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Ideal Relatively Uniform Convergence with Korovkin and Voronovskaya types Approximation Theorems

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Abstract. We introduce the notion of ideally relative uniform convergence of sequences of real valued functions. We then apply this notion to prove Korovkin-type approximation theorem, and then construct an illustrative example by taking (p, q)-Bernstein operators which proves that our Korovkin theorem is stronger than its classical version as well as statistical relative uniform convergence. The rate of ideal relatively uniform convergence of positive linear operators by means of modulus of continuity is calculated. Finally, the Voronovskaya-type approximation theorem is also investigated.

1. Introduction and preliminaries

Moore [36] was the first who introduced the notion of relative uniform convergence of sequence of functions. Thereafter, Chittenden [13, 14] studied this notion (which is equivalent to Moore's definition) as follows: A sequence of function (g_n), defined on $J = [a, b]$ converges relatively uniformly to a limit function g if there is a function $\gamma(t)$, called a scale function such that for every $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that

$$|g_n(t) - g(t)| < \varepsilon|\gamma(t)| \text{ uniformly in } t \text{ on } J, \quad \forall n > m.$$

Based on this definition, Demirci and Orhan [15] and Dirik and Şahin [16] introduced the concept of statistical relatively uniform convergence and statistical relatively equal convergence, respectively, and the authors of both the papers used their notions of convergence to prove approximation results.

The asymptotic density or density of a subset U of $\mathbb{N}$, denoted by $\delta(U)$, is given by

$$\delta(U) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in U\}|,$$

if this limit exists, where $|\{k \le n : k \in U\}|$ denotes the cardinality of the set $\{k \le n : k \in U\}$. With the help of asymptotic density, Fast [19] introduced the notion of statistical convergence as follows: A sequence $x = (x_n)$

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is said to be statistically convergent to ℓ if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\}$ has density zero. For more details on these type of convergence and their application, we refer to [8, 10, 23–25, 31, 33, 34, 38, 48].

The notion of ideal convergence is the dual (equivalent) to the notion of filter convergence which was introduced by Cartan [12]. The filter convergence is a generalization of the classical notion of convergence of sequence and it has been an important tool in general topology and functional analysis. Kostyrko et al., [29] and Nuray and Ruckle [41] independently discussed about the ideal convergence which is based on the structure of the admissible ideal $\mathcal{I}$ of subsets of natural numbers $\mathbb{N}$ (a similar notion was given by Katětov [26]). It was further investigated by many authors, e.g. Šalát et al. [43], Mursaleen and Mohiuddine [39], Hazarika and Mohiuddine [35] and references therein.

A non empty class $\mathcal{I}$ of power sets of a non empty set X is called an *ideal* on X if and only if (i) $\phi \in \mathcal{I}$ (ii) $\mathcal{I}$ is additive under union (iii) $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$. An ideal $\mathcal{I}$ is called *non trivial* if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$. A non-empty family $\mathcal{F}$ of power sets of X is called a *filter* on X if and only if (i) $\phi \notin \mathcal{F}$ (ii) $\mathcal{F}$ is additive under intersection (iii) for each $U \in \mathcal{F}$ and $V \supset U$, implies $V \in \mathcal{F}$. A non-trivial ideal $\mathcal{I}$ is said to be (i) an *admissible ideal* on X if and only if it contains all singletons (ii) *maximal*, if there cannot exists any non-trivial ideal $\mathcal{K} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset (iii) is said to be a *translation invariant ideal* if $\{n+1 : n \in U\} \in \mathcal{I}$, for any $U \in \mathcal{I}$. We consider the ideals are proper ($\neq P(\mathbb{N})$) and contain all finite sets. We denote Fin for set of ideals which consists of all finite sets.

We recall that a real sequence $x = (x_k)$ is called ideal convergent (in short $\mathcal{I}$-convergent) to the number l (denoted by $\mathcal{I}\text{-}\lim x_k = l$) if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ is in $\mathcal{I}$.

Katětov [27] gave a generalization of this notion by assuming sequences of functions into his account as follows: A sequence $g_k : (Y, \rho_1) \to (Y, \rho_2)$ of functions ((Y, ρ_1) and (Y, ρ_2) are metric spaces) is said to be uniform $\mathcal{I}$-convergent to g if

$$(\forall \varepsilon > 0) \quad \left\{k \in \mathbb{N} : \sup_{y \in Y} \rho_2\left(g_k(y) - g(y)\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

The statistical convergence and ideal convergence for sequences of functions in metric space were studied by Balcerzak et al. [7] while the idea of $\mathcal{I}$-uniformly convergent sequence (g_k) of real-valued functions was discussed by Filipów and Staniszewski [20].

2. Ideal relatively uniform convergence

We assume that $C(J)$ ($J = [a, b]$) denotes the space of all continuous real-valued functions defined on a compact subset J of real numbers and is also a Banach space. For $g \in C(J)$, one obtains

$$||g||_{C(J)} = \sup_{y \in J} |g(y)|.$$

We are now introducing the notion of ideal relatively uniform convergence of sequences of real-valued functions as follows:

Definition 2.1. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. A sequence (g_k) of real valued functions defined on $J \subset \mathbb{R}$, is said to be ideally relative uniform convergent to a function g on J, if there is a function $\gamma(t)$, called a scale function $|\gamma(t)| > 0$, such that for every $\varepsilon > 0$,*

$$\left\{k \in \mathbb{N} : \sup_{t \in J} \left|\frac{\bar{g}_k(t) - \bar{g}(t)}{\gamma(t)}\right| \geq \varepsilon\right\} \in \mathcal{I}.$$

We write $(\bar{g}_k) \xrightarrow{\mathcal{I}_u^{r,\gamma}} \bar{g}$ on J. We denote by $\mathcal{I}_u^{r,\gamma}$ the set of all ideally relative uniform convergent sequences.

Note that if we take $\mathcal{I} = \text{Fin}$, then we obtain the usual notions of relatively uniform convergence due to Chittenden [13, 14]. Also, if $I = I_\delta = \{B \subseteq \mathbb{N} : \delta(B) = 0\}$ then Definition 2.1 reduces to statistically relative uniform convergence which was given by Demirci and Orhan [15]. Since ideal $\mathcal{I}$ is admissible, the convergence in Definition 2.1 is implied by the classical uniform convergence

$$(g_k) \xrightarrow{u} g \text{ (in classical sense of uniform convergence)} \Longrightarrow (g_k) \xrightarrow{\mathcal{I}_u^{r,\gamma}} g. \tag{1}$$

But the converse of implication (1) is not true in general. In order to prove this assertion, we construct the following example.

Example 2.2. *We define* $h_k : [0,1] \to \mathbb{R}$ *by*

$$h_k(t) = \begin{cases} \frac{kt}{1+k^2t^2} & (t \neq 0) \\ 0 & (t = 0). \end{cases}$$

Then we have $(h_k) \xrightarrow{\mathcal{I}_u^{r,\gamma}} h = 0$ *relative to the scale function [15] defined by*

$$\gamma(t) = \begin{cases} \frac{1}{t} & (0 < t \leq 1) \\ 1 & (t = 0). \end{cases}$$

But (h_k) *is not uniformly convergent to* $h = 0$ *on* $[0,1]$.

3. Korovkin-type approximation theorem via ideal relatively uniform convergence

For any sequence (T_k) of positive linear operators on $C(J)$, Korovkin [28] was the first who investigated the sufficient conditions for the uniform convergence of a sequence $T_k(g)$ $(k \in \mathbb{N})$ to a function g by considering the test function e_i which is defined by $e_i(s) = s^i$, where $i = 0,1,2$. The Korovkin result, in statistical sense, firstly was proved in [21]. By considering various convergence methods, Korovkin-type approximation theorem studied by many researchers (for example see [3, 5, 6, 9, 11, 18, 30, 42, 44–46, 49]).

Theorem 3.1. *Suppose that* $T_k : C(J) \to C(J)$ *is a sequence of positive linear operators. Then, for any function* $g \in C(J)$,

$$T_k(g;t) \xrightarrow{\mathcal{I}_u^{r,\gamma}} g(t) \tag{2}$$

if and only if

$$T_k(e_j;t) \xrightarrow{\mathcal{I}_u^{r,\gamma}} e_j(t) \tag{3}$$

where

$$e_j(s) = s^j \quad \text{and} \quad \gamma(t) = \max\{|\gamma_j(t)| : |\gamma_j(t)| > 0\} \quad (j = 0,1,2).$$

Proof. Since each of the functions e_j $(j = 0,1,2)$ belongs to $C(J)$, then condition (3) follows immediately from (2). In order to prove the converse part, suppose that (3) holds. By the continuity of g on J, we have $|g(t)| \leq M$, where $M = \sup_{t\in J} |g(t)|$. Therefore, we find that $|g(s) - g(t)| \leq 2M$. Also, since g is continuous on J, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|g(s) - g(t)| < \varepsilon$ whenever $|s - t| < \delta$ for all $t \in J$. We thus find that

$$|g(s) - g(t)| < \varepsilon + \frac{2M}{\delta^2}\mu(s),$$

or,

$$-\varepsilon - \frac{2M}{\delta^2}\mu(s) < g(s) - g(t) < \varepsilon + \frac{2M}{\delta^2}\mu(s). \tag{4}$$

where $\mu(s) = (s-t)^2$. Since $T_k(g;t)$ is monotone and linear, we obtain by operating $T_k(e_0;t)$ in (4) that

$$T_k(e_0;t)\left(-\varepsilon - \frac{2M}{\delta^2}\mu(s)\right) < T_k(e_0;t)(g(s)-g(t)) < T_k(e_0;t)\left(\varepsilon + \frac{2M}{\delta^2}\mu(s)\right). \tag{5}$$

Here t is fixed and so $g(t)$ is a constant number. Therefore

$$-\varepsilon T_k(e_0;t) - \frac{2M}{\delta^2}T_k(\mu(s);t) < T_k(g;t) - g(t)T_k(e_0;t) < \varepsilon T_k(e_0;t) + \frac{2M}{\delta^2}T_k(\mu(s);t) \tag{6}$$

and

$$T_k(g;t) - g(t) = T_k(g;t) - g(t)T_k(e_0;t) + g(t)\{T_k(e_0;t) - e_0\}. \tag{7}$$

It follows from (6) and (7) that

$$T_k(g;t) - g(t) < \varepsilon T_k(e_0;t) + \frac{2M}{\delta^2}T_k(\mu;t) + g(t)\{T_k(e_0;t) - e_0\}. \tag{8}$$

We see that

$$T_k(\mu;t) = \{T_k(e_2;t) - e_2(t)\} - 2t\{T_k(e_1;t) - e_1(t)\} + t^2\{T_k(e_0;t) - e_0(t)\}. \tag{9}$$

Employing (9) in the earlier inequality (8), we obtain

$$\begin{aligned} T_k(g;t) - g(t) &< \varepsilon T_k(e_0;t) + \frac{2M}{\delta^2}\Big[\Big\{T_k(e_2;t) - e_2(t)\Big\} - 2t\{T_k(e_1;t) - e_1(t)\} \\ &\quad + t^2\{T_k(e_0;t) - e_0(t)\Big\}\Big] + g(t)\{T_k(e_0;t) - e_0(t)\} \\ &= \varepsilon + \varepsilon\{T_k(e_0;t) - e_0(t)\} + \frac{2M}{\delta^2}\Big[\Big\{T_k(e_2;t) - e_2(t)\Big\} - 2t\{T_k(e_1;t) - e_1(t)\} \\ &\quad + t^2\{T_k(e_0;t) - e_0(t)\Big\}\Big] + g(t)\{T_k(e_0;t) - e_0(t)\}. \end{aligned}$$

which gives

$$\begin{aligned} |T_k(g;t) - g(t)| &\leq \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2}\|e_2\|_{C(J)}\right)|T_k(e_0;t) - e_0(t)| \\ &\quad + \frac{4M}{\delta^2}\|e_1\|_{C(J)}|T_k(e_1;t) - e_1(t)| + \frac{2M}{\delta^2}|T_k(e_2;t) - e_2(t)| \\ &\leq \varepsilon + L[|T_k(e_0;t) - e_0(t)| + |T_k(e_1;t) - e_1(t)| + |T_k(e_2;t) - e_2(t)|], \end{aligned}$$

where

$$L = \varepsilon + M + \frac{2M}{\delta^2}\left\{\|e_2\|_{C(J)} + 2\|e_1\|_{C(J)} + 1\right\}.$$

We thus have

$$\begin{aligned} \sup_{t\in J}\left|\frac{T_k(g;t) - g(t)}{\gamma(t)}\right| &\leq \sup_{t\in J}\frac{\varepsilon}{\gamma(t)} + L\left[\sup_{t\in J}\left|\frac{T_k(e_0;t) - e_0(t)}{\gamma_0(t)}\right| + \sup_{t\in J}\left|\frac{T_k(e_1;t) - e_1(t)}{\gamma_1(t)}\right|\right. \\ &\quad \left. + \sup_{t\in J}\left|\frac{T_k(e_2;t) - e_2(t)}{\gamma_2(t)}\right|\right]. \end{aligned} \tag{10}$$

For a given $q > 0$, choose $\varepsilon > 0$ such that $\varepsilon_1 = \sup_{t\in J}\frac{\varepsilon}{|\gamma(t)|} < q$. Then, upon setting

$$H = \left\{k \in \mathbb{N} : \sup_{t\in J}\left|\frac{T_k(g;t) - g(t)}{\gamma(t)}\right| \geq q\right\}$$

$$H_0 = \left\{k \in \mathbb{N} : \sup_{t \in J} \left| \frac{T_k(e_0;t) - e_0(t)}{\gamma_0(t)} \right| \geq \frac{q - \varepsilon_1}{3M} \right\}$$

$$H_1 = \left\{k \in \mathbb{N} : \sup_{t \in J} \left| \frac{T_k(e_1;t) - e_1(t)}{\gamma_1(t)} \right| \geq \frac{q - \varepsilon_1}{3M} \right\}$$

and

$$H_2 = \left\{k \in \mathbb{N} : \sup_{t \in J} \left| \frac{T_k(e_2;t) - e_2(t)}{\gamma_2(t)} \right| \geq \frac{q - \varepsilon_1}{3M} \right\},$$

We thus have $H \subseteq \bigcup_{j=0}^{2} H_j$. It follows from our assumption (3) that $H_j \in \mathcal{I}$ for $j = 0, 1, 2$, that is, $\bigcup_{j=0}^{2} H_j$ is in $\mathcal{I}$. Consequently, from the definition of ideal, we have $H \in \mathcal{I}$. Hence, the proof is complete. □

We are now going to construct an example in support of our Theorem 3.1 with the help of (p, q)-Bernstein operators which were constructed by Mursaleen et al. [37]. For recent work on quantum and post-quantum calculus and related operators, one may refer to [1, 2, 4, 32, 40, 47]

Example 3.2. *Assume that $J = [0, 1]$. For any given $k \in \mathbb{N}$, let us consider the (p, q)-Bernstein operators as*

$$B_{k,p,q}(g;t) = \sum_{n=0}^{k} \mathcal{B}_{k,n,p,q}(t)\, g\left(\frac{p^{k-n}[n]_{p,q}}{[k]_{p,q}}\right) \qquad (t \in [0, 1]),$$

where $0 < q < p \leq 1$ and

$$\mathcal{B}_{k,n,p,q}(t) = \begin{bmatrix} k \\ n \end{bmatrix}_{p,q} p^{\frac{n(n-1)-k(k-1)}{2}}\, t^n \prod_{s=0}^{k-n-1} (p^s - q^s t).$$

Recall, as in [22], that $[k]_{p,q}$ denotes a (p, q)-integer, defined as

$$[k]_{p,q} = \frac{p^k - q^k}{p - q} \qquad (0 < q < p \leq 1)$$

and the (p, q)-binomial coefficient $\left[\begin{smallmatrix} k \\ n \end{smallmatrix}\right]_{p,q}$ is defined by

$$\begin{bmatrix} k \\ n \end{bmatrix}_{p,q} = \frac{[k]_{p,q}!}{[k-n]_{p,q}![n]_{p,q}!} \qquad (\forall k, n \in \mathbb{N}, k \geq n),$$

where (p, q)-factorial $[k]_{p,q}!$ is given by

$$[k]_{p,q}! = [1]_{p,q}![2]_{p,q}! \ldots [k]_{p,q}! \;\; for \;\; k \geq 1 \quad and \quad [0]_{p,q}! = 1.$$

Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. Suppose that $p = (p_k)$ and $q = (q_k)$ such that $q_k \in (0, 1)$, $p_k \in (q_k, 1]$, and also $p_k \to 1$, $q_k \to 1$, $p_k^k \to a$, $q_k^k \to b$ as $k \to \infty$ $(a \neq b)$. We are now defining the sequence of positive linear operator $\mathcal{P}_k$ on $C[0, 1]$ by

$$\mathcal{P}_k(g;t) = (1 + h_k(t)) B_{k,p_k,q_k}(g;t) \quad (g \in C[0, 1]), \tag{11}$$

where the sequence $(h_k(t))$ of functions as defined in Example 2.2. We thus find that

$$\mathcal{P}_k(e_0;t) = (1 + h_k(t)) e_0(t), \;\; \mathcal{P}_k(e_1;t) = (1 + h_k(t))\, e_1(t)$$

and

$$\begin{aligned}\mathcal{P}_k(e_2;t) &= (1+h_k(t))\left(\frac{p_k^{k-1}}{[k]_{p_k,q_k}}e_1(t)+\frac{q_k[k-1]_{p_k,q_k}}{[k]_{p_k,q_k}}e_2(t)\right)\\ &= (1+h_k(t))\left(\frac{p_k^{k-1}}{[k]_{p_k,q_k}}e_1(t)+\frac{p_k^{k-1}q_k-q_k^k}{p_k-q_k}\frac{e_2(t)}{[k]_{p_k,q_k}}\right)\\ &= (1+h_k(t))\left(\frac{p_k^{k-1}}{[k]_{p_k,q_k}}e_1(t)-\frac{p_k^{k-1}}{[k]_{p_k,q_k}}e_2(t)+e_2(t)\right).\end{aligned}$$

It is easy to see that

$$\frac{p_k^{k-1}}{[k]_{p_k,q_k}}e_1(t)\xrightarrow{u}0 \quad \text{and} \quad \frac{p_k^{k-1}}{[k]_{p_k,q_k}}e_2(t)\xrightarrow{u}0 \qquad (t\in[0,1]).$$

Since

$$(h_k)\xrightarrow{\mathcal{I}_u^{r,\gamma}}h=0$$

for the sequence (h_k) of functions and scale function $\gamma(t)$ as defined in Example 2.2, and together with our assumption that $\mathcal{I}$ is an admissible ideal of $\mathbb{N}$ and so ideal relatively uniform convergence is implied by the uniform (usual) convergence, we thus obtain

$$\mathcal{P}_k(e_j;t)\xrightarrow{\mathcal{I}_u^{r,\gamma}}e_j(t) \qquad (j=0,1,2).$$

We therefore obtain by Theorem 3.1 that

$$\mathcal{P}_k(g;t)\xrightarrow{\mathcal{I}_u^{r,\gamma}}g(t) \quad (\forall g\in C[0,1] \text{ and } t\in[0,1]).$$

Moreover, since (h_k) is not ideally (or classical) uniform convergent to the function $h=0$ on $[0,1]$, we can say that our Theorem 3.1 is a non-trivial generalizations of the classical and ideal cases of the Korovkin results discussed in [17] and [28], respectively. For

$$\mathcal{I}=\mathcal{I}_\delta=\{B\subseteq\mathbb{N}:\delta(B)=0\},$$

$\mathcal{I}_\delta$ is a non-trivial admissible ideal of $\mathbb{N}$, then Theorem 3.1 reduced to statistical relative uniform version of Korovkin result obtained in [15].

4. Rate of ideal relatively uniform convergence

Using the concept of modulus of continuity, we investigate the rate of ideal relatively uniform convergence of a sequence of positive linear operators defined on $C(J)$. We first recall the modulus of continuity of a function $g\in C(J)$ is defined by

$$\omega(g,\delta)=\sup_{|s-t|\le\delta,s,t\in J}|g(s)-g(t)| \quad (\delta>0).$$

We introduce the following definition in order to compute the rate of ideal relatively uniform convergence.

Definition 4.1. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. A sequence (g_k) of functions is said to be ideally relative uniform convergent to g on J with the rate $\nu\in(0,1)$ if there exists a scale function $\gamma(t)$, $|\gamma(t)|>0$, such that for every $\varepsilon>0$, we have*

$$\left\{k\in\mathbb{N}:\sup_{t\in J}\frac{\left|\frac{g_k(t)-g(t)}{\gamma(t)}\right|}{k^{1-\nu}}\ge\varepsilon\right\}\in\mathcal{I}.$$

Here, in this case, we write

$$g_k - g = \mathcal{I}_u^{r,\gamma}\text{-}o(k^{-\nu}) \text{ on } J.$$

Lemma 4.2. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, and let (g_k) and (h_k) are two sequence of functions defined on $C(J)$. Suppose also that*

$$g_k - g = \mathcal{I}_u^{r,\gamma_1}\text{-}o(k^{-\nu_1}) \text{ on } J \tag{12}$$

and

$$h_k - h = \mathcal{I}_u^{r,\gamma_2}\text{-}o(k^{-\nu_2}) \text{ on } J. \tag{13}$$

Let $\nu = \min\{\nu_1, \nu_2\}$ and $\gamma(t) = \max\{|\gamma_i(t)| : |\gamma_i(t)| > 0\}$, $i = 1, 2$. Then each of the following statements holds true:

(i) $(g_k \pm h_k) - (g \pm h) = \mathcal{I}_u^{r,\gamma}\text{-}o(k^{-\nu})$ *on* J;
(ii) $(\lambda(g_k - g)) = \mathcal{I}_u^{r,\gamma_1}\text{-}o(k^{-\nu_1})$ *on* J *for any scalar* λ;
(iii) $(g_k - g)(h_k - h) = \mathcal{I}_u^{r,\gamma_0}\text{-}o(k^{-\nu})$, *where* $\gamma_0(t) = \gamma_1(t)\gamma_2(t)$ *on* J
(iv) $\sqrt{|g_k - g|} = \mathcal{I}_u^{r,\gamma_1}\text{-}o(k^{-\nu_1})$ *on* J.

Proof. Suppose that $g_k - g = \mathcal{I}_u^{r,\gamma_1}\text{-}o(k^{-\nu_1})$ and $h_k - h = \mathcal{I}_u^{r,\gamma_2}\text{-}o(k^{-\nu_2})$ on J. In order to prove (i), we define the following sets for every $\varepsilon > 0$ and $t \in J$:

$$A(t,\epsilon) = \left\{ k \in \mathbb{N} : \sup_{t \in J} \frac{\left| \frac{(g_k \pm h_k)(t) - (g \pm h)(t)}{\gamma(t)} \right|}{k^{1-\nu}} \geq \varepsilon \right\},$$

$$A_1(t,\epsilon) = \left\{ k \in \mathbb{N} : \sup_{t \in J} \frac{\left| \frac{g_k(t) - g(t)}{\gamma_1(t)} \right|}{k^{1-\nu}} \geq \frac{\varepsilon}{2} \right\}$$

and

$$A_2(t,\epsilon) = \left\{ k \in \mathbb{N} : \sup_{t \in J} \frac{\left| \frac{h_k(t) - h(t)}{\gamma_2(t)} \right|}{k^{1-\nu}} \geq \frac{\varepsilon}{2} \right\}$$

where $\gamma(t) = \max\{|\gamma_i(t)| : |\gamma_i(t)| > 0\}$ $(i = 1, 2)$. Clearly, we have $A(t,\epsilon) \subseteq A_1(t,\epsilon) \cup A_2(t,\epsilon)$. Since $\nu = \min\{\nu_1, \nu_2\}$, then we obtain

$$A(t,\epsilon) \subseteq B_1(t,\epsilon) \cup B_2(t,\epsilon) \tag{14}$$

where

$$B_1(t,\epsilon) = \left\{ k \in \mathbb{N} : \sup_{t \in J} \frac{\left| \frac{g_k(t) - g(t)}{\gamma_1(t)} \right|}{k^{1-\nu_1}} \geq \frac{\varepsilon}{2} \right\}$$

and

$$B_2(t,\epsilon) = \left\{ k \in \mathbb{N} : \sup_{t \in J} \frac{\left| \frac{h_k(t) - h(t)}{\gamma_2(t)} \right|}{k^{1-\nu_2}} \geq \frac{\varepsilon}{2} \right\}.$$

By our assumptions (12) and (13), we find that the right-hand side of the inequality (14) is in $\mathcal{I}$ and consequently, from the definition of ideal, we have

$$\left\{k \in \mathbb{N} : \sup_{t\in J} \frac{\left|\frac{(g_k \pm h_k)(t)-(g\pm h)(t)}{\gamma(t)}\right|}{k^{1-\nu}} \geq \varepsilon\right\} \in \mathcal{I},$$

which completes the proof of statement (i). The proof of the other statements of lemma follows in a similar way. □

Theorem 4.3. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. Suppose that $T_k : C(J) \to C(J)$ is a sequence of positive linear operators. Suppose also that*

(i) $T_k(e_0;t) - e_0 = \mathcal{I}_u^{r,\gamma_1}\text{-}o(k^{-\nu_1})$.

(ii) $\omega(g,\delta_k) = \mathcal{I}_u^{r,\gamma_2}\text{-}o(k^{-\nu_2})$, *where* $\delta_k(t) = \sqrt{T_k(\mu^2;t)}$ *with* $\mu(s) = (s-t)$.

Then

$$T_k(g;t) - g = \mathcal{I}_u^{r,\gamma}\text{-}o(k^{-\nu}) \;\; on\; J \quad (\forall g \in C(J)) \tag{15}$$

where

$$\nu = \min\{\nu_1,\nu_2\} \;\; and \;\; \gamma(t) = \max\{|\gamma_i(t) : |\gamma_i(t) > 0|\} \; (i = 1,2).$$

Proof. Let $g \in C(J)$ and $t \in J$. Since (T_k) $(k \in \mathbb{N})$ ia linear and monotone, we can write

$$\begin{aligned}|T_k(g;t) - g(t)| &\leq T_k(|g(s) - g(t)|;t) + |g(t)||T_k(e_0;t) - e_0(t)| \\ &\leq T_k\left(\left(1 + \frac{|\mu(s)|}{\delta}\right)\omega(g,\delta);t\right) + C|T_k(e_0;t) - e_0(t)| \\ &= \omega(g,\delta)T_k(e_0;t) + \frac{\omega(g,\delta)}{\delta}T_k(|\mu(s)|;t) + C|T_k(e_0;t) - e_0(t)|,\end{aligned}$$

where $C = \|g\|$. It follows from Cauchy-Schwarz inequality that

$$T_k(|\mu(s)|;t) \leq \sqrt{T_k(\mu^2(s);t)}\sqrt{T_k(e_0;t)}.$$

We therefore find that

$$|T_k(g;t) - g(t)| \leq \omega(g,\delta)T_k(e_0;t) + \frac{\omega(g,\delta)}{\delta}\sqrt{T_k(\mu^2;t)}\sqrt{T_k(e_0;t)} + C|T_k(e_0;t) - e_0(t)|.$$

If we take

$$\delta := \delta_k(t) = \sqrt{T_k(\mu^2;t)}$$

in the last inequality, we obtain

$$|T_k(g;t) - g(t)| \leq \omega(g,\delta_k)T_k(e_0;t) + \omega(g,\delta_k)\sqrt{T_k(e_0;t)} + C|T_k(e_0;t) - e_0(t)|$$

which yields

$$|T_k(g;t) - g(t)| \leq \omega(g,\delta_k)\left\{|T_k(e_0;t) - e_0(t)| + 2e_0(t) + \sqrt{T_k(e_0;t) - e_0(t)}\right\} + C|T_k(e_0;t) - e_0(t)|$$

Therefore, we have

$$\begin{aligned}\sup_{t\in J}\left|\frac{T_k(g;t) - g(t)}{\gamma(t)}\right| \leq\; & C\sup_{t\in J}\left|\frac{|T_k(e_0;t) - e_0(t)|}{\gamma_1(t)}\right| + 2e_0\sup_{t\in J}\frac{\omega(g,\delta_k)}{|\gamma_2(t)|} \\ &+ \sup_{t\in J}\frac{\omega(g,\delta_k)}{|\gamma_2(t)|}\left|\frac{T_k(e_0;t) - e_0(t)}{|\gamma_1(t)|}\right| + \sup_{t\in J}\frac{\omega(g,\delta_k)}{|\gamma_2(t)|}\sqrt{\left|\frac{T_k(e_0;t) - e_0(t)}{|\gamma_1(t)|}\right|}.\end{aligned} \tag{16}$$

Employing our hypotheses (i) and (ii) together with Lemma 4.2 in the last inequality (16) gives us (15). The proof of the theorem is thus completed. □

5. Voronovskaya-type approximation theorem

We obtain a Voronovskaya-type approximation theorem by taking our positive linear operator $\mathcal{P}_k(g;t)$ defined by (11) through ideal relatively uniform convergence. Before proceeding further, let us prove following lemma which will be used to prove our next approximation result.

Lemma 5.1. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. Assume that the following assumptions hold:*

(C1) Let $p = (p_k)$ and $q = (q_k)$ satisfy $q_k \in (0,1)$, $p_k \in (q_k, 1]$ with $p_k \to 1$, $q_k \to 1$, $p_k^k \to a$, $q_k^k \to b$ as $k \to \infty$, where $a \neq b$.

Assume also that $t \in [0,1]$. Then, for the scale function $\gamma(t)$ such that $|\gamma(t)| > 0$, we have

$$[k]_{p_k,q_k}\mathcal{P}_k\left((s-t)^2;t\right) \xrightarrow{\mathcal{I}_u^{r,\gamma}} p_k^{k-1}t(1-t) \text{ on } [0,1].$$

Proof. We shall obtain by (11) that

$$\begin{aligned}
\mathcal{P}_k\left((s-t)^2;t\right) &= (1+h_k(t))\, B_{k,p_k,q_k}\left(s^2-2st+t^2;t\right)\\
&= (1+h_k(t))\left\{B_{k,p_k,q_k}\left(s^2;t\right) - 2tB_{k,p_k,q_k}(s;t) + t^2B_{k,p_k,q_k}(1;t)\right\}\\
&= (1+h_k(t))\left\{\frac{p_k^{k-1}}{[k]_{p_k,q_k}}t + \frac{q_k[k-1]_{p_k,q_k}}{[k]_{p_k,q_k}}t^2 - t^2\right\}
\end{aligned}$$

which yields

$$\begin{aligned}
[k]_{p_k,q_k}\mathcal{P}_k\left((s-t)^2;t\right) &= (1+h_k(t))\left\{p_k^{k-1}t + \frac{p_k^{k-1}q_k - q_k^k}{p_k-q_k}t^2 - [k]_{p_k,q_k}t^2\right\}\\
&= (1+h_k(t))\left(p_k^{k-1}t - p_k^{k-1}t^2\right)\\
&= (1+h_k(t))\, p_k^{k-1}t(1-t)
\end{aligned}$$

or,

$$[k]_{p_k,q_k}\mathcal{P}_k\left((s-t)^2;t\right) - p_k^{k-1}t(1-t) = p_k^{k-1}t(1-t)h_k(t) \tag{17}$$

In view of implication (1) and Example 2.2, we see that the right-hand side of inequality (17) is ideal relatively uniform convergent to zero on $[0,1]$. Consequently, we have

$$[k]_{p_k,q_k}\mathcal{P}_k\left((s-t)^2;t\right) \xrightarrow{\mathcal{I}_u^{r,\gamma}} p_k^{k-1}t(1-t) \text{ on } [0,1],$$

which proves the lemma completely. □

Corollary 5.2. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. Assume also that $t \in [0,1]$. Then there exist a positive constant $C(t)$ depending only on t, such that*

$$[k]^2_{p_k,q_k}P_k\left((s-t)^4;t\right) \xrightarrow{\mathcal{I}_u^{r,\gamma}} C(t) \text{ on } [0,1],$$

where $\gamma(t)$ is a scale function satisfying $|\gamma(t)| > 0$.

Theorem 5.3. *Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. Assume that condition (C1) holds. If for every $g \in C[0,1]$ such that g' and g'' in $C[0,1]$ then*

$$[k]_{p_k,q_k}\left(\mathcal{P}_k\left(g(s);t\right) - g(t)\right) \xrightarrow{\mathcal{I}_u^{r,\gamma}} \frac{p_k^{k-1}t(1-t)}{2}g''(t) \text{ on } [0,1],$$

where $\gamma(t) = \max\{|\gamma_i(t)| : i = 1,2\}$, $|\gamma_i(t)| > 0$.

Proof. Let $g, g', g'' \in C[0,1]$ and $t \in [0,1]$. Define

$$\Omega_t(s) = \begin{cases} \frac{g(s)-g(t)-(s-t)g'(t)-\frac{1}{2}(s-t)^2 g''(t)}{(s-t)^2} & (s \neq t) \\ 0 & (s = t). \end{cases}$$

We then have $\Omega_t(s) = 0$ and the function Ω_t belonging to $C[0,1]$. By Taylor's formula, one writes

$$g(s) = g(t) + (s-t)\, g'(t) + \frac{1}{2}(s-t)^2 g''(t) + (s-t)^2 \Omega_t(s). \tag{18}$$

Since the operator $\mathcal{P}_k$ is linear, by operating this operator on both sides of last inequality, we have

$$\begin{aligned} \mathcal{P}_k(g(s);t) &= \mathcal{P}_k(g(t);t) + g'(t)\mathcal{P}_k(s-t;t) + \frac{1}{2}g''(t)\mathcal{P}_k((s-t)^2,t) + \mathcal{P}_k\left((s-t)^2\Omega_t(s);t\right) \\ &= g(t)(1+h_k(t))B_{k,p_k,q_k}(1;t) + g'(t)(1+h_k(t))\left[B_{k,p_k,q_k}(s;t) - tB_{k,p_k,q_k}(1;t)\right] \\ &\quad + \frac{1}{2}g''(t)(1+h_k(t))\left[B_{k,p_k,q_k}(s^2;t) - 2tB_{k,p_k,q_k}(s;t) + t^2 B_{k,p_k,q_k}(1;t)\right] \\ &\quad + \mathcal{P}_k\left((s-t)^2\Omega_t(s);t\right), \end{aligned}$$

or,

$$\begin{aligned} \mathcal{P}_k(g(s);t) - g(t) &= g(t)h_k(t) + \frac{1}{2}g''(t)(1+h_k(t))\left\{\frac{p_k^{k-1}}{[k]_{p_k,q_k}}t + \frac{q_k[k-1]_{p_k,q_k}}{[k]_{p_k,q_k}}t^2 - t^2\right\} \\ &\quad + \mathcal{P}_k\left((s-t)^2\Omega_t(s);t\right) \end{aligned}$$

Multiplying above equality by $[k]_{p_k,q_k}$, we obtain

$$\begin{aligned} [k]_{p_k,q_k}(\mathcal{P}_k(g(s);t) - g(t)) &= [k]_{p_k,q_k}g(t)h_k(t) + \frac{1}{2}g''(t)(1+h_k(t))\left(p_k^{k-1}t - p_k^{k-1}t^2\right) \\ &\quad + [k]_{p_k,q_k}\mathcal{P}_k\left((s-t)^2\Omega_t(s);t\right) \end{aligned}$$

which gives

$$\begin{aligned} \left|[k]_{p_k,q_k}(\mathcal{P}_k(g(s);t) - g(t)) - \frac{p_k^{k-1}t(1-t)}{2}g''(t)\right| &\leq M\left([k]_{p_k,q_k} + p_k^{k-1}\right)h_k(t) \\ &\quad + [k]_{p_k,q_k}\left|\mathcal{P}_k\left((s-t)^2\Omega_t(s);t\right)\right|, \end{aligned} \tag{19}$$

where $M = \|g(t)\|_{C[0,1]} + \|g''(t)\|_{C[0,1]}$. Employing the Cauchy-Schwarz inequality in the last term on the right-hand side of (19), we obtain

$$[k]_{p_k,q_k}\left|\mathcal{P}_k((s-t)^2\Omega_t(s);t)\right| \leq \sqrt{P_k\left(\Omega_t^2(s);t\right)}\sqrt{[k]_{p_k,q_k}^2 P_k\left((s-t)^4;t\right)}. \tag{20}$$

Let

$$\theta_t(s) = \Omega_t^2(s).$$

Clearly, we see that

$$\theta_t(t) = 0 \quad \text{and} \quad \theta_t(.) \in C[0,1].$$

It follows from Theorem 3.1 that

$$\mathcal{P}_k(\theta_t(s);t) \xrightarrow{\mathcal{I}_u^{r,\gamma}} \theta_t(t) = 0 \ \text{ on } \ [0,1]. \tag{21}$$

We thus have from (20) that

$$[k]_{p_k,q_k}\mathcal{P}_k\left((s-t)^2\,\Omega_t(s)\,;t\right)\xrightarrow{\mathcal{I}_u^{r,\gamma}}0. \tag{22}$$

Now, for given $\epsilon > 0$, we define

$$J_k(t)=\left\{k\in\mathbb{N}:\sup_{t\in[0,1]}\left|\frac{[k]_{p_k,q_k}\left(\mathcal{P}_k(g(s)\,;t)-g(t)\right)-\frac{p_k^{k-1}t(1-t)}{2}g''(t)}{\gamma(t)}\right|\geq\epsilon\right\}$$

$$J_k'(t)=\left\{k\in\mathbb{N}:\sup_{t\in[0,1]}\left|\frac{\left([k]_{p_k,q_k}+p_k^{k-1}\right)h_k(t)}{\gamma_1(t)}\right|\geq\frac{\epsilon}{2M}\right\}$$

and

$$J_k''(t)=\left\{k\in\mathbb{N}:\sup_{t\in[0,1]}\left|\frac{[k]_{p_k,q_k}\mathcal{P}_k\left((s-t)^2\,\Omega_t(s)\,;t\right)}{\gamma_2(t)}\right|\geq\frac{\epsilon}{2}\right\},$$

where $\gamma(t)=\max\{|\gamma_i(t)|:i=1,2\}$. We then find from (19) that $J_k(t)\subset J_k'(t)\cup J_k''(t)$. Since $(h_k)\xrightarrow{\mathcal{I}_u^{r,\gamma}}h=0$, we have

$$\left([k]_{p_k,q_k}+p_k^{k-1}\right)h_k(t)\xrightarrow{\mathcal{I}_u^{r,\gamma}}0\qquad(t\in[0,1]). \tag{23}$$

Finally, by using the (22) and (23) together with the definition of ideal, we have

$$[k]_{p_k,q_k}\left(\mathcal{P}_k(g(s)\,;t)-g(t)\right)\xrightarrow{\mathcal{I}_u^{r,\gamma}}\frac{p_k^{k-1}t(1-t)}{2}g''(t)\quad\text{on}\quad[0,1].$$

Hence, the proof is complete. □

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References

[1] T. Acar, A. Aral, S. A. Mohiuddine, On Kantorovich modification of (p,q)-Baskakov operators, J. Inequal. Appl. (2016) 2016:98.
[2] T. Acar, A. Aral, S. A. Mohiuddine, Approximation by bivariate (p,q)-Bernstein-Kantorovich operators, Iran. J. Sci. Technol. Trans. Sci. 42 (2018) 655–662.
[3] T. Acar, S. A. Mohiuddine, Statistical $(C,1)(E,1)$ summability and Korovkin's theorem, Filomat 30(2) (2016) 387–393.
[4] T. Acar, S. A. Mohiuddine, M. Mursaleen, Approximation by (p,q)-Baskakov-Durrmeyer-Stancu operators, Complex Anal. Oper. Theory 12 (2018) 1453–1468.
[5] S. Akdağ, Weighted equi-statistical convergence of the Korovkin type approximation theorems, Results Math. 72(3) (2017) 1073–1085.
[6] H. Aktuğlu, M. A. Özarslan, Korovkin type approximation theorem for BBH type operators via ideal convergence, Math. Slovaca 60 (2010) 865–876.
[7] M. Balcerzak, K. Dem, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (2007) 715–729.
[8] C. Belen, S. A. Mohiuddine, Generalized weighted statistical convergence and application, Appl. Math. Comput. 219 (2013) 9821–9826.
[9] N. L. Braha Some weighted equi-statistical convergence and Korovkin type-theorem, Results Math. 70 (2016) 433–446.
[10] N. L. Braha, V. Loku, H. M. Srivastava, Λ^2-weighted statistical convergence and Korovkin and Voronovskaya type theorems, Appl. Math. Comput. 266 (2015) 675–686.
[11] N. L. Braha, H. M. Srivastava, S. A. Mohiuddine, A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean, Appl. Math. Comput. 228 (2014) 162–169.

[12] H. Cartan, Filters et ultrafilters, C. R. Acad. Sci. Paris 205 (1937) 777–779.
[13] E. W. Chittenden, Relatively uniform convergence of sequences of functions, Trans. Amer. Math. Soc. 20 (1914) 197–201.
[14] E. W. Chittenden, On the limit functions of sequences of continuous functions converging relatively uniformly, Trans. Amer. Math. Soc. 20 (1919) 179–184.
[15] K. Demirci, S. Orhan, Statistically relatively uniform convergence of positive linear operators, Results Math. 69 (2016) 359–367.
[16] F. Dirik, P. O. Şahin, Statistical relatively equal convergence and Korovkin-type approximation theorem, Results Math. 72 (2017) 1613–1621.
[17] O. Duman, A Korovkin type approximation throrems via I-convergence, Czechoslovak Math. J. 75(132) (2007) 367–375.
[18] O. H. H. Edely, S. A. Mohiuddine, A. K. Noman, Korovkin type approximation theorems obtained through generalized statistical convergence, Appl. Math. Lett. 23 (2010) 1382–1387.
[19] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[20] R. Filipów, M. Staniszewski, On ideal equal convergence, Cent. Eur. J. Math. 12(6)(2014) 896–910.
[21] A. D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, Rocky Mt. J. Math. 32 (2002) 129–138.
[22] R. Jagannathan, K. S. Rao, Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, In: Proceedings of the International Conference on Number Theory and Mathematical Physics, pp. 20–21 (2005).
[23] U. Kadak, On weighted statistical convergence based on (p, q)-integers and related approximation theorems for functions of two variables, J. Math. Anal. Appl. 443 (2016) 752–764.
[24] U. Kadak, N. L. Braha, H. M. Srivastava, Statistical weighted B-summability and its applications to approximation theorems, Appl. Math. Comput. 302 (2017) 80–96.
[25] U. Kadak, S. A. Mohiuddine, Generalized statistically almost convergence based on the difference operator which includes the (p, q)-gamma function and related approximation theorems, Results Math. 73(1) (2018), Article 9.
[26] M. Katětov, Products of filters, Comment. Math. Univ. Carolin. 9 (1968) 173–189.
[27] M. Katětov, On descriptive classification of functions, in: Proc. Third Prague Topological Sympos., 1971, pp. 235–242.
[28] P. P. Korovkin, Linear operators and the theory of approximation, India, Delhi, 1960.
[29] P. Kostyrko, T. Šalát, W. Wilczyśki, On I-convergence, Real Analysis Exchange 26(2) (2000-2001) 669–686.
[30] S. A. Mohiuddine, An application of almost convergence in approximation theorems, Appl. Math. Lett. 24 (2011) 1856–1860.
[31] S. A. Mohiuddine, Statistical weighted A-summability with application to Korovkin's type approximation theorem, J. Inequal. Appl. (2016) 2016:101.
[32] S. A. Mohiuddine, T. Acar, A. Alotaibi, Construction of a new family of Bernstein-Kantorovich operators, Math. Meth. Appl. Sci. 40 (2017) 7749–7759.
[33] S. A. Mohiuddine, B. A. S. Alamri, Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems, Rev. R. Acad. Cienc. Exactas Fs. Nat., Ser. A Mat., RACSAM 113 (3) (2019) 1955–1973.
[34] S. A. Mohiuddine, A. Asiri, B. Hazarika, Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems, Int. J. Gen. Syst. 48(5) (2019) 492–506.
[35] S. A. Mohiuddine, B. Hazarika, Some classes of ideal convergent sequences and generalized difference matrix operator, Filomat 31(6) (2017) 1827–1834.
[36] E. H. Moore, An introduction to a form of general analysis, The New Haven Mathematical Colloquium, Yale University Press, New Haven (1910).
[37] M. Mursaleen, K. J. Ansari, A. Khan, On (p, q)-analogue of Bernstein operators, Appl. Math. Comput. 266 (2015) 874–882 [Erratum: Appl. Math. Comput. 278 (2106) 70–71].
[38] M. Mursaleen, V. Karakaya, M. Ertürk, F. Gürsoy, Weighted statistical convergence and its application to Korovkin type approximation theorem, Appl. Math. Comput. 218 (2012) 9132–9137.
[39] M. Mursaleen, S. A. Mohiuddine, On ideal convergence in probabilistic normed spaces, Math. Slovaca 62(1) (2012) 49–62.
[40] M. Mursaleen, Md. Nasiruzzaman, H. M. Srivastava, Approximation by bicomplex beta operators in compact $\mathbb{BC}$-disks, Math. Meth. Appl. Sci. 39 (2016) 2916–2929.
[41] F. Nuray, W. H. Ruckle, Generalized statistical convergence and convergence free spaces, J. Math. Anal. Appl. 245 (2000) 513–527.
[42] M. A. Özarslan, O. Duman, H. M. Srivastava, Statistical approximation results for Kantorovich-type operators involving some special polynomials, Math. Comput. Model. 48 (2008) 388–401.
[43] T. Šalát, B. C. Tripathy, M. Ziman, On some properties of I-convergence, Tatra Mt. Math. Publ., 28(2004) 279–286.
[44] H. M. Srivastava, B. B. Jena, S. K. Paikray, U. K. Misra, A certain class of weighted statistical convergence and associated Korovkin-type approximation theorems involving trigonometric functions, Math. Meth. Appl. Sci. 41 (2018) 671–683.
[45] H. M. Srivastava, B. B. Jena, S. K. Paikray, U. K. Misra, Deferred weighted A-statistical convergence based upon the (p, q)-Lagrange polynomials and its applications to approximation theorems, J. Appl. Anal. 24 (2018) 1–16.
[46] H. M. Srivastava, B. B. Jena, S. K. Paikray, U. K. Misra, Generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems, Rev. Real Acad. Cienc. Exactas Fs. Natur. Ser. A Mat. 112 (2018) 1487–1501.
[47] H. M. Srivastava, M. Mursaleen, A. M. Alotaibi, Md. Nasiruzzaman, A. A. H. Al-Abied, Some approximation results involving the q-Szász-Mirakjan-Kantorovich type operators via Dunkl's generalization, Math. Meth. Appl. Sci. 40 (2017) 5437–5452.
[48] H. M. Srivastava, M. Mursaleen, A. Khan, Generalized equi-statistical convergence of positive linear operators and associated approximation theorems, Math. Comput. Model. 55 (2012) 2041–2051.
[49] H. M. Srivastava, B. B. Jena, S. K. Paikray, U. Misra, Statistically and relatively modular deferred-weighted summability and Korovkin-type approximation theorems, Symmetry 11 (2019), Article 448.