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The Properties of $[\infty,C]$-Isometric Operators

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Abstract. In this paper we introduce the class of $[\infty,C]$-isometric operators and study various properties of this class. In particular, we show that if T is an $[\infty,C]$-isometric operator and Q is a quasi-nilpotent operator, then $T+Q$ is an $[\infty,C]$-isometric operator under suitable conditions. Also, we show that the class of $[\infty,C]$-isometric operators is norm closed. Finally, we examine properties of products of $[\infty,C]$-isometric operators.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on a separable complex Hilbert space H, and let $\mathbb{N}$, $\mathbb{C}$ be the sets of natural numbers and complex numbers, respectively.

In 1990s, Agler and Stankus [1] studied the theory of m-isometric operators which are connected to Toeplitz operators, ordinary differential equations, classical function theory, classical conjugate point theory, distributions, Fejer-Riesz factorization, stochastic processes, and other topics. For a fixed $m \in \mathbb{N}$, an operator $T \in B(H)$ is said to be an m-isometric operator if it satisfies an identity;

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^{m-j} = 0.$$

Several authors have studied the m-isometric operator. We refer the reader to [2–6, 10, 11] for further details.

An antilinear operator C on H is said to be conjugation if C satisfies $C^2 = I$ and $(Cx, Cy) = (y, x)$ for all $x, y \in H$. In [7], M. Chō, E. Ko and J. Lee introduced (m, C)-isometric operators with conjugation C as follows; For an operator $T \in B(H)$ and an integer $m \geq 1$, T is said to be an (m, C)-isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}.CT^{m-j}C = 0.$$

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In [8], M. Chō, E. Ko and J. Lee introduced (∞, C)-isometric operators with conjugation C as follows; For an operator $T \in B(H)$, T is said to be an (∞, C)-isometric operator if there exists some conjugation C such that

$$\limsup_{m\to\infty} \| \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j}.CT^{m-j}C\|^{\frac{1}{m}} = 0.$$

In [9], M. Chō, J. Lee and H. Motoyoshi introduced $[m, C]$-isometric operators with conjugation C as follows; For an operator $T \in B(H)$ and an integer $m \geq 1$, T is said to be an $[m, C]$-isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j} = 0.$$

For an operator $T \in B(H)$ and a conjugation C, define the operator $\lambda_m(T)$ by

$$\lambda_m(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j}.$$

Then T is an $[m, C]$-isometric operator if and only if

$$\lambda_m(T) = 0.$$

Moreover,

$$CTC.\lambda_m(T).T - \lambda_m(T) = \lambda_{m+1}(T)$$

holds. Hence, an $[m, C]$-isometric operator is an $[n, C]$-isometric operator for every $n \geq m$.

According to the definitions of m-isometric, (m, C)-isometric, (∞, C)-isometric and $[m, C]$-isometric operators, we introduce $[\infty, C]$-isometric operators T as follows; An operator T is said to be an $[\infty, C]$-isometric operator if

$$\limsup_{m\to\infty} \|\lambda_m(T)\|^{\frac{1}{m}} = 0.$$

An operator $T \in B(H)$ is called a finite $[m, C]$-isometric operator with conjugation C if T is an $[m, C]$-isometric operator for some $m \geq 1$. The class of $[\infty, C]$-isometric operators is a large class which contains finite $[m, C]$-isometric operators with conjugation C.

In this paper we introduce the class of $[\infty, C]$-isometric operators and study various properties of this class. In particular, we show that if T is an $[\infty, C]$-isometric operator, Q is a quasi-nilpotent operator which satisfy $TQ = QT$, then $T + Q$ is an $[\infty, C]$-isometric operator. Also, we prove that the class of $[\infty, C]$-isometric operators is norm closed. Finally, we investigate properties of products of $[\infty, C]$-isometric operators.

2. $[\infty, C]$-isometric operators

We next investigate the properties of $[\infty, C]$-isometric operators.

Theorem 2.1. *Let $T \in B(H)$. Then the following statements hold:*
(i) If T satisfies $T = CTC$, then

$$\limsup_{m\to\infty} \|\lambda_m(T)\|^{\frac{1}{m}} = r(T^2 - I)$$

where $r(A)$ denotes the spectral radius of A. In particular, if $r(T^2 - I) = 0$, the T is an $[\infty, C]$-isometric operator.
(ii) If T is a strict contraction, i.e., $\|T\| < 1$, then T is not an $[\infty, C]$-isometric operator.

Proof. (i) Since $T = CTC$, we have

$$\lambda_m(T) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C.T^{m-j} = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{2(m-j)},$$

and therefore

$$\|\lambda_m(T)\| = \|\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{2(m-j)}\| = \|(T^2 - I)^m\|,$$

and hence

$$\limsup_{m\to\infty}\|\lambda_m(T)\|^{\frac{1}{m}} = \limsup_{m\to\infty}\|(T^2 - I)^m\|^{\frac{1}{m}} = r(T^2 - I).$$

In particular, if $r(T^2 - I) = 0$, then T is an $[\infty,C]$-isometric operator.

(ii) Suppose that T is an $[\infty,C]$-isometric operator. Then $CTC.T \neq I$. Indeed, if T is a $[1,C]$-isometric operator, then

$$1 > \|T\|^2 = \|C\|\|T\|\|C\|\|T\| \geq \|CTC.T\| = \|I\| = 1,$$

which is a contradiction. Since

$$CTC.\lambda_m(T).T - \lambda_m(T) = \lambda_{m+1}(T),$$

we have

$$\|\lambda_m(T)\| \leq \|T\|^2\|\lambda_m(T)\| + \|\lambda_{m+1}(T)\|.$$

Thus

$$(1 - \|T\|^2)\|\lambda_m(T)\| \leq \|\lambda_{m+1}(T)\|$$

for $m \in \mathbb{N}$. Therefore, we obtain that

$$(1 - \|T\|^2)^m\|\lambda_1(T)\| \leq \|\lambda_{m+1}(T)\|,$$

and so

$$(1 - \|T\|^2)^{\frac{m}{m+1}}\|\lambda_1(T)\|^{\frac{1}{m+1}} \leq \|\lambda_{m+1}(T)\|^{\frac{1}{m+1}}.$$

Since T is an $[\infty,C]$-isometric operator and $\lambda_1(T) \neq 0$, by taking $\limsup_{m\to\infty}$, we derive that $1 - \|T\|^2 \leq 0$. Thus $\|T\| \geq 1$. So we have a contradiction. □

Lemma 2.2. *Let $T, Q \in B(H)$ satisfy $TQ = QT$. Then, for $m \geq 2$,*

$$\|\lambda_m(T + Q)\| \leq K^m(\max_{l\leq n\leq m}\|\lambda_n(T)\| + \max_{l\leq n\leq m}\|Q^n\|)$$

where $K = 2((\|T\| + \|Q\|)^2 + \|T\| + \|Q\| + 1)$ and $l = [\frac{m}{3}]$ is the integer part of $\frac{m}{3}$.

Proof. Since

$$\begin{aligned}[(a + b)(c + d) - 1]^m &= [(ac - 1) + (a + b)d + bc]^m \\ &= \sum_{m_1+m_2+m_3=m}\binom{m}{m_1,m_2,m_3}(a + b)^{m_1}b^{m_2}(ac - 1)^{m_3}c^{m_2}d^{m_1},\end{aligned}$$

we have

$$\lambda_m(T + Q) = \sum_{m_1+m_2+m_3=m}\binom{m}{m_1,m_2,m_3}(CTC + CQC)^{m_1}CQ^{m_2}C.\lambda_{m_3}(T)T^{m_2}Q^{m_1}.$$

Suppose that $l = [\frac{m}{3}]$ is the integer part of $\frac{m}{3}$. Put

$$M_i = \sum_{m_1+m_2+m_3=m, m_i\geq l}\binom{m}{m_1,m_2,m_3}\|(CTC + CQC)^{m_1}CQ^{m_2}C.\lambda_{m_3}(T)T^{m_2}Q^{m_1}\|$$

for $i = 1, 2, 3$. Since $m_1 + m_2 + m_3 = m$, it follows that $m_j \geq l$ for some $j = 1, 2, 3$. Therefore, we obtain that

$$\begin{aligned}
&\|\lambda_m(T+Q)\| \\
&\quad \leq \sum_{m_1+m_2+m_3=m} \binom{m}{m_1,m_2,m_3} \|(CTC + CQC)^{m_1} CQ^{m_2} C.\lambda_{m_3}(T) T^{m_2} Q^{m_1}\| \\
&\quad \leq M_1 + M_2 + M_3.
\end{aligned}$$

On the other hand, since $\|C\| = 1$, we get that

$$\begin{aligned}
M_3 &= \sum_{m_1+m_2+m_3=m, m_3 \geq l} \binom{m}{m_1,m_2,m_3} \|(CTC + CQC)^{m_1} CQ^{m_2} C.\lambda_{m_3}(T) T^{m_2} Q^{m_1}\| \\
&\leq \sum_{m_1+m_2+m_3=m, m_3 \geq l} \binom{m}{m_1,m_2,m_3} (\|T\| + \|Q\|)^{m_1} \|Q\|^{m_2} . \|\lambda_{m_3}(T)\| \|T\|^{m_2} \|Q\|^{m_1} \\
&\leq \max_{l \leq n \leq m} \|\lambda_n(T)\| . \sum_{m_1+m_2+m_3=m} \binom{m}{m_1,m_2,m_3} (\|T\| + \|Q\|)^{m_1} \|Q\|^{m_2} \|T\|^{m_2} \|Q\|^{m_1} \\
&= \max_{l \leq n \leq m} \|\lambda_n(T)\| . ((\|T\| + \|Q\|)\|Q\| + \|T\|\|Q\| + 1)^m \\
&\leq \max_{l \leq n \leq m} \|\lambda_n(T)\| . (\frac{K}{2})^m.
\end{aligned}$$

Since $\|\lambda_k(T)\| \leq (\|T\|^2 + 1)^k$ for all $k \in \mathbb{N}$, by the similar fashion, we obtain

$$\begin{aligned}
M_1 &\leq \max_{l \leq n \leq m} \|Q^n\| . ((\|T\| + \|Q\|) + \|T\|\|Q\| + (\|T\|^2 + 1))^m \\
&\leq \max_{l \leq n \leq m} \|Q^n\| . (\frac{K}{2})^m
\end{aligned}$$

and

$$\begin{aligned}
M_2 &\leq \max_{l \leq n \leq m} \|Q^n\| . ((\|T\| + \|Q\|)\|Q\| + \|T\| + (\|T\|^2 + 1))^m \\
&\leq \max_{l \leq n \leq m} \|Q^n\| . (\frac{K}{2})^m,
\end{aligned}$$

then

$$\begin{aligned}
\|\lambda_m(T + Q)\| &\leq \max_{l \leq n \leq m} \|\lambda_n(T)\| . (\frac{K}{2})^m + 2 \max_{l \leq n \leq m} \|Q^n\| . (\frac{K}{2})^m \\
&\leq K^m (\max_{l \leq n \leq m} \|\lambda_n(T)\| + \max_{l \leq n \leq m} \|Q^n\|).
\end{aligned}$$

Hence this completes the proof. □

Theorem 2.3. *Let $T \in B(H)$ and let C be a conjugation on H. Then the following assertions hold:*
(i) If T is an $[\infty,C]$-isometric operator and Q is a quasi-nilpotent operator which satisfy $TQ = QT$, then $T + Q$ is an $[\infty,C]$-isometric operator.
(ii) If T_n is a sequence of commuting $[\infty,C]$-isometric operators such that $\lim_{n\to\infty} \|T_n - T\| = 0$, then T is an $[\infty,C]$-isometric operator.

Proof. (i) Since T is an $[\infty,C]$-isometric operator and Q is a quasi-nilpotent operator, it follows that for given $0 < \varepsilon < 1$, there exists $\mathbb{N}$ such that

$$\|\lambda_n(T)\| \leq \varepsilon^n \text{ and } \|Q^n\| \leq \varepsilon^n$$

for all $n \geq \mathbb{N}$. By Lemma 2.2, for $m \geq 3\mathbb{N}$ and $l = [\frac{m}{3}] \geq \mathbb{N}$, we have that

$$\begin{aligned}\|\lambda_m(T+Q)\|^{\frac{1}{m}} &\leq K(\max_{l\leq n\leq m}\|\lambda_n(T)\| + \max_{l\leq n\leq m}\|Q^n\|)^{\frac{1}{m}} \\ &\leq K(2\varepsilon^n)^{\frac{1}{m}} \leq K(2\varepsilon^l)^{\frac{1}{m}} \\ &= 2^{\frac{1}{m}}K\varepsilon^{\frac{l}{m}} = 2^{\frac{1}{m}}K\varepsilon^{\frac{1}{m}[\frac{m}{3}]}.\end{aligned}$$

Since ε is arbitrary, $\limsup_{m\to\infty}\|\lambda_m(T+Q)\|^{\frac{1}{m}} = 0$. Hence $T+Q$ is an $[\infty,C]$- isometric operator.

(ii) If $T_nT_k = T_kT_n$ for all $k, n \in \mathbb{N}$, then $TT_n = T_nT$ for all $n \geq 1$. For a given $0 < \varepsilon < 1$, there exists n_0 such that

$$\|T - T_{n_0}\| \leq \varepsilon \text{ and } \|\lambda_n(T_{n_0})\| \leq \varepsilon^n$$

for all $n \geq n_0$. By Lemma 2.2, for $m \geq 3n_0$ and $l = [\frac{m}{3}] \geq n_0$, we obtain that

$$\begin{aligned}\|\lambda_m(T)\|^{\frac{1}{m}} &= \|\lambda_m(T_{n_0} + T - T_{n_0})\|^{\frac{1}{m}} \\ &\leq K(\max_{l\leq n\leq m}\|\lambda_n(T_{n_0})\| + \max_{l\leq n\leq m}\|T - T_{n_0}\|^n)^{\frac{1}{m}} \\ &\leq 2^{\frac{1}{m}}K\varepsilon^{\frac{l}{m}} = 2^{\frac{1}{m}}K\varepsilon^{\frac{1}{m}[\frac{m}{3}]}.\end{aligned}$$

Since ε is arbitrary, it follows that $\limsup_{m\to\infty}\|\lambda_m(T)\|^{\frac{1}{m}} = 0$. Hence T is an $[\infty,C]$-isometric operator. □

We illustrate the following example by Theorem 2.3 (ii).

Example 2.4. *Let C_n be the conjugation on $\mathbb{C}^n$ defined by*

$$C_n(z_1, z_2, \cdots, z_n) := (\overline{z_1}, \overline{z_2}, \cdots, \overline{z_n}).$$

Assume that $T = \oplus_{n=1}^{\infty} T_n$, where T_n is an $n \times n$ matrix;

$$T_n = I_n + N_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{n} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n} & 0 \end{pmatrix}.$$

Since N_n is nilpotent of order n, then T_n is a $[2n-1, C_n]$-isometric operator by [9]. Hence T is an $[\infty, C]$-isometric operator with a conjugation $C = \oplus_{n=1}^{\infty} C_n$. Indeed, if $R_n = T_1 \oplus \cdots \oplus T_n \oplus I \oplus I \oplus \cdots$, then R_n is a $[2n-1, C]$-isometric operator and $R_nR_k = R_kR_n$ for all $n, k \geq 1$. Since $R_n \to T$ in the operator norm, it follows from Theorem 2.3(ii) that T is an $[\infty, C]$-isometric operator with a conjugation $C = \oplus_{n=1}^{\infty} C_n$.

Finally, we study properties of products of $[\infty,C]$-isometric operators.

Lemma 2.5. *Let $T, S \in B(H)$ satisfy $TS = ST$ and $T(CSC) = (CSC)T$. Then*

$$\lambda_m(TS) = \sum_{j=0}^{m} \binom{m}{j} CT^jC\lambda_{m-j}(T)T^j\lambda_j(S)$$

where $\lambda_0(*) = I$.

Proof. Assume that $TS = ST$ and $T(CSC) = (CSC)T$. Since

$$\begin{aligned}(abcd-1)^m &= [(ab-1)+a(cd-1)b]^m \\ &= \sum_{j=0}^{m}\binom{m}{j}a^j(ab-1)^{m-j}b^j(cd-1)^j,\end{aligned}$$

we have

$$\begin{aligned}\lambda_m(TS) &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}C(TS)^{m-j}C(TS)^{m-j} \\ &= \sum_{j=0}^{m}\binom{m}{j}CT^jC\lambda_{m-j}(T)T^j\lambda_j(S)\end{aligned}$$

where $\lambda_0(*) = I$. □

Theorem 2.6. *Let T and S be [∞,C]-isometric operators. Assume that $TS = ST$ and $T(CSC) = (CSC)T$. Then TS is an [∞,C]-isometric operator.*

Proof. Assume that T and S are [∞,C]-isometric operators. Then for a given $0 < \varepsilon < 1$, there exist $\mathbb{N}_1$ and $\mathbb{N}_2$ such that

$$\|\lambda_{n_1}(T)\| \le \varepsilon^n \text{ and } \|\lambda_{n_2}(S)\| \le \varepsilon^n$$

for $n_1 \ge \mathbb{N}_1$ and $n_2 \ge \mathbb{N}_2$. Put $\mathbb{N} = \max\{\mathbb{N}_1, \mathbb{N}_2\}$. Then it's sufficient to show that there exists a constant $K > 0$ such that for $m \ge 2\mathbb{N}$,

$$\|\lambda_m(TS)\| \le K^m\varepsilon^{\frac{m}{2}}.$$

Let $l = [\frac{m}{2}]$ denote the integer part of $\frac{m}{2}$. Then by Lemma 2.5

$$\begin{aligned}\lambda_m(TS) &= \sum_{j=0}^{l}\binom{m}{j}CT^jC\lambda_{m-j}(T)T^j\lambda_j(S) \\ &\quad + \sum_{j=l+1}^{m}\binom{m}{j}CT^jC\lambda_{m-j}(T)T^j\lambda_j(S).\end{aligned}$$

If $j \le l = [\frac{m}{2}]$, then $m - j \ge [\frac{m}{2}] = l \ge \mathbb{N}$, and so $\|\lambda_{m-j}(T)\| \le \varepsilon^{m-j} \le \varepsilon^l$. Since $\|C\| = 1$, it follows that $\|\lambda_j(S)\| \le (\|S\|^2 + 1)^j$ for all $j \ge 1$. Thus we have

$$\begin{aligned}&\|\sum_{j=0}^{l}\binom{m}{j}CT^jC\lambda_{m-j}(T)T^j\lambda_j(S)\| \\ &\le \sum_{j=0}^{l}\binom{m}{j}\|\lambda_{m-j}(T)\|\|CT^jC\|\|T^j\|\|\lambda_j(S)\| \\ &\le \sum_{j=0}^{l}\binom{m}{j}\varepsilon^{m-j}\|T\|^j\|T\|^j(\|S\|^2+1)^j \\ &\le \varepsilon^l\sum_{j=0}^{m}\binom{m}{j}\|T\|^{2j}(\|S\|^2+1)^j \\ &= \varepsilon^l(1+\|T\|^2(\|S\|^2+1))^m.\end{aligned}$$

Similarly, if $j \geq l+1 \geq \mathbb{N}$, then $\|\lambda_j(S)\| \leq \varepsilon^l$ and hence we have

$$\|\sum_{j=l+1}^{m} \binom{m}{j} CT^j C\lambda_{m-j}(T)T^j\lambda_j(S)\| \leq \varepsilon^l(\|T\|^2 + (\|T\|^2+1))^m.$$

Then for $m \geq 2\mathbb{N}$

$$\|\lambda_m(TS)\| \leq \varepsilon^{[\frac{m}{2}]}((1+\|T\|^2(\|S\|^2+1))^m + (\|T\|^2+(\|T\|^2+1))^m).$$

Thus $\limsup_{m\to\infty} \|\lambda_m(TS)\|^{\frac{1}{m}} = 0$. Hence TS is an [∞,C]-isometric operator. □

We illustrate the following example by Theorem 2.6.

Example 2.7. *Let $C : H \to H$ be the conjugation given by*

$$C(\sum_{n=1}^{\infty} x_n e_n) = \sum_{n=1}^{\infty} \overline{x_n} e_n$$

where $\{x_n\}$ is a sequence in $\mathbb{C}$ with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Suppose that $A, B \in B(H)$ satisfy $Ae_n = \alpha e_n$ and $Be_n = \beta_n e_{n+1}$ with $\beta_n = \frac{1}{n}$ for all $n \geq 1$. If $|\alpha|^2 = 1$, then A and $B + I$ are [∞,C]-isometric operators, and it is easy to compute

$$ACBCe_n = ACBe_n = AC(\beta_n e_{n+1}) = A\overline{\beta_n} e_{n+1} = \alpha\overline{\beta_n} e_{n+1}$$

and

$$CBCAe_n = CBC(\alpha e_n) = CB(\overline{\alpha} e_n) = C(\overline{\alpha}\beta_n e_{n+1}) = \alpha\overline{\beta_n} e_{n+1}.$$

Moreover, $ABe_n = A\beta_n e_{n+1} = \beta_n\alpha e_{n+1}$ and $BAe_n = B\alpha e_n = \alpha\beta_n e_{n+1}$. Hence $A(I + B)$ is an [∞,C]- isometric operator from Theorem 2.6.

Corollary 2.8. *Let T and S be [∞,C]-isometric operators with conjugation C. Suppose that $T(CTC) = (CTC)T$. Then the following statements hold.*
(i) If $TS = ST$, $T(CSC) = (CSC)T$ and $S(CSC) = (CSC)S$, then T^kS^j and S^jT^k are [∞,C]-isometric operators for any $k, j \in \mathbb{N}$.
(ii) T^n is an [∞,C]-isometric operator for any $n \in \mathbb{N}$.

Proof. (i) By Theorem 2.6, TS is an [∞,C]-isometric operator. It suffices to show that T^kS is an [∞,C]-isometric operator. Since $TS = ST$, $T(CSC) = (CSC)T$ and $T(CTC) = (CTC)T$, it follows that $T^{k-1}(TS) = (TS)T^{k-1}$ and $T^{k-1}(CTSC) = (CTC)(CSC)T^{k-1} = (CTSC)T^{k-1}$. By Theorem 2.6, $T^{k-1}TS = T^kS$ is an [∞,C]-isometric operator. Similarly, T^kS^j is an [∞,C]-isometric operator. Also, we can show that S^jT^k is an [∞,C]-isometric operator by a similar way.

(ii) It is easy to show T^n is an [∞,C]-isometric operator by (i). □

Theorem 2.9. *Let $T \in B(H)$. Then the following statements hold.*
(i) T is an [∞,C]-isometric operator if and only if T^* is an [∞,C]-isometric operator.
(ii) If T is an invertible [∞,C]-isometric operator, then T^{-1} is an [∞,C]-isometric operator.

Proof. (i) Suppose that T is an [∞,C]-isometric operator. Since $\lambda_m(T^*) = \sum_{j=0}^{m}(-1)^j CT^{*m-j}CT^{*m-j}$, it follows that

$$\begin{aligned} C\lambda_m(T^*)C &= \sum_{j=0}^{m}(-1)^j T^{*m-j}CT^{*m-j}C \\ &= (\lambda_m(T))^* \end{aligned}$$

i..e., $C\lambda_m(T^*)C = (\lambda_m(T))^*$. Therefore, we have

$$\begin{aligned}\limsup_{m\to\infty} \|\lambda_m(T^*)\|^{\frac{1}{m}} &= \limsup_{m\to\infty} \|C\lambda_m(T^*)C\|^{\frac{1}{m}} \\ &= \limsup_{m\to\infty} \|(\lambda_m(T))^*\|^{\frac{1}{m}} \\ &= \limsup_{m\to\infty} \|\lambda_m(T)\|^{\frac{1}{m}} \\ &= 0.\end{aligned}$$

Hence T^* is an $[\infty,C]$-isometric operator. The converse implication holds by the same way.

(ii) Note for any $a, b \in \mathbb{C}$,

$$a^m(1 - a^{-1}b^{-1})^m b^m = (ab - 1)^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} a^{m-j} b^{m-j}.$$

Take $a = CTC$ and $b = T$. Then we get

$$\lambda_m(T) = (-1)^m (CTC)^m \lambda_m(T^{-1}) T^m.$$

Therefore, so

$$(-1)^m (CTC)^{-m} \lambda_m(T) T^{-m} = \lambda_m(T^{-1}).$$

Hence

$$\limsup_{m\to\infty} \|\lambda_m(T^{-1})\|^{\frac{1}{m}} \leq \limsup_{m\to\infty} \|T^{-1}\| \|\lambda_m(T)\|^{\frac{1}{m}} \|T^{-1}\|.$$

So T^{-1} is an $[\infty,C]$-isometric operator. □

Corollary 2.10. *Let $T \in B(H)$ be an invertible $[\infty,C]$-isometric operator and $T(CTC) = (CTC)T$. Then T^{-n} and T^{*-n} are $[\infty,C]$-isometric operators for any $n \in \mathbb{N}$.*

Proof. The proof follows from Theorem 2.9 and Corollary 2.8. □

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