *there exists* $d \in \mathbb{Z}_+$ *such that* $c'_d(T) = 0$ *and* $\mathcal{R}(T^{d+1})$ *is closed*$\}$,
$R_9 = \{T \in \mathcal{L}(X) : c'_n(T) < \infty$ *for every* $n \in \mathbb{Z}_+$ *and* $\mathcal{R}(T)$ *is closed*$\}$,
$R_{10} = \{T \in \mathcal{L}(X) :$ *there exists* $d \in \mathbb{Z}_+$ *such that* $c'_d(T) < \infty$ *and* $\mathcal{R}(T^{d+1})$ *is closed*$\}$,
$R_{11} = \{T \in \mathcal{L}(X) : k(T) = 0$ *and* $\mathcal{R}(T)$ *is closed*$\}$,
$R_{12} = \{T \in \mathcal{L}(X) : k(T) < \infty$ *and* $\mathcal{R}(T)$ *is closed*$\}$,
$R_{13} = \{T \in \mathcal{L}(X) :$ *there exists* $d \in \mathbb{Z}_+$ *such that* $k_n(T) = 0$ *for every* $n \geq d$ *and* $\mathcal{R}(T^{d+1})$ *is closed*$\}$,
$R_{14} = \{T \in \mathcal{L}(X) : k_n(T) < \infty$ *for every* $n \in \mathbb{Z}_+$ *and* $\mathcal{R}(T)$ *is closed*$\}$,
$R_{15} = \{T \in \mathcal{L}(X) :$ *there exists* $d \in \mathbb{Z}_+$ *such that* $k_n(T) < \infty$ *for every* $n \geq d$ *and* $\mathcal{R}(T^{d+1})$ *is closed*$\}$,
$R_{16} = \{T \in \mathcal{L}(X) :$ *there exists* $d \in \mathbb{Z}_+$ *such that* $c_d(T) = 0$ *and* $\mathcal{R}(T) + N(T^d)$ *is closed*$\}$,
$R_{17} = \{T \in \mathcal{L}(X) :$ *there exists* $d \in \mathbb{Z}_+$ *such that* $c_d(T) < \infty$ *and* $\mathcal{R}(T) + N(T^d)$ *is closed*$\}$,
$R_{18} = \{T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+$ *such that* $k_n(T) = 0$ *for every* $n \geq d$ *and* $\mathcal{R}(T) + N(T^d)$ *is closed*$\}$,
$R_{19} = \{T \in \mathcal{L}(X) : \exists d \in \mathbb{Z}_+$ *such that* $k_n(T) < \infty$ *for every* $n \geq d$ *and* $\mathcal{R}(T) + N(T^d)$ *is closed*$\}$.

We have

$$R_1 \subseteq R_2 = R_3 \cap R_4 \subseteq R_3 \cup R_4 \subseteq R_5 \subseteq R_{13},$$

$$R_6 \subseteq R_7 = R_8 \cap R_9 \subseteq R_8 \cup R_9 \subseteq R_{10} \subseteq R_{13},$$

$$R_{11} \subseteq R_{12} = R_{13} \cap R_{14} \subseteq R_{13} \cup R_{14} \subseteq R_{15}.$$

It was proved in [18, Proposition 2.7] that

$$\begin{array}{lcl} R_3 &=& \{T \in \mathcal{L}(X) : dsc(T) < \infty \text{ and } \mathcal{R}(T^{dsc(T)+1}) \text{ is closed}\}, \\ R_5 &=& \{T \in \mathcal{L}(X) : dsc_e(T) < \infty \text{ and } \mathcal{R}(T^{dsc_e(T)+1}) \text{ is closed}\}, \\ R_8 &=& \{T \in \mathcal{L}(X) : asc(T) < \infty \text{ and } \mathcal{R}(T^{asc(T)+1}) \text{ is closed}\}, \\ R_{10} &=& \{T \in \mathcal{L}(X) : asc_e(T) < \infty \text{ and } \mathcal{R}(T^{asc_e(T)+1}) \text{ is closed}\}, \\ R_{13} &=& \{T \in \mathcal{L}(X) : dis(T) < \infty \text{ and } \mathcal{R}(T^{dis(T)+1}) \text{ is closed}\}, \\ R_{15} &=& \{T \in \mathcal{L}(X) : dis_e(T) < \infty \text{ and } \mathcal{R}(T^{dis_e(T)+1}) \text{ is closed}\}. \end{array}$$

The operators of R_1, R_2, R_3, R_4 and R_5 are surjective, lower semi-Browder, right Drazin invertible, lower semi-Fredholm and right essentially Drazin invertible operators, respectively. The operators of R_6, R_7, R_8, R_9 and R_{10} are bounded below, upper semi-Browder, left Drazin invertible, upper semi-Fredholm and left essentially Drazin invertible operators, respectively. The operators of R_{11}, R_{12} and R_{13} are semi-regular, essentially semi-regular and quasi-Fredholm operators. The operators of R_{18} are the operators with eventual topological uniform descent.

3. Main results

The following is our main result.

Theorem 3.1. *Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$. Then*

$$\sigma_{R_i}(AC) \setminus \{0\} = \sigma_{R_i}(BA) \setminus \{0\} \text{ for } 1 \leq i \leq 19.$$

The proof of our main result uses several auxiliary lemmas.

Lemma 3.2. *Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$. Let Q be a polynomial. Then we have*
1) $ABA\mathcal{R}(Q(CA - I)) \subseteq \mathcal{R}(Q(AB - I))$;
2) $ABA\mathcal{N}(Q(CA - I) \subseteq \mathcal{N}(Q(AB - I))$;
3) $ACA\mathcal{R}(Q(BA - I)) \subseteq \mathcal{R}(Q(AC - I))$;
4) $ACA\mathcal{N}(Q(BA - I)) \subseteq \mathcal{N}(Q(AC - I))$.

Proof. It is easy to see that for each $k \in \mathbb{Z}_+$,

$$ABA(CA - I)^k = (AB - I)^k ABA \text{ and } ACA(BA - I)^k = (AC - I)^k ACA. \tag{2}$$

Then

$$ABAQ(CA - I) = Q(AB - I)ABA \text{ and } ACAQ(BA - I) = Q(AC - I)ACA. \tag{3}$$

1) Let x belongs to $\mathcal{R}(Q(CA - I))$. Then there exists some $y \in X$ such that $Q(CA - I)y = x$. Hence it follows from (2) that $ABAx = ABAQ(CA - I)x = Q(AB - I)ABAx$ which belongs to $\mathcal{R}(Q(AB - I))$. Thus $ABA\mathcal{R}(Q(CA - I)) \subseteq \mathcal{R}(Q(AB - I))$.

2) Let $x \in \mathcal{N}(Q(CA-I))$. Then $Q(CA-I)x = 0$. It follows from (2) that $Q(AB-I)ABAx = ABAQ(CA-I)x = 0$. Thus $ABAx \in \mathcal{N}(Q(AB - I))$.

Using (3), 3) and 4) go similarly. □

Lemma 3.3. *Let $A \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$. Then*

$$c_n(AC - I) = c_n(BA - I) \text{ for all } n \in \mathbb{Z}_+.$$

In particular, $c(AC - I) = c(BA - I)$.

Proof. Let

$$\Gamma_{ACA} : \mathcal{R}((BA - I)^n)/\mathcal{R}((BA - I)^{n+1}) \to \mathcal{R}((AC - I)^n)/\mathcal{R}((AC - I)^{n+1})$$

be the linear application defined by

$$\Gamma_{ACA}(x + \mathcal{R}((BA - I)^{n+1})) = ACAx + \mathcal{R}((AC - I)^{n+1}).$$

Since $ACA\mathcal{R}((BA - I)^n) \subseteq \mathcal{R}((AC - I)^n)$ by Lemma 3.2, part 3), then Γ_{ACA} is well defined. We shall show that Γ_{ACA} is injective.

Let $x \in \mathcal{R}((BA - I)^n)$ such that $\Gamma_{ACA}(x) = 0$. Then $ACAx \in \mathcal{R}((AC - I)^{n+1})$. Hence $CACAx \in \mathcal{R}((CA - I)^{n+1})$. From Lemma 3.2, part 1), we have $ABACACAx \in \mathcal{R}((AB - I)^{n+1})$. Then

$$(BA)^4x = BABACACAx \in \mathcal{R}((BA - I)^{n+1}).$$

Since $x \in \mathcal{R}((BA - I)^n)$ then $x = (BA - I)^n z$ for some $z \in X$. Hence

$$\begin{aligned} x &= (BA)^4x - ((BA)^4 - I)x \\ &= (BA)^4x - ((BA)^3 + (BA)^2 + (BA) + I)(BA - I)x \\ &= (BA)^4x - ((BA)^3 + (BA)^2 + (BA) + I)(BA - I)^{n+1}z \\ &= (BA)^4x - (BA - I)^{n+1}(((BA)^3 + (BA)^2 + (BA) + I)z) \in \mathcal{R}((BA - I)^{n+1}). \end{aligned}$$

Thus Γ_{ACA} is injective and consequently

$$c_n(BA - I) \le c_n(AC - I). \tag{4}$$

In similar way, we show that

$$c_n(CA - I) \le c_n(AB - I). \tag{5}$$

Finally,

$$\begin{aligned} c_n(BA - I) &\le c_n(AC - I) \\ &= c_n(CA - I) \text{ ([18, Lemma 3.9]} \\ &\le c_n(AB - I) \text{ by (5)} \\ &= c_n(BA - I) \text{ ([18, Lemma 3.9].} \end{aligned}$$

Therefore $c_n(BA - I) = c_n(AC - I)$ for all $n \in \mathbb{Z}_+$. In particular, $c(AC - I) = c(BA - I)$. □

For $T \in \mathcal{L}(X)$, let $\sigma_{dsc}(T)$ and $\sigma^e_{dsc}(T)$ be, respectively, the *descent spectrum* and the *essential descent spectrum* of T defined by

$$\sigma_{dsc}(T) = \{\lambda \in \mathbb{C} : dsc(T) = \infty\} \text{ and } \sigma^e_{dsc}(T) = \{\lambda \in \mathbb{C} : dsc_e(T) = \infty\}.$$

The following is an immediate consequence of Lemma 3.3.

Corollary 3.4. *Let $A \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$. Then*

$$\sigma_*AC \setminus \{0\} = \sigma_*BA \setminus \{0\}, \text{ for } \sigma_* \in \{\sigma_{dsc}, \sigma^e_{dsc}\}.$$

Lemma 3.5. *Let $A \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$. Then*

$$c'_n(AC - I) = c'_n(BA - I) \text{ for all } n \in \mathbb{Z}_+.$$

In particular, $c'(AC - I) = c'(BA - I)$.

Proof. Let

$$\Psi_{ACA} : \mathcal{N}((BA-I)^{n+1})/\mathcal{N}((BA-I)^n) \to \mathcal{N}((AC-I)^{n+1})/\mathcal{N}((AC-I)^n)$$

be the linear application defined by

$$\Psi_{ACA}(x + \mathcal{N}((BA-I)^n)) = ACAx + \mathcal{N}((AC-I)^n).$$

Since $ACA\mathcal{N}((BA-I)^{n+1}) \subseteq \mathcal{N}((AC-I)^{n+1})$ by Lemma 3.2, part 4), then Ψ_{ACA} is well defined.

Now we show that Ψ_{ACA} is injective. Let $x \in \mathcal{N}((BA-I)^{n+1})$ such that $\Psi_{ACA}(x) = 0$, which means that $ACAx \in \mathcal{N}((AC-I)^n)$. Hence $CACAx \in \mathcal{N}((CA-I)^n)$. It follows from Lemma 3.2, part ii), that $ABACACAx \in \mathcal{N}((AB-I)^n)$. Then

$$(BA)^4x = BABACACAx \in \mathcal{N}((BA-I)^n).$$

Hence

$$\begin{aligned} x &= (BA)^4x - ((BA)^4 - I)x \\ &= (BA)^4x - [(BA)^3 + (BA)^2 + (BA) + I](BA-I)x \in \mathcal{N}((BA-I)^n). \end{aligned}$$

Which implies that Ψ_{ACA} is injective and then

$$c'_n(BA-I) \le c'_n(AC-I). \tag{6}$$

Similarly, we prove that

$$c'_n(CA-I) \le c'_n(AB-I). \tag{7}$$

Finally,

$$\begin{aligned} c'_n(BA-I) &\le c'_n(AC-I) \\ &= c'_n(CA-I) \text{ ([18, Lemma 3.10]} \\ &\le c'_n(AB-I) \text{ by (7)} \\ &= c'_n(BA-I) \text{ ([18, Lemma 3.10];} \end{aligned}$$

Therefore $c'_n(BA-I) = c'_n(AC-I)$ for all $n \in \mathbb{Z}_+$. In particular, $c'(AC-I) = c'(BA-I)$. □

For $T \in \mathcal{L}(X)$ let $\sigma_{asc}(T)$ and $\sigma^e_{asc}(T)$ be respectively the *ascent spectrum* and the *essential ascent spectrum* of T defined by

$$\sigma_{asc}(T) = \{\lambda \in \mathbb{C} : asc(T) = \infty\} \text{ and } \sigma^e_{asc}(T) = \{\lambda \in \mathbb{C} : asc_e(T) = \infty\}.$$

Then the following is an immediate consequence of Lemma 3.5

Corollary 3.6. *Let $A \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$. Then*

$$\sigma_*AC \setminus \{0\} = \sigma_*BA \setminus \{0\}, \text{ for } \sigma_* \in \{\sigma_{asc}, \sigma^e_{asc}\}.$$

Lemma 3.7. *Let $A \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$. Then*

$$k_n(AC-I) = k_n(BA-I) \text{ for all } n \in \mathbb{Z}_+.$$

In particular, $k(AC-I) = k(BA-I)$.

Proof. Let Φ_{ACA} be the linear application from $\mathcal{R}(BA-I) + \mathcal{N}((BA-I)^{n+1})/\mathcal{R}(BA-I) + \mathcal{N}((BA-I)^n)$ to $\mathcal{R}(AC-I) + \mathcal{N}((AC-I)^{n+1})/\mathcal{R}(AC-I) + \mathcal{N}((AC-I)^n)$ defined by

$$\Phi_{ACA}(x + \mathcal{R}(BA-I) + \mathcal{N}((BA-I)^n)) = ACAx + \mathcal{R}(BA-I) + \mathcal{N}((AC-I)^n).$$

Since, by Lemme 3.2, parts 3) and 4),

$$ACA(\mathcal{R}(BA-I))+\mathcal{N}((BA-I)^{n+1})\subseteq\mathcal{R}(BA-I))+\mathcal{N}((BA-I)^{n+1}),$$

then Φ_{ACA} is well defined.

We prove that Φ_{ACA} is injective. Let $x\in\mathcal{R}(BA-I)+\mathcal{N}((BA-I)^{n+1})$ such that $\Phi_{ACA}(x)=0$. Then $ACAx\in\mathcal{R}(AC-I)+\mathcal{N}((AC-I)^n)$. So, there exist some $y\in\mathcal{R}(BA-I)$ and $z\in\mathcal{N}((AC-I)^n)$ such that $ACAx=y+z$. Then $CACAx=Cy+Cz\in\mathcal{R}(CA-I)+\mathcal{N}((CA-I)^n)$. Thus by Lemma 3.2, parts 1) and 2), we get that $ABACACAx\in\mathcal{R}(AB-I)+\mathcal{N}((AB-I)^n)$ and consequently $(BA)^4x=BABACACAx\in\mathcal{R}(BA-I)+\mathcal{N}((BA-I)^n)$. Thus

$$\begin{aligned}x&=(BA)^4x-((BA)^4-I)x\\&=(BA)^4x-(BA-I)((BA)^3+(BA)^2+(BA)+I)x\in\mathcal{R}(BA-I)+\mathcal{N}((BA-I)^n).\end{aligned}$$

Hence Φ_{ACA} is injective. Thus

$$k_n(BA-I)\leq k_n(AC-I). \tag{8}$$

In similar way, we show that

$$k_n(CA-I)\leq k_n(AB-I). \tag{9}$$

Therefore,

$$\begin{aligned}k_n(BA-I)&\leq k_n(AC-I)\\&=k_n(CA-I)\text{ ([18, Lemma 3.8]}\\&\leq k_n(AB-I)\text{ by (9)}\\&=k_n(BA-I)\text{ ([18, Lemma 3.8].}\end{aligned}$$

□

Lemma 3.8. *Let $A\in\mathcal{L}(X,Y)$ and $B,C\in\mathcal{L}(Y,X)$ such that $A(BA)^2=ABACA=ACABA=(AC)^2A$. Then for all $n\in\mathbb{Z}_+$, $\mathcal{R}((AC-I)+\mathcal{N}((AC-I)^n)$ is closed if and only if $\mathcal{R}(BA-I)+\mathcal{N}((BA-I)^n)$ is closed.*

In particular $\mathcal{R}(AC-I)$ is closed if and only if $\mathcal{R}(BA-I)$ is closed.

Proof. Assume that $\mathcal{R}(AC-I)+\mathcal{N}((AC-I)^n)$ is closed. Let $\{x_p\}$ be a sequence in $\mathcal{R}(BA-I)+\mathcal{N}((BA-I)^n)$ which converges to $x\in X$. Then $ACAx_p$ converge to $ACAx$. Since $ACA(\mathcal{R}(BA-I)+\mathcal{N}((BA-I)^n))\subset\mathcal{R}(AC-I)+\mathcal{N}((AC-I)^n)$ by Lemma 3.2, part 3) and 4), then $ACAx_p$ belongs to $\mathcal{R}((AC-I)+\mathcal{N}((AC-I)^n)$. Since $\mathcal{R}(AC-I)+\mathcal{N}((AC-I)^n)$ is closed and $ACAx_p$ converges to $ACAx$.

$$\begin{aligned}&\Longrightarrow ACAx\in\mathcal{R}(AC-I)+\mathcal{N}((AC-I)^n)\\&\Longrightarrow CACAx\in\mathcal{R}(CA-I)+\mathcal{N}((CA-I)^n)\\&\Longrightarrow ABA(CACAx)\in\mathcal{R}(AB-I)+\mathcal{N}((AB-I)^n)\quad\text{(by Lemma 3.2)}\\&\Longrightarrow (BA)^4x=ABA(CACAx)\in\mathcal{R}(AB-I)+\mathcal{N}((AB-I)^n).\end{aligned}$$

Thus

$$\begin{aligned}x&=(BA)^4x-((BA)^4-I)x\\&=(BA)^4x-(BA-I)((BA)^3+(BA)^2+(BA)+I)x\in\mathcal{R}(BA-I)+\mathcal{N}((BA-I)^n).\end{aligned}$$

Therefore $\mathcal{R}(BA-I))+\mathcal{N}((BA-I)^n)$ is closed.

The opposite implication goes similarly. □

Lemma 3.9. *Let $A\in\mathcal{L}(X,Y)$ and $B,C\in\mathcal{L}(Y,X)$ such that $A(BA)^2=ABACA=ACABA=(AC)^2A$. Then for all $n\in\mathbb{N}$, $\mathcal{R}((AC-I)^n)$ is closed if and only if $\mathcal{R}(BA-I)^n)$ is closed.*

Proof. As in the presentation before [2, Proposition], for each $n \in \mathbb{N}$ there exists B_n and $C_n \in \mathcal{L}(Y, X)$ such that

$$(I - AC)^n = I - AC_n \text{ and } (I - BA)^n = I - B_n A.$$

Indeed, we have $B_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} B(AB)^{k-1}$ and $C_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (CA)^{k-1} C$. It is easy to check that

$$A(B_n A)^2 = AB_n AC_n A = AC_n AB_n A = (AC_n)^2 A.$$

Then it follows from Lemma 3.8 that $\mathcal{R}((AC - I)^n)$ is closed if and only if $\mathcal{R}((BA - I)^n)$ is closed. □

Proof of Theorem 3.1 : The proof follows at once from Lemmas 3.2-3.9.

4. Applications and concluding remarks

A bounded operator $T \in \mathcal{L}(X)$ is said to be *upper semi-Weyl* operator if T is upper semi-Fredholm with $ind(T) \leq 0$, and T is said to be *lower semi-Weyl* operator if T is lower semi-Fredholm with $ind(T) \geq 0$. If T is both upper and lower semi-Fredholm then T is said to *Weyl* operator. Then T is weyl operator precisely when T is a Fredholm operator with index zero. The *upper semi-Weyl spectrum* $\sigma_{uw}(T)$, the *lower semi-Weyl spectrum* $\sigma_{lw}(T)$ and the *Weyl spectrum* $\sigma_w(T)$ of T are defined by

$$\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Weyl}\},$$

$$\sigma_{lw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Weyl}\},$$

$$\sigma_w(T) = \sigma_{uw}(T) \cup \sigma_{lw}(T).$$

From Lemma 3.3 and Lemma 3.5 we deduce the following result

Proposition 4.1. *Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2 A$. Then*

$$\sigma_*(AC) \setminus \{0\} = \sigma_*(BA) \setminus \{0\} \text{ for } \sigma_* \in \{\sigma_{uw}, \sigma_{lw}, \sigma_w\}.$$

An operator $T \in \mathcal{L}(X)$ is said to be *Riesz* operator if $T - \lambda I$ is a Fredholm operator for all $0 \neq \lambda \in \mathbb{C}$. Then the following proposition is an immediate consequence of Theorem 3.1

Proposition 4.2. *Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2 A$. Then AC is a Riesz operator if and only if BA is a Riesz operator.*

Following [21], an operator $T \in \mathcal{L}(X)$ is said to be *generalized Drazin-Riesz* operator if there exists $S \in \mathcal{L}(X)$ such that

$$TS = ST, \; STS = S \text{ and } T^2 S - T \text{ is a Riesz operator.}$$

The operator S is called a *generalized Drazin-Riesz inverse* of T.

Theorem 4.3. *Let $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2 A$. Then AC is generalized Drazin-Riesz invertible if and only if BA is generalized Drazin-Riesz invertible. In this case, if S is a generalized Drazin-Riesz inverse of AC then $BS^2 A$ is a generalized Drazin-inverse of BA.*

Proof. Assume that AC is generalized Drazin-Riesz invertible. then there exists $S \in \mathcal{L}(X)$ such that $S(AC) = (AC)S$, $S(AC)S = S$ and $(AC)^2 S - AC$ is Riesz. Set $T = BS^2 A$ and we shall show that

$$T(BA) = (BA)T, \; T(BA)T = T \text{ and } (BA)^2 T - BA \text{ is Riesz operator.}$$

For the first equality, we have

$$\begin{aligned} T(BA) &= BS^2A(BA) \\ &= BS^2(AC)S^2(AC)A(BA) \\ &= BS^4(AC)^2A(CA) \\ &= B(AC)^3S^4A \\ &= B(AB)S^2A \\ &= BAT. \end{aligned}$$

For the second,

$$\begin{aligned} T^2(BA) &= BS^2ABS^2ABA \\ &= BS^2ABS^2(AC)S^2(AC)ABA \\ &= BS^2ABS^2(AC)S^2(AC)ACA \\ &= BS^2AB(AC)(AC)S^4ACA \\ &= BS^2AC(AC)(AC)S^4ACA \\ &= BS^2ACS^2ACA \\ &= BS^2A \\ &= T. \end{aligned}$$

Set $P = ACS - I = SAC - I$. Then

$$\begin{aligned} T(BA)^2 - BA &= BS^2A(BA)^2 - BA \\ &= BS^2(AC)^2A - BA \\ &= BS^ACA - BA \\ &= B(SAC - I)A \\ &= BPA. \end{aligned}$$

Hence it remains to show that BPA is a Riesz operator. We have

$$\begin{aligned} (PA)B(PA)B(PA) &= (SACA - A)B(SACA - A)B(ACSA - A) \\ &= (SACA - A)B(SACABA - ABA)(CSA - A) \\ &= (SACA - A)B(SACACA - ABA)(CSA - A) \\ &= [(SACA - A)B(SACACA) - (SACA - A)BABA](CSA - A) \\ &= [(SACA - A)B(SACACA) - (SACA - A)BACA](CSA - A) \\ &= (SACA - A)B(SACACA - ACA)(CSA - A) \\ &= (SACA - A)B(SACA - A)C(ACSA - A) \\ &= (PA)B(PA)C(PA). \end{aligned}$$

In the same way, one can prove that

$$(PA)B(PA)B(PA) = (PA)B(PA)C(PA) = (PA)C(PA)B(PA) = (PA)C(PA)C(PA).$$

Since $(PA)C = (AC)^2S - AC$ is a Riesz operator by assumption, then it follows from Proposition 4.2 that $B(PA)$ is a Riesz operator. Therefore BA is generalized Drazin-Riesz invertible and BS^2A is a generalized Drazin-inverse of BA.

In similar way, we prove the opposite implication. □

Remark 4.4. *If A and $B \in \mathcal{L}(X)$ such that $ABA = A^2$ and $BAB = B^2$, then*

$$A(BA)^2 = ABAIA = AIABA = (AI)^2A \tag{10}$$

and

$$B(AB)^2 = BABIB = BIBAB = (BI)^2B. \tag{11}$$

Then it follows from (10) and (11 that A, B, BA and AB share above spectral properties. So we retrieve the results of [7].

In the following two examples, the common spectral properties for AC and BA can only followed directly from the above results, but not from the corresponding ones in [7, 9, 15, 16, 19].

Example 4.5. *Let P be a non trivial idempotent on X. Let A, B and C defined on $X \oplus X \oplus X$ by*

$$A = \begin{pmatrix} 0 & I & 0 \\ 0 & P & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}.$$

Then $A(BA)^2 = ABACA = ACABA = (AC)^2A$, while $ABA \neq ACA$ and $BAB \neq B^2$.

Example 4.6. *Let A and B be as in Example 4.5 and let C be defined on $X \oplus X \oplus X$ by*

$$C = \begin{pmatrix} 0 & 0 & 0 \\ P & 0 & 0 \\ 0 & I & 0 \end{pmatrix}.$$

Then $A(BA)^2 = ABACA = ACABA = (AC)^2A$, while $ABA \neq ACA$ and $BAB \neq B^2$.

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