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# Summability of Subsequences of a Divergent Sequence by Regular Matrices II

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*Dedicated to the memory of Professor Harry I. Miller*

**Abstract.** C. Stuart proved in [27, Proposition 7] that the Cesàro matrix $C_1$ cannot sum almost every subsequence of a bounded divergent sequence. At the end of the paper he remarked *'It seems likely that this proposition could be generalized for any regular matrix, but we do not have a proof of this'*. In [4, Theorem 3.1] Stuart's conjecture is confirmed, and it is even extended to the more general case of divergent sequences. In this note we show that [4, Theorem 3.1] is a special case of Theorem 3.5.5 in [24] by proving that the set of all index sequences with positive density is of the second category. For the proof of that a decisive hint was given to the author by Harry I. Miller a few months before he passed away on 17 December 2018.

## 1. Introduction

Throughout this note we assume familarity with summability and the standard sequence spaces (see e. g. [3, 28]). So we denote by $\omega$ , $\ell_\infty$, $c$, $c_0$, and $\ell$ the set of all sequences in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), of all bounded sequences, of all convergent sequences, of all sequences converging to 0, and of all absolutely summable sequences, respectively.

If $A = (a_{nk})$ is an infinite matrix with scalar entries, then we consider the *domain*

$$c_A := \left\{ (x_k) \in \omega \;\middle|\; \sum_{k=1}^{\infty} a_{nk}\, x_k \text{ converges for each } n \in \mathbb{N} \text{ and } Ax := \left( \sum_{k=1}^{\infty} a_{nk}\, x_k \right)_n \in c \right\}$$

of $A$. The matrix (method) $A$ is called *regular*, if $c \subseteq c_A$ and $\lim_A x := \lim Ax = \lim x$ ($x \in c$). A characterization of regular matrices is contained in the wellknown Theorem of Toeplitz, Silverman, Kojima and Schur (cf. [3, Th. 2.3.7 II]). The *Cesàro matrix* $C_1 = (c_{nk})$ with $c_{nk} := \frac{1}{n}$ if $1 \le k \le n$ ($k, n \in \mathbb{N}$) and $c_{nk} := 0$ otherwise is certainly the most famous example of a regular matrix.

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## 2. Preliminary considerations and known results

Steinhaus stated in [26] that a regular matrix cannot sum all sequences of $0's$ and $1's$ for which Connor gave in [7] a very short proof based on the Baire Classification Theorem. In particular, the Steinhaus Theorem implies obviously that a regular matrix cannot sum all bounded sequences which is also a corollary of the Schur Theorem (cf. [25] and [3, Corollary 2.4.2]). Moreover, the Hahn Theorem (cf. [10], [3, Theorem 2.4.5]) says that a matrix sums all bounded sequences if it sums all sequences of $0's$ and $1's$.

More general we may determine (small) subsets $Q$ of $\ell_\infty \setminus c$ or of $\omega \setminus c$ such that each regular matrix cannot sum all $x \in Q$.

A related problem is based on the question *'How many subsequences of a given divergent sequence can be summed by a given regular matrix (or by any regular matrix)?'*. This question makes sense as the following result shows.

**Proposition 2.1 (cf. [5, Theorem], [27, Theorem 5]).** *If $x$ is any bounded divergent sequence, then each regular matrix cannot sum all subsequences of $x$.* (Note, by [24, 3.5.5], the statement holds even for divergent sequences.)

Now, let $\mathcal{I}$ denote the set of all index sequences[1)] $(n_i)$ and $x = (x_n)$ be any bounded divergent or divergent sequence. Then we ask for (small) subsets $\mathbf{Q}$ of $\mathcal{I}$ such that a special regular matrix like $C_1$ or, more generally, each regular matrix cannot sum all subsequences $(x_{n_i})$ of $x$ with $(n_i) \in \mathbf{Q}$.

Following Stuart in [27] we consider any set of subsequences (of a bounded divergent sequence) that have index sets with positive density:

**Definition 2.2 (positive density).** Given a set $S \subseteq \mathbb{N}$ and let $S_n := S \cap \mathbb{N}_n$, then the *density of $S$* is defined by[2)] $d(S) := \limsup_n \frac{|S_n|}{n}$ where $|Y|$ denotes the cardinality of any set $Y$. (Note $0 \le d(S) \le 1$.) A property holds for ***almost every*** *subsequence of a given sequence* if it holds for all the subsequences that have index sets with positive density.[3)] For any $(n_i) \in \mathcal{I}$ and $n \in \mathbb{N}$ we set $S_{(n_i),n} := \{i \mid n_i \le n\}$.

In this sense we consider in the following the set

$$\mathbf{Q} := \left\{ (n_i) \in \mathcal{I} \;\middle|\; d(\{n_i \mid i \in \mathbb{N}\}) > 0 \right\}. \tag{1}$$

In particular, we get

$${}^c\mathbf{Q} = \left\{ (n_i) \in \mathcal{I} \;\middle|\; d(\{n_i \mid i \in \mathbb{N}\}) = 0 \right\} = \left\{ (n_i) \in \mathcal{I} \;\middle|\; \left( \frac{|S_{(n_i),n}|}{n} \right) \in c_0 \right\}. \tag{2}$$

Stuart presented following two results for the case $\mathbf{Q}$:

**Proposition 2.3 (cf. [27, Proposition 6]).** *The matrix $C_1$ cannot sum almost every subsequence of any sequence of $0's$ and $1's$.*

**Proposition 2.4 (cf. [27, Proposition 7]).** *The matrix $C_1$ cannot sum almost every subsequence of any bounded divergent sequence.*

Boos and Zeltser proved in [4] that Stuarts result holds in general for any regular matrix and any divergent sequence:

**Theorem 2.5 (cf. [4, Theorem 3.1]).** *Let $A = (a_{nk})$ be a regular matrix. Then $A$ cannot sum almost every subsequence of any divergent sequence $x = (x_k)$.*

In this note we'll show that the last Theorem is a special case of Theorem 3.5.5 in [24].

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[1)] By definition, an index sequence is a strictly increasing sequence of natural numbers. For the sake of simplicity we consider also finite strictly increasing sequences as members of $\mathcal{I}$

[2)] Note that $d(S)$ is defined in [27] by $d(S) := \frac{1}{n} \limsup_n |S_n|$ which is with certainty an oversight.

[3)] Note that in some papers $d(S)$ is denoted as *'upper (asymptotic) density'* (cf. [2]).

## 3. A general result due to Keogh and Petersen

To quantify the set $Q$ of index sequences we used the positive density of the members of $Q$ which leads us —in case of a given sequence $x$— to the notion of 'almost all subsequences'[4)] of $x$.

In 1958 Keogh and Petersen trod another path to quantify sets $Q$ of index sequences:

**Definition 3.1.** Let $Q$ be any set of index sequences. Furthermore, if $(x_{n_k})$ is a subsequence of $(x_n)$, then we identify the index sequence $(n_k)$ with the number $\alpha \in [0,1]$ with binary code $.\alpha_1\alpha_2\alpha_3\dots$ defined by

$$\alpha_r := \begin{cases} 1 & \text{if } r = n_i \ (i \in \mathbb{N}), \\ 0 & \text{otherwise} \end{cases} \qquad (r \in \mathbb{N}). \tag{3}$$

In this way we identify a set $\widehat{Q}$ of subsequences of $(x_n)$ with the set $Q$ of the corresponding index sequences and —applying correspondingly (3)— $Q$ with the set

$$Q^* := \left\{ \alpha = .\alpha_1\alpha_2\alpha_3\dots \;\middle|\; \exists\,(n_k) \in Q \,:\, \alpha_r \text{ defined by (3)} \right\}$$

of the corresponding members of $[0,1]$ in binary code. Now we define $\widehat{Q}$ to be *of the second category* if $Q^*$ is of the second category in the set $[0,1]$; otherwise we call it *of the first category*.

**Theorem 3.2 (cf. [24, Theorem 3.5.5] and [15, Theorem on page 1]).** *Let $x = (x_n) \in \omega$ and $A$ be any given regular matrix. Then $A$ limits a set of subsequences of $x$, that is of the second category, if and only if $x \in c$.* (More exactly: If $\widehat{Q}$ is a set of subsequences of $x$, that is of the second category, then $A$ limits all members of $\widehat{Q}$ if and only if $x \in c$.)

**Remark 3.3.** The non-trivial part of the proof of Theorem 3.2 is done in the following way: *If $x$ is divergent and $A$ is regular, then*

$$F := \left\{ y \in c_A \;\middle|\; y \text{ is a subsequence of } x \right\}$$

*is of the first category.* (Thus, $F$ cannot contain any set of the second category.)

In Section 5 we proof that the result in Theorem 2.5 is contained in that of Theorem 3.2. More exactly, we show that, if $Q$ is the set of index sequences defined in (1), then the corresponding set $\widehat{Q}$ of subsequences of $x$ is of the second category (in the above sense).

## 4. Some density properties of $Q$ and ${}^cQ$

First of all, we remark that the set $\chi$ of all sequences of 0's and 1's is obviously a closed subset of the FK–space $\omega$ because convergence of sequences in $\chi$ is equivalent to coordinatewise convergence.

**Proposition 4.1.** *The set $Q$, identified as set of sequences of 0's and 1's, is dense in $\chi$ (provided with the topology induced by the FK–topology $\tau_\omega$ of $\omega$).*

*Proof.* For any $x \in \chi \setminus Q$ we define a sequence $(x^{(n)})$ with $x^{(n)} = (x^{(n)}_k) \in Q$ converging to an $x = (x_k)$ in $(\chi, \tau_\omega|_\chi)$. Let $x \in \chi$ be given, then we set

$$x^{(n)}_k := \begin{cases} x_k & \text{if } k \le n, \\ 1 & \text{if } k > n \end{cases} \qquad (k, n \in \mathbb{N}).$$

Obviously, $x^{(n)}$ is (coordinatewise) convergent to $x$, and $x^{(n)} \in Q$ for all $n \in \mathbb{N}$ since $x^{(n)}_k = 1$ if $k > n$. □

[4)] Probably, this notion of 'almost all subsequences' is not equivalent to that used in [6].

In the following considerations we identify the complement

$$^{c}\mathcal{Q} = \Big\{(n_i) \in \mathcal{I} \;\Big|\; d(\{n_i \mid i \in \mathbb{N}\}) = 0\Big\}$$

of $\mathcal{Q}$ in $\mathcal{I}$ with the corresponding set of sequences of 0's and 1's and —applying (3)— $^{c}\mathcal{Q}$ again with the set

$$^{c}\mathcal{Q}^{*} = \Big\{\alpha = .\alpha_1\alpha_2\alpha_3\ldots \;\Big|\; \exists\,(n_k) \in {}^{c}\mathcal{Q} \,:\, \alpha_r \text{ defined by (3)}\Big\}$$

of the corresponding members of $[0,1]$ in binary code.

**Proposition 4.2.** *Both,* $\mathcal{Q}^{*}$ *and* $^{c}\mathcal{Q}^{*}$ *are dense*[5)]*, thus somewhere dense, in* $[0,1]$.

*Proof.* For that we prove that the closure of $\mathcal{Q}^{*}$ as well as $^{c}\mathcal{Q}^{*}$ equals $[0,1]$.
Let $\alpha = .\alpha_1\alpha_2\alpha_3\ldots \in {}^{c}\mathcal{Q}^{*}$ be given and $v(\alpha) = \sum_r 2^{-r}\alpha_r$ denote the real value of $\alpha$. For any $t \in \mathbb{N}$ we consider $\beta^{(t)} = .\beta_1^{(t)}\beta_2^{(t)}\beta_3^{(t)}\ldots \in \mathcal{Q}^{*}$ defined by

$$\beta_i^{(t)} := \begin{cases} \alpha_i & \text{if } i \le t\,, \\ 1 & \text{otherwise} \end{cases} \qquad (i \in \mathbb{N}).$$

Then we have

$$\Big|v(\alpha) - v(\beta^{(t)})\Big| \;\le\; 2\cdot\sum_{i=t+1}^{\infty} 2^{-i} \;=\; 2^{-t+1} \;\overset{t\to\infty}{\longrightarrow}\; 0,$$

consequently $\alpha \in \overline{\mathcal{Q}^{*}}$, thus $\overline{\mathcal{Q}^{*}} = [0,1]$.
Now let $\alpha = .\alpha_1\alpha_2\alpha_3\ldots \in \mathcal{Q}^{*}$ be given and $v(\alpha) = \sum_r 2^{-r}\alpha_r$ denote the real value of $\alpha$. For any $t \in \mathbb{N}$ we consider $\beta^{(t)} = .\beta_1^{(t)}\beta_2^{(t)}\beta_3^{(t)}\ldots \in {}^{c}\mathcal{Q}^{*}$ defined by

$$\beta_i^{(t)} := \begin{cases} \alpha_i & \text{if } i \le t\,, \\ 0 & \text{otherwise} \end{cases} \qquad (i \in \mathbb{N}).$$

Then we have

$$\Big|v(\alpha) - v(\beta^{(t)})\Big| \;\le\; \sum_{i=t+1}^{\infty} 2^{-i} \;=\; 2^{-t} \;\overset{t\to\infty}{\longrightarrow}\; 0,$$

thus $\alpha \in \overline{{}^{c}\mathcal{Q}^{*}}$, therefore $\overline{{}^{c}\mathcal{Q}^{*}} = [0,1]$. □

**Proposition 4.3.** *For any* $s \in ]0,1]$ *let be*[6)]

$$\mathcal{D}_s := \Big\{(n_i) \in \mathcal{I} \;\Big|\; d(\{n_i \mid i \in \mathbb{N}\}) > s\Big\}, \quad \widetilde{\mathcal{D}_s} := \Big\{(n_i) \in \mathcal{I} \;\Big|\; d(\{n_i \mid i \in \mathbb{N}\}) \ge s\Big\}$$

*and, correspondingly,*

$$\begin{aligned} \mathcal{D}_s^{*} &:= \Big\{\alpha = .\alpha_1\alpha_2\alpha_3\ldots \;\Big|\; \exists\,(n_k) \in \mathcal{D}_s \,:\, \alpha_r \textit{ defined by } (3)\Big\}, \\ \widetilde{\mathcal{D}_s}^{*} &:= \Big\{\alpha = .\alpha_1\alpha_2\alpha_3\ldots \;\Big|\; \exists\,(n_k) \in \widetilde{\mathcal{D}_s} \,:\, \alpha_r \textit{ defined by } (3)\Big\}. \end{aligned}$$

*Then* $\mathcal{D}_s^{*}$, $\widetilde{\mathcal{D}_s}^{*}$, $^{c}\mathcal{D}_s^{*}$ *and* $^{c}\widetilde{\mathcal{D}_s}^{*}$ *are (everywhere) dense in* $]0,1]$.

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5) In [24] 'everywhere dense' is used instead of 'dense'

6) Obviously $\mathcal{D}_1 = \emptyset$ and $\widetilde{\mathcal{D}_1} = \{(n_i) \in \mathcal{I} \mid d(\{n_i \mid i \in \mathbb{N}\}) = 1\}$

*Proof.* Obviously,

$$^{c}\mathcal{D}_s = \Big\{(n_i) \in \mathcal{I} \ \Big| \ d(\{n_i \mid i \in \mathbb{N}\}) \leq s\Big\}.$$

Let $\alpha = .\alpha_1\alpha_2\alpha_3 \ldots \in {}^{c}\mathcal{D}_s^*$ be given and $v(\alpha) = \sum_r 2^{-r}\alpha_r$ denote the real value of $\alpha$. For any $t \in \mathbb{N}$ we consider $\beta^{(t)} = .\beta_1^{(t)}\beta_2^{(t)}\beta_3^{(t)} \ldots \in \mathcal{D}_s^*$ defined by

$$\beta_i^{(t)} := \begin{cases} \alpha_i & \text{if } i \leq t\,, \\ 1 & \text{otherwise} \end{cases} \qquad (i \in \mathbb{N}).$$

Then we have

$$\left|v(\alpha) - v(\beta^{(t)})\right| \ \leq \ 2 \cdot \sum_{i=t+1}^{\infty} 2^{-i} \ = \ 2^{-t+1} \ \overset{t\to\infty}{\longrightarrow} \ 0,$$

thus $\alpha \in \overline{\mathcal{D}_s^*}$, therefore $\overline{\mathcal{D}_s^*} = [0, 1]$.

The proofs of the density of $\widetilde{\mathcal{D}_s}^*$, ${}^{c}\mathcal{D}_s^*$ and ${}^{c}\widetilde{\mathcal{D}_s}^*$ are quite similar. □

**Remark 4.4.** Note

$$\mathcal{Q} = \bigcup_{s\in]0,1[} \mathcal{D}_s = \bigcup_{k=2}^{\infty} \mathcal{D}_{1/k} = \bigcup_{k=2}^{\infty} \widetilde{\mathcal{D}_{1/k}}, \quad \text{thus} \quad {}^{c}\mathcal{Q} = \bigcap_{s\in]0,1[} {}^{c}\mathcal{D}_s = \bigcap_{k=2}^{\infty} {}^{c}\mathcal{D}_{1/k} = \bigcap_{k=2}^{\infty} {}^{c}\widetilde{\mathcal{D}_{1/k}}.$$

## 5. Proof of the main result

Using similar constructions as in the not yet published draft [21] (cf. also [20] and [19]), we'll prove the main result of this note.

**Theorem 5.1.** *The set* ${}^{c}\mathcal{Q}^*$ *is of the first category in* $]0, 1[$. *In particular,* ${}^{c}\mathcal{Q}$ *is of the first category in the sense of Definition 3.1.*

**Corollary 5.2 (cf. Theorem 2.5).** *Let* $A = (a_{nk})$ *be a regular matrix. Then* $A$ *cannot sum almost every subsequence of any divergent sequence* $x = (x_k)$*.*

*Proof.* Apply Theorem 3.2 to the set $\widehat{Q} := \widehat{\mathcal{Q}}$ (cf. also Definition 3.1). □

*Proof of Theorem 5.1 .* First of all, we remark

$$\begin{aligned} {}^{c}\mathcal{Q}^* \ &= \ \{\alpha = .\alpha_1\alpha_2\alpha_3 \ldots \mid \exists\,(n_k) \in {}^{c}\mathcal{Q} : \ \alpha_r \text{ defined by (3)}\} \\ &\widehat{=} \ \left\{(n_j) \in \mathcal{I} \ \Big| \ \limsup_n \frac{|S_{(n_j),n}|}{n} = 0\right\} \\ &= \ \left\{(n_j) \in \mathcal{I} \ \Big| \ \lim_n \frac{|S_{(n_j),n}|}{n} = 0\right\} \quad \text{(cf. (2))} \\ &= \ \left\{(n_j) \in \mathcal{I} \ \Big| \ \forall\, k \geq 2\ \exists\,\Theta_k \geq 1\ \forall\, n \geq \Theta_k : \ \frac{|S_{(n_j),n}|}{n} < \frac{1}{k}\right\} \\ &= \ \bigcap_{k=2}^{\infty}\bigcup_{\Theta=1}^{\infty}\bigcap_{n=\Theta}^{\infty} \left\{(n_j) \in \mathcal{I} \ \Big| \ \frac{|S_{(n_j),n}|}{n} < \frac{1}{k}\right\}. \end{aligned}$$

Motivated by that, we define for $\Theta$, $k \in \mathbb{N}$, $k \geq 2$ the sets

$$\mathcal{P}_{\Theta,k} := \bigcap_{n=\Theta}^{\infty} \left\{ (n_j) \in \mathcal{I} \;\middle|\; \frac{|S_{(n_j),n}|}{n} < \frac{1}{k} \right\} = \left\{ (n_j) \in \mathcal{I} \;\middle|\; \forall n \geq \Theta : \frac{|S_{(n_j),n}|}{n} < \frac{1}{k} \right\},$$

thus

$$^{c}\mathcal{P}_{\Theta,k} = \left\{ (n_j) \in \mathcal{I} \;\middle|\; \exists\, n \geq \Theta : \frac{|S_{(n_j),n}|}{n} \geq \frac{1}{k} \right\},$$

and

$$\mathcal{P}^{*}_{\Theta,k} := \left\{ \alpha = .\alpha_1\alpha_2\alpha_3 \ldots \;\middle|\; \exists\,(n_j) \in \mathcal{P}_{\Theta,k} : \alpha_r \text{ defined by (3)} \right\}.$$

Now, we are going to prove that $\mathcal{P}^{*}_{\Theta,k}$ is nowhere dense for all $\Theta, k \in \mathbb{N}$, $k \geq 2$. (Consequently, as a countable union of nowhere dense subsets,

$$\bigcup_{\Theta=1}^{\infty} \mathcal{P}^{*}_{\Theta,k} \quad (k \geq 2)$$

is of the first category in $[0,1]$ (cf. [3, Remark 6.2.27]) and therefore, as a countable intersection of these sets of the first category, $^{c}\mathcal{Q}^{*} = \bigcap_{k=2}^{\infty} \bigcup_{\Theta=1}^{\infty} \mathcal{P}^{*}_{\Theta,k}$ is also of the first category in $]0,1[$.) For that, it is sufficient to prove that for all $\Theta,\ k \in \mathbb{N},\ k \geq 2$, and each Interval $I = ]a,b[ \subset [0,1]$ there exists an Interval $J \subseteq I$ such that $J \cap P^{*}_{\Theta,k} = \emptyset$.

Now, let any $k \in \mathbb{N},\ k \geq 2$, and $\Theta \in \mathbb{N}$ be given, and let $I = ]a,b[ \subset [0,1]$ be an interval with midpoint $\alpha^0 := .\alpha_1\alpha_2\alpha_3 \ldots$. Then we choose a $p \in \mathbb{N}$ such that all extensions of $\widetilde{\alpha^0} := .\alpha_1 \ldots \alpha_p 0 \ldots$ are members of $I$ (note $\widetilde{\alpha^0} \leq \alpha^0$). Now we extend $\widetilde{\alpha^0}$ to $\widetilde{\beta^0} := .\beta_1 \ldots \beta_p\beta_{p+1} \ldots \beta_{p+q} 0 \ldots$ where

$$\beta_r := \begin{cases} \alpha_r & \text{if } 1 \leq r \leq p, \\ 0 & \text{if } r = p+1 \text{ or } r > p+q, \\ 1 & \text{if } p+1 < r \leq p+q \end{cases} \qquad (r \in \mathbb{N})$$

such that

$$n := p + q > \Theta \quad \text{and} \quad \frac{|\{r \in \mathbb{N}_n \mid \beta_r = 1\}|}{n} > \frac{3}{4}.$$

After this, we consider the set

$$J := \left\{ \widetilde{\gamma} = .\gamma_1\gamma_2 \ldots \gamma_\nu \ldots \;\middle|\; \gamma_\nu := \beta_\nu \ (\nu \in \mathbb{N}_n) \text{ and } \gamma_\nu \in \{0,1\} \ (\nu > n) \right\}$$

of all extensions of $\widetilde{\beta^0}$. Obviously, $J$ is a subinterval of $I$ (containing more than one element) and each $\widetilde{\gamma} \in J$ fulfills obviously

$$\frac{|\{\nu \in \mathbb{N}_n \mid \gamma_\nu = 1\}|}{n} > \frac{3}{4} \qquad (n \in \mathbb{N}).$$

Therefore, $J \subseteq {}^{c}P^{*}_{\Theta,k}$, thus $J \cap P^{*}_{\Theta,k} = \emptyset$. Alltogether, $P^{*}_{\Theta,k}$ is nowhere dense in $]0,1[$. □

We close this section with some remarks and a question.

**Remarks 5.3.** Given a set $S \subseteq \mathbb{N}$ and let $S_n := S \cap \mathbb{N}_n$, then we may set $d_u(S) := d(S)$ (upper density), $d_l(S) := \liminf_n \frac{|S_n|}{n}$ (lower density), and

$$d_c(S) := \lim_n \frac{|S_n|}{n} \quad \text{provided that the limit exists (convergent density).}$$

Moreover, we may consider $\mathcal{Q}_u := \mathcal{Q}$,

$$\mathcal{Q}_l := \left\{(n_i) \in \mathcal{I} \;\middle|\; d_l(\{n_i \mid i \in \mathbb{N}\}) > 0\right\},$$

and

$$\mathcal{Q}_c := \left\{(n_i) \in \mathcal{I} \;\middle|\; d_c(\{n_i \mid i \in \mathbb{N}\}) \text{ exists and } d_c(\{n_i \mid i \in \mathbb{N}\}) > 0\right\}.$$

Obviously, $\mathcal{Q}_c \subsetneq \mathcal{Q}_l \subsetneq \mathcal{Q}_u$.

**(a)** The set ${}^c\mathcal{Q}_u^*$ is dense in $[0,1]$ by 4.2, and therefore ${}^c\mathcal{Q}_r^*$ ($r = c$ or $r = l$) are also dense in $[0,1]$.

**(b)** The set $\mathcal{Q}_c^*$, thus ${}^c\mathcal{Q}_r^*$ ($r = l$ and $r = u$), are dense in $[0,1]$.

*Proof.* **(b)** Let us prove that the closure of $\mathcal{Q}_c^*$ equals $[0,1]$. Obviously,

$${}^c\mathcal{Q}_c = {}^c\mathcal{Q}_u \cup \mathcal{P} \quad \text{with} \quad \mathcal{P} := \left\{(n_i) \in \mathcal{Q}_u \;\middle|\; d_u(\{n_i \mid i \in \mathbb{N}\}) > d_l(\{n_i \mid i \in \mathbb{N}\})\right\},$$

therefore

$$\begin{aligned} {}^c\mathcal{Q}_c^* = {}^c\mathcal{Q}_u^* \cup \mathcal{P}^* &= \left\{\alpha = .\alpha_1\alpha_2\alpha_3\ldots \;\middle|\; \exists\,(n_k) \in {}^c\mathcal{Q}_u : \alpha_r \text{ defined by (3)}\right\} \\ &\cup \left\{\alpha = .\alpha_1\alpha_2\alpha_3\ldots \mid \exists\,(n_k) \in \mathcal{P} : \alpha_r \text{ defined by (3)}\right\}. \end{aligned}$$

In the first step of the proof we consider $\alpha = .\alpha_1\alpha_2\alpha_3... \in {}^c\mathcal{Q}_u^*$ with corresponding $(n_i) \in {}^c\mathcal{Q}_u$ and $\alpha_r$ defined by (3).

Let $\alpha = .000\ldots$. Then $v(\alpha) = 0$, $\alpha \in {}^c\mathcal{Q}_u^*$, and $\beta^{(r)} = .\beta_1^{(r)}\beta_2^{(r)}\beta_3^{(r)}\ldots$ with

$$\beta_k^{(r)} := \begin{cases} 0 & \text{if } 1 \le k \le r, \\ 1 & \text{if } r < k < \infty \end{cases} \qquad (r, k \in \mathbb{N})$$

satisfies $\beta^{(r)} \in \mathcal{Q}_u^*$ and

$$\left|v(\alpha) - v(\beta^{(r)})\right| = \sum_{k=r+1}^{\infty} 2^{-k} = 2^{-r} \overset{r\to\infty}{\longrightarrow} 0.$$

Consequently, $\alpha = .000\ldots$ is a member of the closure of $\mathcal{Q}_c^*$ in $[0,1]$.

Now, let $\alpha = .\alpha_1\alpha_2\alpha_3... \in {}^c\mathcal{Q}_u^*$ with $v(\alpha) \in ]0,1[$ be given. Moreover, we choose $r \in \mathbb{N}$ with $2^{-r} < \min\{1 - v(\alpha), v(\alpha)\}$ and define $\beta^{(r)} = .\beta_1^{(r)}\beta_2^{(r)}\beta_3^{(r)}\ldots$ with

$$\beta_k^{(r)} := \begin{cases} \alpha_k & \text{if } 1 \le k \le r, \\ 1 & \text{if } r < k < \infty \end{cases} \qquad (k \in \mathbb{N}) \tag{4}$$

satisfies $\beta^{(r)} \in \mathcal{Q}_u^*$ and

$$\left|v(\alpha) - v(\beta^{(r)})\right| \le 2 \cdot \sum_{k=r+1}^{\infty} 2^{-k} = 2^{-r+1} \overset{r\to\infty}{\longrightarrow} 0. \tag{5}$$

Therefore, the chosen $\alpha = .\alpha_1\alpha_2\alpha_3...$ is a member of the closure of $\mathcal{Q}_c^*$ in $[0,1]$.

In the second step of the proof we consider $\alpha = .\alpha_1\alpha_2\alpha_3... \in \mathcal{P}^*$ with corresponding $(n_i) \in \mathcal{Q}_u$ satisfying $d_u(\{n_i \mid i \in \mathbb{N}\}) > d_l(\{n_i \mid i \in \mathbb{N}\})$, and $\alpha_r$ defined by (3). Note, $v(\alpha) \in ]0,1[$. Choosing as above $r \in \mathbb{N}$ with $2^{-r} < \min\{1 - v(\alpha), v(\alpha)\}$ and define $\beta^{(r)} = .\beta_1^{(r)}\beta_2^{(r)}\beta_3^{(r)}\ldots \in \mathcal{Q}_u^*$ as in (4), then by (5) we get again that $\alpha = .\alpha_1\alpha_2\alpha_3...$ is a member of the closure of $\mathcal{Q}_c^*$ in $[0,1]$. □

**Problem 5.4.** Does Theorem 5.1 also hold in the cases ${}^c\mathcal{Q}_r^*$ ($r = c$ or $r = l$)?

## 6. Further Problems

Buck and Pollard proved:

**Theorem 6.1 (cf. [6, Theorem 4]).** *A bounded sequence $x$ is $(C,1)$ summable*[7] *if and only if almost all of its subsequences are $(C,1)$ summable, that is, the set of all subsequences of $x$ being not $(C,1)$ summable is a Lebesgue nullset.* (Thereby 'almost all of its subsequences' is used in the sense of the Lebesgue measure, cf. [6, page 1].)

**Problems 6.2. (a)** Does the statement in Theorem 6.1 hold if we consider unbounded sequences instead of bounded sequences?

**(b)** Is the statement in Theorem 6.1 true if we consider *almost every subsequence* (in the sense of *positive density*)?

**(c)** Does the statement in Theorem 6.1 hold in the case of bounded or even unbounded sequences $x$ if we consider any regular matrix method instead of $(C,1)$? Note that Keogh and Petersen proved (cf. [16, Theorem 7]) that Theorem 6.1 holds too for a special matrix $G$ introduced by Garreau in [9].

**(d)** The corresponding question as in (c), but by considering *almost every subsequence* (in the sense of *positive density*) instead *almost all of its subsequences* used in the *sense of the Lebesgue measure*.

**(e)** Does ${}^c\boldsymbol{Q}^*$ (or $\boldsymbol{Q}^*$) have Lebesque measure 0?

**(f)** Let $Q$ be a set of subsequences of a sequence $x$. Is it of the second category in the sense of Theorem 3.2 if and only if the complement of it has Lebesque measure 0? (Cf. [23, Chapter 1].)

Concerning Problem 6.2(f) we should take into account the following results.

**Theorem 6.3 (cf. [23, Theorem 1.6]).** *The (real) line can be decomposed into two complementary sets $V$ and $W$ such that $V$ is of the first category and $W$ is of measure zero.*

**Theorem 6.4 (cf. [23, Corollary 1.7]).** *Every subset of the line, in particular the interval $[0,1]$, can be represented as the union of a Lebesgue nullset $V$ and a set $W$ of the first category.* In particular, $V$ is of the second category.

In the following remark we show that the answer to the question in Problem 6.2(f) is negative.

**Remark 6.5.** Let $V$ and $W$ be a decomposition of $[0,1]$ such that $V$ is a Lebesgue nullset and $W$ is of the first category (and thus $V$ is of the second category). We identify $V$ with the set $\mathcal{V}^*$ of the members of $V$ in binary code. Then the corresponding set $\mathcal{V}$ of index sequences and thus for any divergent sequence $x$ the corresponding set $\mathcal{V}(x)$ of subsequences of $x$ are of the second category. Consequently, any regular matrix sums all elements of $\mathcal{V}(x)$ if and only if $x \in c$ (cf. Theorem 3.2).

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[7]Note, $(C,1) = C_1$, the Cesàro matrix of order 1.

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