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# Selection Principles and Games in Bitopological Function Spaces

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**Abstract.** For a Tychonoff space $X$, we denote by $(C(X), \tau_k, \tau_p)$ the bitopological space of all real-valued continuous functions on $X$, where $\tau_k$ is the compact-open topology and $\tau_p$ is the topology of pointwise convergence. In the papers [6, 7, 13] variations of selective separability and tightness in $(C(X), \tau_k, \tau_p)$ were investigated. In this paper we continue to study the selective properties and the corresponding topological games in the space $(C(X), \tau_k, \tau_p)$.

## 1. Introduction

In the papers [1, 3, 4, 9–12, 14] the authors investigated the selectors of dense subsets of the space $C(X)$ of all real-valued continuous functions on a Tychonoff space $X$ with the topology $\tau_p$ of pointwise convergence and with the compact-open topology $\tau_k$. For a Tychonoff space $X$, we denote by $(C(X), \tau_k, \tau_p)$ the bitopological space. In the articles [6, 7, 13] variations of selective separability and tightness in $(C(X), \tau_k, \tau_p)$ were investigated. In this paper, we continue to study the selective properties and the corresponding topological games in the space $(C(X), \tau_k, \tau_p)$. The following selection properties for $(C(X), \tau_k, \tau_p)$ are considered.

$$S_1(\mathcal{D}^k, \mathcal{S}^p) = S_{fin}(\mathcal{D}^k, \mathcal{S}^p) \Rightarrow S_1(\mathcal{D}^k, \mathcal{D}^p) \Rightarrow S_{fin}(\mathcal{D}^k, \mathcal{D}^p)$$

For example, a space $(C(X), \tau_k, \tau_p)$ satisfies $S_1(\mathcal{D}^k, \mathcal{S}^p)$ (resp., $S_{fin}(\mathcal{D}^k, \mathcal{S}^p)$) if whenever $(D_n : n \in \mathbb{N})$ is a sequence of dense subsets of $C_k(X)$, one can take points $f_n \in D_n$ (resp., finite $F_n \subset D_n$) such that $\{f_n : n \in \mathbb{N}\}$ (resp., $\bigcup\{F_n : n \in \mathbb{N}\}$) is sequentially dense in $C_p(X)$. There is a topological game, denoted by $G_*(\mathcal{A}, \mathcal{B})$, corresponding to $S_*(\mathcal{A}, \mathcal{B})$.

In this paper, we give characterizations for the bitopological space $(C(X), \tau_k, \tau_p)$ to satisfy the selection properties and the corresponding games.

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## 2. Main Definitions and Notation

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set $X$. Then many topological properties are characterized in terms of the following classical selection principles:

$S_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n$, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each $n$, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

The following prototype of many classical properties is called "$\mathcal{A}$ choose $\mathcal{B}$" in [15].

$\binom{\mathcal{A}}{\mathcal{B}}$ : For each $\mathcal{U} \in \mathcal{A}$ there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$. In this paper we accept that $|\mathcal{V}| = \aleph_0$.

Then $S_{fin}(\mathcal{A},\mathcal{B})$ implies $\binom{\mathcal{A}}{\mathcal{B}}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of $X$ if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space $X$ is called:

• an $\omega$-cover (a $k$-cover) if each finite (compact) subset $C$ of $X$ is contained in an element of $\mathcal{U}$.

• a $\gamma$-cover (a $\gamma_k$-cover) if $\mathcal{U}$ is infinite and for each finite (compact) subset $C$ of $X$ the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

For a topological space $X$ we denote:

• $\mathcal{O}$ — the family of all open covers of $X$;
• $\Gamma$ — the family of all open $\gamma$-covers of $X$;
• $\Gamma_k$ — the family of all open $\gamma_k$-covers of $X$;
• $\Omega$ — the family of all open $\omega$-covers of $X$;
• $\mathcal{K}$ — the family of all open $k$-covers of $X$;
• $\mathcal{D}^k$ — the family of all dense subsets of $C_k(X)$;
• $\mathcal{D}^p$ — the family of all dense subsets of $C_p(X)$;
• $\mathcal{S}^k$ — the family of all sequentially dense subsets of $C_k(X)$;
• $\mathcal{S}^p$ — the family of all sequentially dense subsets of $C_p(X)$;
• $\mathbb{K}(X)$ — the family of all non-empty compact subsets of $X$;
• $\mathbb{F}(X)$ — the family of all non-empty finite subsets of $X$.

A space $X$ is said to be a $\gamma_k$-set if each $k$-cover $\mathcal{U}$ of $X$ contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_k$-cover of $X$ [5].

If $X$ is a space and $A \subseteq X$, then the sequential closure of $A$, denoted by $[A]_{seq}$, is the set of all limits of sequences from $A$. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space $X$ is called sequentially separable if it has a countable sequentially dense set. Clearly, every sequentially separable space is separable.

Let $X$ be a topological space, and $x \in X$. A subset $A$ of $X$ *converges* to $x$, $x = \lim A$, if $A$ is infinite, $x \notin A$, and for each neighborhood $U$ of $x$, $A \setminus U$ is finite. Consider the following collections:

• $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
• $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to $x$. So, simply $\Gamma_x$ may be the set of non-trivial convergent sequences to $x$.

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ without specifying $x$, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

So we have three types of topological properties of $(C(X), \tau_k, \tau_p)$ described through the selection principles of $X$ where the index $k$ means the compact-open topology and the index $p$ - the topology of pointwise convergence:

• local properties of the form $S_*(\Phi^k_x, \Psi^p_x)$;
• global properties of the form $S_*(\Phi^k, \Psi^p)$;
• semi-local properties of the form $S_*(\Phi^k, \Psi^p_x)$.

There is a game, denoted by $G_{fin}(\mathcal{A},\mathcal{B})$, corresponding to $S_{fin}(\mathcal{A},\mathcal{B})$; two players, ONE and TWO, play a round for each natural number $n$. In the $n$-th round ONE chooses a set $A_n \in \mathcal{A}$ and TWO responds with a finite subset $B_n$ of $A_n$. A play $A_1, B_1; ...; A_n, B_n; ...$ is won by TWO if $\bigcup\limits_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise, ONE wins.

A strategy of a player is a function $\sigma$ from the set of all finite sequences of moves of the opponent into the set of (legal) moves of the strategy owner.

If ONE does not have a winning strategy in the game $G_*(\mathcal{A}, \mathcal{B})$, then the selection hypothesis $S_*(\mathcal{A}, \mathcal{B})$ is true; it is easy to prove. The converse implication is not always true.

Similarly, one defines the game $G_1(\mathcal{A}, \mathcal{B})$, associated with $S_1(\mathcal{A}, \mathcal{B})$.

So we have three types of topological games on $(C(X), \tau_k, \tau_p)$ described through the selection principles (or topological games) of $X$:

- local games of the form $G_*(\Phi^k_x, \Psi^p_x)$;
- global games of the form $G_*(\Phi^k, \Psi^p)$;
- semi-local games of the form $G_*(\Phi^k, \Psi^p_x)$.

The symbol $\mathbf{0}$ denotes the constantly zero function in $C(X)$. Since the compact-open topology coincides with the topology of uniform convergence on compact subsets of $X$, we can represent a basic neighborhood of the point $f \in C_k(X)$ as $\langle f, A, \epsilon\rangle$ where $\langle f, A, \epsilon\rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall x \in A\}$, $A$ is a compact subset of $X$ and $\epsilon > 0$.

## 3. $S_1(\mathcal{D}^k, \mathcal{D}^p)$ and $G_1(\mathcal{D}^k, \mathcal{D}^p)$

**Theorem 3.1.** ([13, Theorem 3.7] for $\lambda = k$ and $\mu = p$) *For a space X the following are equivalent:*

1. *$(C(X), \tau_k, \tau_p)$ has the property $S_1(\Omega^k_{\mathbf{0}}, \Omega^p_{\mathbf{0}})$;*
2. *$X$ has the property $S_1(\mathcal{K}, \Omega)$.*

Recall that the $i$-weight $iw(X)$ of a space $X$ is the smallest infinite cardinal number $\tau$ such that $X$ can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than $\tau$. Note that a space $X$ has a coarser second countable topology iff $iw(X) = \aleph_0$.

**Theorem 3.2.** (Noble [8]) *A space $C_k(X)$ is separable if and only if $iw(X) = \aleph_0$.*

Recall that a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is bidense (double dense or short $d$-dense) in $X$ if $A$ is dense in both $(X, \tau_1)$ and $(X, \tau_2)$ ([2]). $(X, \tau_1, \tau_2)$ is *$d$-separable* if there is a countable set $A$ which is $d$-dense in $X$. Note that if $iw(X) = \aleph_0$, then $(C(X), \tau_k, \tau_p)$ is $d$-separable.

**Theorem 3.3.** *Let X be a space with a coarser second countable topology. The following assertions are equivalent:*

1. *$(C(X), \tau_k, \tau_p)$ has the property $S_1(\mathcal{D}^k, \mathcal{D}^p)$;*
2. *$(C(X), \tau_k, \tau_p)$ has the property $S_1(\mathcal{D}^k, \Omega^p_0)$;*
3. *$(C(X), \tau_k, \tau_p)$ has the property $S_1(\Omega^k_0, \Omega^p_0)$;*
4. *$X$ has the property $S_1(\mathcal{K}, \Omega)$;*
5. *$(C(X), \tau_k, \tau_p)$ has the property $\binom{\mathcal{D}^k}{\mathcal{D}^p}$;*
6. *ONE has no winning strategy in the game $G_1(\mathcal{K}, \Omega)$;*
7. *ONE has no winning strategy in the game $G_1(\mathcal{D}^k, \mathcal{D}^p)$;*
8. *ONE has no winning strategy in the game $G_1(\Omega^k_{\mathbf{0}}, \Omega^p_0)$;*
9. *ONE has no winning strategy in the game $G_1(\mathcal{D}^k, \Omega^p_0)$.*

*Proof.* (1) $\Rightarrow$ (4). Let $(\mathcal{U}^k_i : i \in \mathbb{N})$ be a sequence of $k$-covers of $X$ and let $D = \{f_s : s \in \mathbb{N}\}$ be a countable dense set in $C_k(X)$. Consider $P_i := \{h^i_{L,W,f_s} \in C(X) : h^i_{L,W,f_s} \upharpoonright L = f_s \upharpoonright L, L \in \mathbb{K}(X), L \subset W, W \in \mathcal{U}^k_i, h^i_{L,W,f_s} \upharpoonright (X \setminus W) = 1, f_s \in D\}$. Note $P_i$ is a dense subset of $C_k(X)$ for each $i \in \mathbb{N}$. Indeed fix $f \in C(X), K \in \mathbb{K}(X), \epsilon > 0$. For $\langle f, K, \epsilon\rangle$ there exists $W_k \in \mathcal{U}^k_i$ and $f_s \in D$ such that $K \subset W_k$ and $f_s \in \langle f, K, \epsilon\rangle$. Take $h^i_{K,W_k,f_s} \in \langle f, K, \epsilon\rangle$.

Since $\{P_i : i \in \mathbb{N}\}$ is a countable set of dense sets of $C_k(X)$, by (1), there exists $\{p_i : i \in \mathbb{N}\}$ such that $p_i \in P_i$ and $\{p_i : i \in \mathbb{N}\}$ is a dense subset of $C_p(X)$. For $\{p_i = h^i_{L_i,W_i,f_{s_i}} : i \in \mathbb{N}\}$, we have that $\{W_i : i \in \mathbb{N}\}$ is an $\omega$-cover of $X$. Indeed, let $M = \{x_1, x_2, ..., x_k\} \in \mathbb{F}(X)$. Consider $U = \langle \mathbf{0}, M, (-\frac{1}{2}; \frac{1}{2})\rangle$, then there exists $i'$ such that $p_{i'} \in U$. It follows that $M \subset W_{i'}$.

(3) ⇒ (2) is immediate.

(4) ⇒ (3). By Theorem 3.1.

(2) ⇒ (1). Let $\{D_{i,j} : i, j \in \mathbb{N}\}$ be a countable set of dense sets in $C_k(X)$. Let $D = \{d_i : i \in \mathbb{N}\}$ be a countable dense set in $C_k(X)$. By $S_1(D^k, \Omega^p_{d_i})$ there exists $\{d_{i,j} : j \in \mathbb{N}\}$ such that $d_{i,j} \in D_{i,j}$ and $\{d_{i,j} : j \in \mathbb{N}\} \in \Omega^p_{d_i}$. Consider $M = \{d_{i,j} : i, j, \in \mathbb{N}\}$. The set $M$ is dense in $C_p(X)$. Fix $f \in C(X)$. Let $L = \{x_1, x_2, ..., x_n\} \in \mathbb{F}(X)$ and $\epsilon > 0$. The set $\langle f, L, \epsilon\rangle$ is a neighborhood of $f$, then there is $d_{i'} \in D$ such that $d_{i'} \in \langle f, L, \epsilon\rangle$, than there is $j'$ such that $d_{i',j'} \in \langle f, L, \epsilon\rangle$, hence $M \in D^p$

(6) ⇒ (4) is immediate.

(4) ⇒ (6). Let $\sigma$ be a strategy for ONE in $G_1(\mathcal{K}, \Omega)$ and let the first move of ONE be a $k$-cover $\sigma(\emptyset) = \{U_{(\alpha^1)} : \alpha^1 \in \Lambda^1\}$. Suppose that for each finite sequence $s$ of numbers $\alpha^i \in \Lambda^i$ of length at most $m$, $U_s$ has been already defined. Then define $\{U_{(\alpha^1,...,\alpha^m,\alpha^k)} : \alpha^k \in \Lambda^k\}$ to be the set $\sigma(U_{(\alpha^1)}, U_{(\alpha^1,\alpha^2)}, ..., U_{(\alpha^1,...,\alpha^m)}) \setminus \{U_{(\alpha^1)}, U_{(\alpha^1,\alpha^2)}, ..., U_{(\alpha^1,...,\alpha^m)}\}$. Because each compact subset of $X$ belongs to infinitely many elements of a $k$-cover, we have that, for each $s$, a finite sequence of numbers $\alpha^i \in \Lambda^i$, the set $\{U_{s\frown(\alpha^n)} : \alpha^n \in \Lambda^n\}$ is a $k$-cover. Apply (4) and, for each $s$, choose $\alpha^s \in \Lambda^s$ such that $\{U_{s\frown(\alpha^s)} : s$ a finite sequence of numbers $\alpha^i \in \Lambda^i, i \in \mathbb{N}\}$ is a $\omega$-cover of $X$. Then inductively define a sequence $\alpha^1 = \alpha^\emptyset, \alpha^{k+1} = \alpha^{(\alpha^1,...,\alpha^k)}$ for $k \geq 1$. Then $U_{\alpha^1}, U_{\alpha^1,\alpha^2}, ..., U_{\alpha^1,...,\alpha^k}, ...$ is a $\omega$-cover, and because it is, in fact, a sequence of moves TWO in a play of game $G_1(\mathcal{K}, \Omega)$, $\sigma$ is not a winning strategy for ONE.

Similarly to (4) ⇔ (6) we have that (1) ⇔ (7), (2) ⇔ (9) and (3) ⇔ (8). □

## 4. $S_{fin}(\mathcal{D}^k, \mathcal{D}^p)$ and $G_{fin}(\mathcal{D}^k, \mathcal{D}^p)$

**Theorem 4.1.** ([13, Theorem 3.9] for $\lambda = k$ and $\mu = p$) *For a space X the following are equivalent:*

1. *$(C(X), \tau_k, \tau_p)$ has the property $S_{fin}(\Omega^k_{\mathbf{0}}, \Omega^p_{\mathbf{0}})$;*
2. *X has the property $S_{fin}(\mathcal{K}, \Omega)$.*

**Theorem 4.2.** *Let X be a space with a coarser second countable topology. The following assertions are equivalent:*

1. *$(C(X), \tau_k, \tau_p)$ has the property $S_{fin}(\mathcal{D}^k, \mathcal{D}^p)$;*
2. *$(C(X), \tau_k, \tau_p)$ has the property $S_{fin}(\mathcal{D}^k, \Omega^p_0)$;*
3. *$(C(X), \tau_k, \tau_p)$ has the property $S_{fin}(\Omega^k_0, \Omega^p_0)$;*
4. *X satisfies the selection principle $S_{fin}(\mathcal{K}, \Omega)$;*
5. *ONE has no winning strategy in the game $G_{fin}(\mathcal{K}, \Omega)$;*
6. *ONE has no winning strategy in the game $G_{fin}(\mathcal{D}^k, \mathcal{D}^p)$;*
7. *ONE has no winning strategy in the game $G_{fin}(\Omega^k_{\mathbf{0}}, \Omega^p_0)$;*
8. *ONE has no winning strategy in the game $G_{fin}(\mathcal{D}^k, \Omega^p_0)$.*

*Proof.* The implications are proved similarly to the proof of Theorem 3.3. □

## 5. $S_1(\mathcal{D}^k, \mathcal{S}^p)$ and $G_1(\mathcal{D}^k, \mathcal{S}^p)$

**Theorem 5.1.** ([3, Theorem 15]) *For a space X the following are equivalent:*

1. *$(C(X), \tau_k, \tau_p)$ has the property $S_1(\Omega^k_{\mathbf{0}}, \Gamma^p_{\mathbf{0}})$;*
2. *X has the property $S_1(\mathcal{K}, \Gamma)$.*

**Theorem 5.2.** ([4, Theorem 10]) *For a space X the following are equivalent:*

1. *X has the property $S_{fin}(\mathcal{K}, \Gamma)$;*
2. *X has the property $S_1(\mathcal{K}, \Gamma)$;*
3. *ONE has no winning strategy in the game $G_1(\mathcal{K}, \Gamma)$.*

**Theorem 5.3.** *Let $X$ be a space with a coarser second countable topology. The following assertions are equivalent:*

1. $(C(X), \tau_k, \tau_p)$ *has the property* $S_1(\mathcal{D}^k, \mathcal{S}^p)$;
2. $(C(X), \tau_k, \tau_p)$ *has the property* $\binom{\mathcal{D}^k}{\mathcal{S}^p}$;
3. *$X$ has the property* $S_1(\mathcal{K}, \Gamma)$;
4. $(C(X), \tau_k, \tau_p)$ *has the property* $S_{fin}(\mathcal{D}^k, \mathcal{S}^p)$;
5. *$X$ has the property* $S_{fin}(\mathcal{K}, \Gamma)$;
6. *Each finite power of $X$ has the property* $S_1(\mathcal{K}, \Gamma)$;
7. $(C(X), \tau_k, \tau_p)$ *has the property* $S_1(\Omega^k_{\mathbf{0}}, \Gamma^p_{\mathbf{0}})$;
8. $(C(X), \tau_k, \tau_p)$ *has the property* $S_1(\mathcal{D}^k, \Gamma^p_{\mathbf{0}})$;
9. *$X$ has the property* $\binom{\mathcal{K}}{\Gamma}$;
10. *ONE has no winning strategy in the game* $G_1(\mathcal{K}, \Gamma)$;
11. *ONE has no winning strategy in the game* $G_1(\mathcal{D}^k, \mathcal{S}^p)$;
12. *ONE has no winning strategy in the game* $G_1(\Omega^k_{\mathbf{0}}, \Gamma^p_{\mathbf{0}})$;
13. *ONE has no winning strategy in the game* $G_1(\mathcal{D}^k, \Gamma^p_{\mathbf{0}})$.

*Proof.* By Theorem 5.1 ([3, Theorem 15]), (3) $\Leftrightarrow$ (7).

By Theorem 5.2 (Theorem 10 in [4]), (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (10).

By Theorem 14 in [3], (3) $\Leftrightarrow$ (9).

(3) $\Leftrightarrow$ (6) (Proposition 13 and Theorem 10 in [4]).

(1) $\Rightarrow$ (4) is immediate.

(7) $\Rightarrow$ (8) is immediate.

Similarly to (3) $\Leftrightarrow$ (10) (the implication (2) $\Rightarrow$ (3) in Theorem 10 in [4]) we have that (1) $\Leftrightarrow$ (11), (7) $\Leftrightarrow$ (12) and (8) $\Leftrightarrow$ (13).

(4) $\Rightarrow$ (2). Let $D$ be a dense subset of $C_k(X)$. By the property $S_{fin}(\mathcal{D}^k, \mathcal{S}^p)$, for sequence $(D_i : D_i = D$ and $i \in \mathbb{N})$ there is a sequence $(K_i : i \in \mathbb{N})$ such that for each $i$, $K_i$ is finite, $K_i \subset D_i$, and $\bigcup_{i\in\mathbb{N}} K_i$ is a countable sequentially dense subset of $C_p(X)$.

(2) $\Rightarrow$ (9). Let $\mathcal{U}$ be an open $k$-cover of $X$. Note that the set $\mathcal{D} := \{f \in C(X) : f \upharpoonright (X \setminus U) \equiv 1$ for some $U \in \mathcal{U}\}$ is dense in $C_k(X)$ and, hence, $\mathcal{D}$ contains a countable sequentially dense set $A$ in $C_p(X)$ . Take $\{f_n : n \in \mathbb{N}\} \subset A$ such that $f_n \mapsto \mathbf{0}$ $(n \mapsto \infty)$ in $C_p(X)$. Let $f_n \upharpoonright (X \setminus U_n) \equiv 1$ for some $U_n \in \mathcal{U}$. Then $\{U_n : n \in \mathbb{N}\}$ is a $\gamma$-subcover of $\mathcal{U}$, because of $f_n \mapsto \mathbf{0}$. Hence, $X$ satisfies $\binom{\mathcal{K}}{\Gamma}$.

(3) $\Rightarrow$ (1). Let $(D_{i,j} : i, j \in \mathbb{N})$ be a sequence of dense subsets of $C_k(X)$ and let $D = \{f_i : i \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$.

For every $f_i \in D$ and $j \in \mathbb{N}$ consider $\mathcal{U}_{i,j} = \{U_{h,i,j} : U_{h,i,j} = (f_i - h)^{-1}(-\frac{1}{j}, \frac{1}{j}) \wedge (U_{h,i,j} \neq \emptyset)$ for $h \in D_{i,j}\}$. Note that $\mathcal{U}_{i,j}$ is a $k$-cover of $X$ for every $i, j \in \mathbb{N}$. Since $X$ satisfies $S_1(\mathcal{K}, \Gamma)$, there is a sequence $(U_{h(i,j),i,j} : i, j \in \mathbb{N})$ such that $U_{h(i,j),i,j} \in \mathcal{U}_{i,j}$, and $\phi := \{U_{h(i,j),i,j} : i, j \in \mathbb{N}\}$ is an element of $\Gamma$.

We claim that $\{h(i, j) : i, j \in \mathbb{N}\}$ is a sequentially dense subset of $C_p(X)$.

Fix $g \in C(X)$. There exists $(f_{i_k} : k \in \mathbb{N})$ such that $f_{i_k} \to g$ $(k \to \infty)$ in $\tau_p$. Then $(g - f_{i_k}) \to \mathbf{0}$ in $\tau_p$. Show that $h(i_k, j) \to g$ in $\tau_p$. Let $W = \langle g, A, \epsilon\rangle$ be a base neighborhood of $g$ in $C_p(X)$, where $A \in \mathbb{F}(X)$ and $\epsilon > 0$. Since $\phi$ is a $\gamma$-cover of $X$, then $\{U_{h(i_k,j),i_k,j} : k, j \in \mathbb{N}\}$ is a $\gamma$-cover of $X$, too. There exists $k', j'$ such that $\frac{1}{j'} < \frac{\epsilon}{2}$ and for every $k > k'$, $j > j'$ the following statements are true: $(g - f_{i_k})(A) \subset (-\frac{\epsilon}{2}; \frac{\epsilon}{2})$ and $(f_{i_k} - h(i_k, j))(A) \subset (-\frac{1}{j'}; \frac{1}{j'}) \subset (-\frac{\epsilon}{2}; \frac{\epsilon}{2})$. Notice, that $((g - f_{i_k}) + (f_{i_k} - h(i_k, j)))(A) = (g - h(i_k, j))(A) \subset (-\epsilon; \epsilon)$. Then $h(i_k, j) \subset W$ for every $k > k'$, $j > j'$.

(8) $\Rightarrow$ (3). Let $\{\mathcal{U}_i : i \in \mathbb{N}\} \subset \mathcal{K}$ and let $D = \{d_j : j \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$. Consider $D_i = \{f_{K,U,i,j} \in C(X)$ : such that $f_{K,U,i,j} \upharpoonright K \equiv d_j$, $f_{K,U,i,j} \upharpoonright (X \setminus U) \equiv 1$ where $K \in \mathbb{K}(X)$, $K \subset U \in \mathcal{U}_i\}$ for every $i \in \mathbb{N}$. Since $D$ is a dense subset of $C_k(X)$, then $D_i$ is a dense subset of $C_k(X)$ for every $i \in \mathbb{N}$. By (8), there is a set $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\}$ such that $f_{K(i),U(i),i,j(i)} \in D_i$ and $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma^p_{\mathbf{0}}$. Claim that the set $\{U(i) : i \in \mathbb{N}\} \in \Gamma$. Let $K \in \mathbb{F}(X)$ and let $W = \langle \mathbf{0}, K, \frac{1}{2}\rangle$ be a base neighborhood of $\mathbf{0}$. Since $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma^p_{\mathbf{0}}$, there is $i' \in \mathbb{N}$ such that $f_{K(i),U(i),i,j(i)} \in W$ for every $i > i'$. It follows that $K \subset U(i)$ for every $i > i'$ and hence $\{U(i) : i \in \mathbb{N}\} \in \Gamma$. $\square$

We can summarize the relationships between considered notions in next diagrams.

$$\begin{array}{ccccccc}
G_1(\mathcal{D}^k,\Gamma^p_{\mathbf{0}}) & \Leftrightarrow & G_{fin}(\mathcal{D}^k,\Gamma^p_{\mathbf{0}}) & \Rightarrow & G_1(\mathcal{D}^k,\Omega^p_0) & \Rightarrow & G_{fin}(\mathcal{D}^k,\Omega^p_0)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
G_1(\Omega^k_0,\Gamma^p_{\mathbf{0}}) & \Leftrightarrow & G_{fin}(\Omega^k_0,\Gamma^p_{\mathbf{0}}) & \Rightarrow & G_1(\Omega^k_0,\Omega^p_0) & \Rightarrow & G_{fin}(\Omega^k_0,\Omega^p_0)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
G_1(\mathcal{D}^k,\mathcal{S}^p) & \Leftrightarrow & G_{fin}(\mathcal{D}^k,\mathcal{S}^p) & \Rightarrow & G_1(\mathcal{D}^k,\mathcal{D}^p) & \Rightarrow & G_{fin}(\mathcal{D}^k,\mathcal{D}^p)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
S_1(\mathcal{D}^k,\mathcal{S}^p) & \Leftrightarrow & S_{fin}(\mathcal{D}^k,\mathcal{S}^p) & \Rightarrow & S_1(\mathcal{D}^k,\mathcal{D}^p) & \Rightarrow & S_{fin}(\mathcal{D}^k,\mathcal{D}^p)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
S_1(\mathcal{D}^k,\Gamma^p_{\mathbf{0}}) & \Leftrightarrow & S_{fin}(\mathcal{D}^k,\Gamma^p_{\mathbf{0}}) & \Rightarrow & S_1(\mathcal{D}^k,\Omega^p_0) & \Rightarrow & S_{fin}(\mathcal{D}^k,\Omega^p_0)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
S_1(\Omega^k_0,\Gamma^p_{\mathbf{0}}) & \Leftrightarrow & S_{fin}(\Omega^k_0,\Gamma^p_{\mathbf{0}}) & \Rightarrow & S_1(\Omega^k_0,\Omega^p_0) & \Rightarrow & S_{fin}(\Omega^k_0,\Omega^p_0)
\end{array}$$

Fig. 1. The Diagram of games and selectors of $(C(X),\tau_k,\tau_p)$.

$$\begin{array}{ccccccc}
G_1(\mathcal{K},\Gamma) & \Leftrightarrow & G_{fin}(\mathcal{K},\Gamma) & \Rightarrow & G_1(\mathcal{K},\Omega) & \Rightarrow & G_{fin}(\mathcal{K},\Omega)\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow\\
S_1(\mathcal{K},\Gamma) & \Leftrightarrow & S_{fin}(\mathcal{K},\Gamma) & \Rightarrow & S_1(\mathcal{K},\Omega) & \Rightarrow & S_{fin}(\mathcal{K},\Omega)
\end{array}$$

Fig. 2. The Diagram of games and selection principles for a space $X$ with $iw(X)=\aleph_0$ corresponding to selectors of $(C(X),\tau_k,\tau_p)$.

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