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On Coincidence Point and Fixed Point Theorems for a General Class of Multivalued Mappings in Incomplete Metric Spaces with an Application

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Abstract. In this paper, we prove existence of fixed and coincidence points for a general class of multivalued mappings satisfying a new generalized contractive condition in incomplete metric spaces which generalize a number of published results in the last decades. In addition, this article not only brings a new approaches on the subject and but also involves several non-trivial examples which demonstrate the significance of the motivation. Finally, the obtained results of this paper provide a result on the convergence of successive approximations for certain operators (not necessarily linear) on a norm space (not necessarily a Banach space). In particular, we conclude that the renowned Kelisky-Rivlin theorem works on iterates of the Bernstein operators on an incomplete subspace of $C[0, 1]$.

1. Introduction and preliminaries

In 1969, Nadler [22] proved the analog of renowned Banach fixed point results for multivalued mappings and initiated a trend of researching on fixed point theorems for multivalued mappings. For the sake of completeness, we recall the main theorem of Nadler [22] here:

Theorem 1.1. *Let T be a mapping from a complete metric space (X, d) into the set of all nonempty closed and bounded subsets of X. Suppose that there exists $q \in [0, 1)$ such that $H(Tx, Ty) \leq qd(x, y)$ for all $x, y \in X$, where H is Hausdorff metric induced by d, that is,*

$$H(U, V) = \max\{\sup_{u\in U} D(u, V), \sup_{v\in V} D(v, U)\}. \tag{1}$$

$$D(x, U) = \inf\{d(x, u); u \in U\},$$

for all $x \in X$ and for all nonempty closed and bounded subsets U, V of X. Then there exists $z \in X$ such that $z \in T(z)$.

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A number of generalizations of Nadler's fixed point theorem in various directions have been examined and improved by several authors, see e.g. Reich [25], Berinde-Berinde [9], Baghani *et al.* [4, 5, 16], Mizoguchi and Takahashi [21], Du [15], Daffer and Kaneko [12], Daffer *et al.* [13], Pathak *et al.* [23], Amini-Harandi [1], Boonsri and Saejung [10] and the related references therein.

We, first, recall the famous Reich's result in this direction.

Theorem 1.2. *Let T be a mapping from a complete metric space (X, d) into the class of all nonempty compact subsets of X. Suppose that $\beta : [0,\infty) \to (0,1)$ is a function such that $\limsup_{s\to t^+} \beta(s) < 1$ for each $t > 0$. Assume that*

$$H(Tx, Ty) \leq \beta(d(x,y))d(x,y)$$

for each $x, y \in X$. Then T possess a fixed point.

Notice that it is in the setting of compact subsets. Mizoguchi and Takahashi [21] refined the result of Reich for the frameworks of closed and bounded subsets.

Theorem 1.3. *Let T be a mapping from a complete metric space (X, d) into the set of all nonempty closed and bounded subsets of X. Suppose that $\beta : [0,\infty) \to [0,1)$ is a function such that $\limsup_{s\to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*

$$H(Tx, Ty) \leq \beta(d(x,y))d(x,y)$$

for each $x, y \in X$. Then $\mathrm{Fix}(T) \neq \emptyset$.

In the following, we state Berinde-Berinde's fixed point theorem [9].

Theorem 1.4. *Let T be a mapping from a complete metric space (X, d) into the set of all nonempty closed and bounded subsets of X. Suppose that $\beta : [0,\infty) \to [0,1)$ is a function such that $\limsup_{s\to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*

$$H(Tx, Ty) \leq \beta(d(x,y))d(x,y) + LD(y, Tx)$$

for each $x, y \in X$, where $L \geq 0$. Then $\mathrm{Fix}(T) \neq \emptyset$.

It is clear that for $L = 0$ in Theorem 1.4 we deduce Mizoguchi-Takahashi's fixed point theorem [21].

Also, Du [15] improved the main results of [9] as follows:

Theorem 1.5. *Let T be a mapping from a complete metric space (X, d) into the set of all nonempty closed and bounded subsets of X. Suppose that $f : X \to X$ is a continuous self-mapping and $\beta : [0,\infty) \to [0,1)$ be a function such that $\limsup_{s\to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy : y \in Tx\} \subseteq Tx$;
(a_2) there exists a function $\hat{h} : X \to [0,\infty)$ such that

$$H(Tx, Ty) \leq \beta(d(x,y))d(x,y) + \hat{h}(fy)D(fy, Tx)$$

for each $x, y \in X$. Then, the intersection of the set of fixed point of T and the set of coincidence points of f and T is non-empty.

In 2011, Amini-Harandi [1] expanded the well-known quasi-contraction in the setting of set-valued mappings.

Theorem 1.6. *Let T be a mapping from a complete metric space (X, d) into the set of all nonempty closed and bounded subsets of X. Assume that*

$$H(Tx, Ty) \leq k \max\{d(x,y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}$$

for each $x, y \in X$, where $0 < k < \frac{1}{2}$. Then $\mathrm{Fix}(T) \neq \emptyset$.

Among all, we focus on the very recent result given by Pathak *et al.* [23].

Theorem 1.7. *Let T be a mapping from a complete metric space (X, d) into the set of all nonempty closed and bounded subsets of X. Suppose that f, g are continuous self-mappings and $\beta : [0, \infty) \to [0, 1)$ is a function with $\limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0, \infty)$ such that

$$H(Tx, Ty) \leq \beta(d(x, y))\Big(\frac{D(x, Ty) + D(y, Tx)}{2}\Big) + \hat{h}(fy)D(fy, Tx) + \hat{k}(gy)D(gy, Tx)$$

for each $x, y \in X$. Then, the intersection of the set of fixed point of T and the set of coincidence points of f, g and T is non-empty.

On the other hand, Boonsri and Saejung in [10] showed that the conclusion of Daffer and Kaneno[12] remains true without assuming the lower semicontinuity of the function $x \mapsto D(x, Tx)$. In the following, we state Boonsri-Saejung's fixed point theorem.

Theorem 1.8. *Let T be a mapping from a complete metric space (X, d) into the set of all nonempty closed and bounded subsets of X. Suppose that*

$$H(Tx, Ty) \leq k \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}$$

for each $x, y \in X$, where $0 < k < 1$. Then $\mathrm{Fix}(T) \neq \emptyset$.

Very recently, Eshaghi Gordji *et al.* [17] and Baghani *et al.* [3] introduced the notation of the orthogonal sets and gave a real generalization of the Banach fixed point theorem in incomplete metric spaces. The notion helps them to find the solution of a integral equation in incomplete metric spaces, see e.g.[2, 4, 6–8, 24].

As motivated by these works, we define a new type of monotone multivalued mappings and prove some coincidence point and fixed point theorems under a new generalized contractive condition which is different not only from Nadler's fixed point theorem but also the other results in the literature, e.g. [23], [15], [9], [21], [25]. The presented results extend and improve several well-known fixed point theorems for multivalued contractive mappings.

To set up our results, we need to collect some basic definitions and fundamental results that will be used in further sections. Throughout the paper, we shall use the standard notations in the literature, see e.g. [23]. In particular, we shall denote nonnegative real number by $\mathbb{R}_0^+$.

Let (X, d) be a metric space. We denote the set of all nonempty subsets of X by $\mathcal{P}^*(X)$. Further, we shall reserve the letters $K(X)$ (respectively, $CB(X)$) to denote the class of all nonempty compact (respectively, closed and bounded) subsets of X.

We shall say that $x \in X$ is a coincidence point of $f, g : X \to X$ and $T : X \to CB(X)$ whenever $fx = gx \in Tx$. Moreover, if $f = g = id$, the identity mapping, then $x = fx = gx \in Tx$, that is, x a fixed point of T. We shall use the letters $Fix(T)$ and $COP(f, g, T)$ for the set of fixed points of T and the set of coincidence points of f, g and T, respectively.

Let A, B be subset of a nonempty set X and $\mathfrak{R}$ be an arbitrary binary relation on X. We say that A and B has a *strong relation* (briefly, $\mathfrak{R}_S$) if $a \ \mathfrak{R} \ b$ for all $a \in A$ and $b \in B$. In this case, we write $A \ (\mathfrak{R}_S) \ B$. Moreover, We say that A and B has a *weak relation* (briefly, $\mathfrak{R}_W$) if for each $a \in A$ there exists $b \in B$ such that $a \ \mathfrak{R} \ b$ and we write $A \ \mathfrak{R}_W \ B$. It is clear that the relation $\mathfrak{R}_S$ implies the relation $\mathfrak{R}_W$. Example 1.10 shows that the converse of the statement is not true in general.

In what follows, we introduce a new type monotone multivalued mapping by using the relation $\mathfrak{R}_S$.

Definition 1.9. *Let T be a mapping from a metric space (X, d) into the set of all nonempty closed and bounded subsets of X. Let $\mathfrak{R}$ be a relation on X. We say that T is a monotone mapping of type $\mathfrak{R}_S$ if*

$$x, y \in X, x \ \mathfrak{R} \ y \Longrightarrow Tx \ (\mathfrak{R}_S) \ Ty.$$

Example 1.10. *Let $X = \{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots\} \cup \{0, 1\}$ be equipped with the standard metric $d(x, y) = |x - y|$ for $x, y \in X$. Consider a relation $\Re$ on X that is defined as*

$$x \ \Re \ y \iff \begin{cases} \frac{y}{x} \in \mathbb{N}, \\ or\ x = y = 0. \end{cases}$$

Define a multivalued mapping $T : X \to CB(X)$ as follows

$$Tx = \begin{cases} \{\frac{1}{2^n}, \frac{1}{2^{n+1}}\}, & x = \frac{1}{2^n}, n = 1, 2, \cdots, \\ \{0\}, & x = 0, \\ \{1, \frac{1}{4}\}, & x = 1. \end{cases}$$

Note that T is not monotone of type $\Re_S$.

Example 1.11. *Let $X = [0, 1)$ be equipped with the standard metric $d(x, y) = |x - y|$ for $x, y \in X$. Consider a relation $\Re$ on $\Re$ on X that is defined by*

$x \Re y$ if and only if either $x = 0$ or $y = 0$.

Define a multivalued mapping $T : X \to CB(X)$ as follows

$$T(x) = \begin{cases} \{x^2, \frac{1}{4}e^x\}, & x \in \mathbb{Q} \cap X, \\ \{0\}, & x \in \mathbb{Q}^c \cap X. \end{cases}$$

Note that T is monotone of type $\Re_S$.

Definition 1.12. *Let Φ denote the class of those functions $\phi(\vec{t_1^5}) : \mathbb{R}_+^5 \to \mathbb{R}_0^+$ that satisfy:*
(Φ_1) ϕ is increasing in t_2, t_3, t_4 and t_5;
(Φ_2) $t < \phi(s, s, t, s + t, 0)$ implies that $t < s$ for each $s, t \in \mathbb{R}_0^+$;
(Φ_3) if $t_n, s_n \to 0$ and $u_n \to \gamma > 0$, as $n \to \infty$, then $\limsup_{n\to\infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1}) \leq \gamma$;
(Φ_4) $\phi(s, s, s, 2s, 0) \leq s$ for each $s \in \mathbb{R}_0^+$.

The class Φ is very wide class as it is shown by the following examples.

Example 1.13. *Let $(\vec{t_1^5}) := (t_1, t_2, t_3, t_4, t_5) \in (\mathbb{R}_0^+)^5$ for simplicity. Then, we express the following functions:*

(I) $\phi_1(\vec{t_1^5}) = \hat{\alpha} t_1 + \hat{\beta} t_2 + \hat{\gamma} t_3 + \hat{\delta} t_4 + L t_5$, *where $\hat{\gamma} \neq 1$, $\hat{\alpha} + \hat{\beta} + \hat{\gamma} + 2\hat{\delta} = 1$ and $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, L \geq 0$.*

(II) $\phi_2(\vec{t_1^5}) = \frac{1}{2} \max\{t_1, t_2, t_3, t_4, t_5\} + L t_5, \ L \geq 0.$

(III) $\phi_3(\vec{t_1^5}) = \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\} + L t_5, \ L \geq 0.$

Definition 1.14. *Let Λ denote the class of those functions $\lambda(\vec{t_1^5}) : (\mathbb{R}_0^+)^5 \to \mathbb{R}_0^+$ which satisfy the following conditions:*
(Λ_1) λ is increasing in t_2, t_3, t_4 and t_5;
(Λ_2) $\lambda(s, s, t, s + t, 0) \leq t$ for all $s, t \in \mathbb{R}_0^+$;
(Λ_3) λ is continuous at $(0, 0, t, t, 0)$ for all $t \in \mathbb{R}_0^+$.

The class Λ is also very rich class as shown by the following examples.

Example 1.15. **(I)**

$$\lambda_1(\vec{t_1^5}) = \frac{t_3(1 + t_2)(1 + t_5)}{1 + t_1}.$$

(II)

$$\lambda_2(\vec{t_1^5}) = \frac{t_3(1+t_2)}{1+t_1}.$$

(III)

$$\lambda_3(\vec{t_1^5}) = t_3 + Lt_5, \ \ L \geq 0.$$

(IV)

$$\lambda_4(\vec{t_1^5}) = \begin{cases} \frac{t_2 t_3}{t_1}, & t_1 > 0, \\ \\ 0, & t_1 = 0. \end{cases}$$

Definition 1.16. *[4] Let X be a non-empty set. Let $\mathfrak{R}$ be a relation on a metric space (X, d). A sequence $\{x_n\}$ is called an $\mathfrak{R}$-sequence if $x_n \ \mathfrak{R} \ x_{n+1}$ for every $n \in \mathbb{N}$. If a Cauchy sequence $\{x_n\}$ forms an $\mathfrak{R}$-sequence, then, we say that it is a Cauchy $\mathfrak{R}$-sequence.*

Moreover, X is called $\mathfrak{R}$-regular if for each convergent $\mathfrak{R}$-sequence $\{x_n\}$, there exists $n_0 \in \mathbb{N}$ such that $x_n \ \mathfrak{R} \ x$ for all $n \geq n_0$. In this case, the triple $(X, d, \mathfrak{R})$ is called an $\mathfrak{R}$-regular metric space. Furthermore, a metric space (X, d) is called $\mathfrak{R}$-complete if every Cauchy $\mathfrak{R}$-sequence is convergent.

Lastly, a mapping $f : X \to X$ is said to be $\mathfrak{R}$-continuous at $a \in X$ if $f(a_n) \to f(a)$ whenever $\{a_n\}$ is an $\mathfrak{R}$-sequence in X and $a_n \to a$.

Example 1.17. *Consider $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 2]$ equipped with the Euclidean metric. Define relation $\mathfrak{R}$ on X by $\mathfrak{R} = \{(0,0), (0,1), (1,0), (1,1), (0,2)\}$. Here we show that $(X, d, \mathfrak{R})$ is an $\mathfrak{R}$-regular metric space. Take $\mathfrak{R}$-sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$. Since $\{x_n\}$ is an $\mathfrak{R}$-sequence, for each $n \in \mathbb{N}$, $(x_n, x_{n+1}) \in \{(0,0), (0,1), (1,0), (1,1)\}$ which gives rise to $\{x_n\} \subseteq \{0, 1\}$. As $\{0, 1\}$ is closed, we have $x_n \ \mathfrak{R} \ x$ for all $n \in \mathbb{N}$. Similarly, we can show that $(X, d, \mathfrak{R})$ is an $\mathfrak{R}$-complete (not complete) metric space.*

Example 1.18. *Let X be a linear subspace of a Hilbert space $(H, \|.\|)$. We define a relation $\mathfrak{R}$ as follows:*

$$x \ \mathfrak{R} \ y \text{ if and only if } | < x, y > | = \|x\|\|y\|,$$

for all $x, y \in X$. We assert that $(X, \|.\|, \mathfrak{R})$ forms an $\mathfrak{R}$-complete metric space. Suppose that the sequence $\{x_n\} \subseteq X$ is Cauchy $\mathfrak{R}$-sequence. Accordingly, the sequence $\{x_n\}$ converges to some $x \in H$. We shall show that x is an element of X. The relation $\mathfrak{R}$ ensures that for all $n \in \mathbb{N}$,

$$\exists \alpha_n \ \ s.t. \quad x_n = \alpha_n x_{n+1} \quad or \quad x_{n+1} = \alpha_n x_n. \tag{2}$$

We examine two cases:

Case 1. *There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = 0$ for all $k \in \mathbb{N}$. This implies that $x = 0 \in X$.*

Case 2. *For all sufficiently large $n \in \mathbb{N}$, $x_n \neq 0$. Take $n_0 \in \mathbb{N}$ such that for all $n \geq n_o$, $x_n \neq 0$. It follows from (2) that for all $n \geq n_0$, there exists $\alpha_n > 0$ such that $x_n = \alpha_n x_{n_0}$. In other words,*

$$|\alpha_n - \alpha_m| \ \|x_{n_0}\| = \|x_n - x_m\| \to 0 \quad as \ \ m, n \to \infty.$$

Therefore, $\{\alpha_n\}$ is a Cauchy sequence in $\mathbb{R}$. Assume that $\alpha_n \to \alpha$ as $n \to \infty$. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \alpha_n x_{n_0} = \alpha x_{n_0}$. This implies that $x \in X$.

Example 1.19. *Let $\{z_n\}$ be a bounded increasing sequence in $\mathbb{R}$. Set $z = \lim_{n\to\infty} z_n$, $X = \Big(z_1, \infty\Big)$ and $\mathfrak{R} = \{(z_i, z_{i+1}) : i \in \mathbb{N}\}$. We observe that X with the standard metric $d(x, y) = |x - y|$ is $\mathfrak{R}$-complete but it is not an $\mathfrak{R}$-regular metric space. To this end, let $\{x_n\}$ be an $\mathfrak{R}$-sequence in X. The definition of $\mathfrak{R}$ implies that $\{x_n\}$ is a subsequence of $\{z_n\}$. This shows that every Cauchy $\mathfrak{R}$-sequence in X converges to $z \in X$. Moreover, z is not $\mathfrak{R}$-relative to elements of the $\mathfrak{R}$-sequence.*

Remark 1.20. *Every complete metric space is $\mathfrak{R}$-complete, but the converse is not need to be true, see e.g. Example 1.18 - Example 1.19 .*

Example 1.21. *Let $X = [0,1] = I$ with the standard metric $d(x,y) = |x-y|$. We define a relation $\mathfrak{R}$ as follows:*

$x \,\mathfrak{R}\, y$ if and only if $xy = 0$.

Define $f : X \to X$ by

$$f(x) = \begin{cases} 1, & x \in I \cap \mathbb{Q}, \\ x, & x \in I \cap \mathbb{Q}^c. \end{cases}$$

Here, f is not continuous but it is $\mathfrak{R}$-continuous. If $\{x_n\}$ is an $\mathfrak{R}$-sequence in X which converges to $x \in X$ Employing definition $\mathfrak{R}$, we find $x_n = 0$ which yields that $1 = f(x_n) \to f(x) = 1$.

2. Main results

In this section, we express our main theorem in $\mathfrak{R}$-complete metric spaces (not need to be complete).

Definition 2.1. *Let T be a mapping from a complete metric space (X,d) into $CB(X)$, $\lambda \in \Lambda$ and $\phi \in \Phi$. We define functions $M^T_\lambda, M^T_\phi : X \times X \to [0,\infty)$ as follows*

$$M^T_\theta(x,y) = \theta(d(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)),$$

for all $x, y \in X$, where $\theta = \lambda$ or $\theta = \phi$.

Theorem 2.2. *Let $(X,d,\mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f, g : X \to X$ be $\mathfrak{R}$-continuous self-mappings and $\alpha, \beta : [0,\infty) \to [0,1)$ be functions such that α is continuous and $\alpha(t) + \limsup_{s\to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0,\infty)$, $\lambda \in \Lambda$ and $\phi \in \Phi$ such that

$$H(Tx,Ty) \leq \alpha(d(x,y))M^T_\lambda(x,y) + \beta(d(x,y))M^T_\phi(x,y) + \hat{h}(fy)D(fy,Tx) + \hat{k}(gy)D(gy,Tx) \tag{3}$$

for each $x \,\mathfrak{R}\, y$, $x \neq y$. Suppose that
(i) T is monotone of type $\mathfrak{R}_S$;
(ii) there exists $x_0 \in X$ such that $\{x_0\} \,\mathfrak{R}_W\, Tx_0$.
Then $\mathrm{COP}(f,g,T) \bigcap \mathrm{Fix}(T) \neq \varnothing$.

Proof: By (a_1), we note that, for each $x \in X$, $D(fy,Tx) = 0$ and $D(gy,Tx) = 0$ for all $y \in Tx$. Also, it is easy to see that, if $x^* \in T(x^*)$, then $x^* \in COP(f,g,T) \bigcap Fix(T)$. For this reason we suppose that T has no fixed point, i.e., $D(x,Tx) > 0$ for all $x \in X$.

By properties of functions α and β, for each $t > 0$, there exist $k(t) > 0$ and $\delta(t) > 0$ such that

$$\frac{\beta(s)}{1-\alpha(s)} \leq k(t) < 1 \quad \text{for all} \quad s \in (t, t+\delta(t)). \tag{4}$$

Since $\{x_0\} \,\mathfrak{R}_W\, Tx_0$, there exists $x_1 \in Tx_0$ such that $x_0 \,\mathfrak{R}\, x_1$. If $x_0 = x_1$, then $x_0 = x_1 \in Tx_0$ and this is a contradiction. So, we may assume that $x_0 \neq x_1$. Moreover, by monotonicity of T, we have $Tx_0 \,(\mathfrak{R}_S)\, Tx_1$. Put $t_1 = D(x_1,Tx_1)$. It is clear that $D(x_1,Tx_1) \leq d(x_1,y)$ for all $y \in Tx_1$. The following cases are considered:

Case 1. $D(x_1,Tx_1) < d(x_1,y)$ for all $y \in Tx_1$. Select positive number $d(t_1)$ such that

$$d(t_1) < \min\{\delta(t_1), (\frac{1}{k(t_1)} - 1)t_1\} \tag{5}$$

and put

$$\epsilon(x_1) = \min\{1, \frac{d(t_1)}{t_1}\}. \tag{6}$$

Then there exists $x_2 \in Tx_1$ such that $x_1 \mathfrak{R} x_2$ and

$$d(x_1, x_2) < D(x_1, Tx_1) + \epsilon(x_1)D(x_1, Tx_1) = (1 + \epsilon(x_1))D(x_1, Tx_1). \tag{7}$$

By the hypotheses that T no fixed point, we have $x_1 \neq x_2$ and by (a_2), we can write

$$\begin{aligned} D(x_2, Tx_2) \leq H(Tx_1, Tx_2) &\leq \alpha(d(x_1, x_2)).M^T_\lambda(x_1, x_2) + \beta(d(x_1, x_2)).M^T_\phi(x_1, x_2) \\ &= \alpha(d(x_1, x_2)).\lambda(d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), D(x_1, Tx_2), D(x_2, Tx_1)) \\ &+ \beta(d(x_1, x_2)).\phi(d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), D(x_1, Tx_2), D(x_2, Tx_1)). \end{aligned} \tag{8}$$

Note that by (Λ_1) and (Λ_2), we have

$$\begin{aligned} M^T_\lambda(x_1, x_2) &\leq \lambda(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), D(x_1, Tx_2), 0) \\ &\leq \lambda(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), d(x_1, x_2) + D(x_2, Tx_2), 0) \\ &\leq D(x_2, Tx_2), \end{aligned}$$

and by (Φ_1), we obtain

$$\begin{aligned} M^T_\phi(x_1, x_2) &\leq \phi(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), D(x_1, Tx_2), 0) \\ &\leq \phi(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), d(x_1, x_2) + D(x_2, Tx_2), 0). \end{aligned}$$

This implies that

$$\begin{aligned} D(x_2, Tx_2) &\leq \frac{\beta(d(x_1, x_2))}{1 - \alpha(d(x_1, x_2))}\phi(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), d(x_1, x_2) + D(x_2, Tx_2), 0) \\ &< \phi(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), d(x_1, x_2) + D(x_2, Tx_2), 0). \end{aligned}$$

Now by above relation, (Φ_2), (Φ_1) and (Φ_4), we conclude that

$$D(x_2, Tx_2) \leq \frac{\beta(d(x_1, x_2))}{1 - \alpha(d(x_1, x_2))}d(x_1, x_2).$$

Thus,

$$\begin{aligned} D(x_1, Tx_1) - D(x_2, Tx_2) &\geq D(x_1, Tx_1) - \frac{\beta(d(x_1, x_2))}{1 - \alpha(d(x_1, x_2))}d(x_1, x_2) \\ &> (\frac{1}{1 + \epsilon(x_1)} - \frac{\beta(d(x_1, x_2))}{1 - \alpha(d(x_1, x_2))})d(x_1, x_2). \end{aligned} \tag{9}$$

By (5), (6) and (7), we have

$$\begin{aligned} t_1 = D(x_1, Tx_1) &< d(x_1, x_2) < D(x_1, Tx_1) + \epsilon(x_1)D(x_1, Tx_1) \\ &\leq t_1 + d(t_1) < t_1 + \delta(t_1). \end{aligned}$$

This implies by (4) that $\frac{\beta(d(x_1,x_2))}{1-\alpha(d(x_1,x_2))} \leq k(t_1) < 1$. Since $\epsilon(x_1) \leq \frac{d(t_1)}{t_1} < \frac{1}{k(t_1)} - 1$, we have

$$\frac{1}{1 + \epsilon(x_1)} - \frac{\beta(d(x_1, x_2))}{1 - \alpha(d(x_1, x_2))} > 0. \tag{10}$$

By relations (9) and (10), $D(x_2, Tx_2) < D(x_1, Tx_1)$.

Case 2. $D(x_1, Tx_1) = d(x_1, x_2)$ for some $x_2 \in Tx_1$. Since $Tx_0\ (\mathfrak{R}_S)\ Tx_1$, we have $x_1\ \mathfrak{R}\ x_2$ and also

$$D(x_1, Tx_1) - D(x_2, Tx_2) \geq (1 - \frac{\beta(d(x_1, x_2))}{1 - \alpha(d(x_1, x_2))})d(x_1, x_2) > 0.$$

Therefore $D(x_2, Tx_2) < D(x_1, Tx_1)$.

Next, let $t_2 = D(x_2, Tx_2)$. Then $D(x_2, Tx_2) \leq d(x_2, y)$ for all $y \in Tx_2$. Again we consider the following two cases:

Case A. $D(x_2, Tx_2) < d(x_2, y)$ for all $y \in Tx_2$. For $\delta(t_2)$ and $k(t_2)$, choose $d(t_2)$ with

$$d(t_2) < \min\{\delta(t_2), (\frac{1}{k(t_2)} - 1)t_2\}$$

and set

$$\epsilon(x_2) = \min\{\frac{d(t_2)}{t_2}, \frac{1}{2}, \frac{t_1}{t_2} - 1\}.$$

By using a similar reason as above, we obtain $x_3 \in Tx_2$ such that $x_2\ \mathfrak{R}\ x_3$, $x_2 \neq x_3$, $d(x_2, x_3) < (1 + \epsilon(x_2))D(x_2, Tx_2)$ and

$$D(x_2, Tx_2) - D(x_3, Tx_3) \geq (\frac{1}{1 + \epsilon(x_2)} - \frac{\beta(d(x_2, x_3))}{1 - \alpha(d(x_2, x_3))})d(x_2, x_3) > 0.$$

Hence $D(x_3, Tx_3) < D(x_2, Tx_2)$. From $\epsilon(x_2) \leq \frac{t_1}{t_2} - 1$, it follows that

$$d(x_2, x_3) < (1 + \epsilon(x_2))D(x_2, Tx_2) \leq D(x_1, Tx_1) \leq d(x_1, x_2).$$

Case B. $D(x_2, Tx_2) = d(x_2, x_3)$ for some $x_3 \in Tx_2$. Since $Tx_1\ (\mathfrak{R}_S)\ Tx_2$, we have $x_2\ \mathfrak{R}\ x_3$ and by using the same method as above, we can show that

$$D(x_2, Tx_2) - D(x_3, Tx_3) \geq (1 - \frac{\beta(d(x_2, x_3))}{1 - \alpha(d(x_2, x_3))})d(x_2, x_3) > 0$$

and

$$d(x_2, x_3) = D(x_2, Tx_2) < D(x_1, Tx_1) \leq d(x_1, x_2).$$

Hence, $D(x_3, Tx_3) < D(x_2, Tx_2)$ and $d(x_2, x_3) < d(x_1, x_2)$. Repeating this process, we find that there exists an $\mathfrak{R}$-sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ such that $\{D(x_n, Tx_n)\}$ and $\{d(x_n, x_{n+1}\}$ are decreasing sequences of positive numbers and for each $n \in \mathbb{N}$,

$$D(x_n, Tx_n) - D(x_{n+1}, Tx_{n+1}) \geq (\frac{1}{1 + \gamma(x_n)} - \frac{\beta(d(x_n, x_{n+1}))}{1 - \alpha(d(x_n, x_{n+1}))})d(x_n, x_{n+1}), \tag{11}$$

where $\gamma(x_n)$ is a real number with $0 \leq \gamma(x_n) \leq \frac{1}{n}$. Since $\{d(x_n, x_{n+1})\}$ is a decreasing sequence, there exists $\hat{t} \in [0, \infty)$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \hat{t}$.

Let $a_n := \frac{1}{1+\gamma(x_n)} - \frac{\beta(d(x_n,x_{n+1}))}{1-\alpha(d(x_n,x_{n+1}))}$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}\liminf_{n\to\infty} a_n &\geq \lim_{n\to\infty} \frac{1}{1 + \gamma(x_n)} - \limsup_{n\to\infty} \frac{\beta(d(x_n, x_{n+1}))}{1 - \alpha(d(x_n, x_{n+1}))}\\ &= 1 - \limsup_{s\to\hat{t}^+} \frac{\beta(s)}{1 - \alpha(s)} > 0.\end{aligned}$$

This implies that from (11) there exists $b > 0$ such that

$$D(x_n, Tx_n) - D(x_{n+1}, Tx_{n+1}) \geq bd(x_n, x_{n+1})$$

for large enough n. Since $\{d(x_n, x_{n+1})\}$ is a decreasing sequence, it is convergent. On the other hand, for each $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \frac{1}{b} \sum_{i=n}^{m-1} \{D(x_i, Tx_i) - D(x_{i+1}, T_{i+1})\} \\ &= \frac{1}{b}\{D(x_n, Tx_n) - D(x_m, Tx_m)\} \to 0 \end{aligned}$$

as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy $\mathfrak{R}$-sequence. So $\{x_n\}$ converges to some $x^* \in X$. Since $x_{n+1} \in Tx_n$, it follows from (a_1) that $fx_{n+1} = gx_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$. Since f, g are $\mathfrak{R}$-continuous and $\lim_{n\to\infty} x_n = x^*$, we have

$$\lim_{n\to\infty} fx_{n+1} = \lim_{n\to\infty} gx_{n+1} = fx^* = gx^*.$$

By $\mathfrak{R}$-regularity assumption of X, since $x_n \,\mathfrak{R}\, x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x^*$ as $n \to \infty$, $x_n \,\mathfrak{R}\, x^*$ for each $n \in \mathbb{N}$. On the other hand, by continuity of λ in $(0, 0, D(x^*, Tx^*), D(x^*, Tx^*), 0)$, (Φ_1) and (Φ_3), we obtain

$$\begin{aligned} D(x^*, Tx^*) &= \limsup_{n\to\infty} D(x_{n+1}, Tx^*) \\ &\leq \limsup_{n\to\infty} H(Tx_n, Tx^*) \\ &\leq \limsup_{n\to\infty} \Big(\alpha(d(x_n, x^*)).M_\lambda^T(x_n, x^*) + \beta(d(x_n, x^*)).M_\phi^T(x_n, x^*) + \\ &\quad \hat{h}(fx^*)d(fx^*, fx_{n+1}) + \hat{k}(gx^*)d(gx^*, gx_{n+1})\Big) \\ &\leq \alpha(0).\lambda(0, 0, D(x^*, Tx^*), D(x^*, Tx^*), 0) \\ &\quad + \limsup_{n\to\infty} \beta(d(x_n, x^*))D(x^*, Tx^*) \\ &\leq \Big(\alpha(0) + \limsup_{s\to 0^+} \beta(s)\Big)D(x^*, Tx^*). \end{aligned}$$

Then $x^* \in Tx^*$ which is a contradiction because it is supposed that T has no fixed point. By (a_1), $fx^* = gx^* \in Tx^*$. Hence $x^* \in COP(f, g, T)$. This completes the proof. □

3. Some extensions of the previous results

Letting

$$\phi_1(\vec{t_1^5}) = \hat{\alpha}t_1 + \hat{\beta}t_2 + \hat{\gamma}t_3 + \hat{\delta}t_4 + Lt_5,$$

where $\hat{\alpha} + \hat{\beta} + \hat{\gamma} + 2\hat{\delta} = 1$, $\hat{\gamma} \neq 1$, and $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, L \geq 0$, we get a generalization of Theorem 2.2 of [15], Theorem 3.2 of [23], Theorem 4 of [9] and Theorem 5 of [21].

Corollary 3.1. *Let $(X, d, \mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f, g : X \to X$ be $\mathfrak{R}$-continuous self-mappings and $\alpha, \beta : [0, \infty) \to [0, 1)$ be functions such that α is continuous and $\alpha(t) + \limsup_{s\to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*

(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0, \infty)$ and $\lambda \in \Lambda$ such that

$$H(Tx, Ty) \leq \alpha(d(x, y))M^T_\lambda(x, y) + \beta(d(x, y))\Big(\hat{\alpha}d(x, y) + \hat{\beta}D(x, Tx) + \hat{\gamma}D(y, Ty) + \hat{\delta}D(x, Ty) + LD(y, Tx)\Big) + \hat{h}(fy)D(fy, Tx) + \hat{k}(gy)D(gy, Tx)$$

for each $x \ \mathfrak{R} \ y$ with $x \neq y$, where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, L \geq 0$ and $\hat{\gamma} \neq 1$, $\hat{\alpha} + \hat{\beta} + \hat{\gamma} + 2\hat{\delta} = 1$. Suppose that
(i) T is monotone of type $\mathfrak{R}_S$;
(ii) there exists $x_0 \in X$ such that $\{x_0\} \ \mathfrak{R}_W \ Tx_0$.
Then $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Proof: The proof follows from Theorem 2.2 and Example 1.13-(I). □

The following corollary for single-valued mappings directly follows from the above Corollary 3.1. This corollary plays main roles in the future sections.

Corollary 3.2. *Let $(X, d, \mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $f : X \to X$ be a single-valued mapping and $\beta : [0, \infty) \to [0, 1)$ be a function such that $\limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*

$$d(fx, fy) \leq \beta(d(x, y))\Big(\hat{\alpha}d(x, y) + \hat{\beta}d(x, fx) + \hat{\gamma}d(y, fy) + \hat{\delta}d(x, fy)\Big)$$

for each $x \ \mathfrak{R} \ y$ with $x \neq y$, where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \geq 0$, $\hat{\alpha} + \hat{\beta} + \hat{\gamma} + 2\hat{\delta} = 1$ and $\hat{\gamma} \neq 1$. Suppose that f is monotone of type $\mathfrak{R}_S$ and there exists $x_0 \in X$ such that $x_0 \ \mathfrak{R} \ fx_0$. Then, $\{f^n(x_0)\}$ converges to a fixed point of f. Moreover, if z_1 and z_2 are two fixed points of f such that $z_1 \ \mathfrak{R} \ z_2$, then $z_1 = z_2$.

Letting

$$\phi_2(\vec{t_1^5}) = \frac{1}{2}\max\{t_1, t_2, t_3, t_4, t_5\} + Lt_5,$$

where $L \geq 0$, we get a generalization of Theorem 2.2 of [1].

Corollary 3.3. *Let $(X, d, \mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f, g : X \to X$ be $\mathfrak{R}$-continuous self-mappings and $\alpha, \beta : [0, \infty) \to [0, 1)$ be functions such that α is continuous and $\alpha(t) + \limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0, \infty)$ and $\lambda \in \Lambda$ such that

$$H(Tx, Ty) \leq \alpha(d(x, y))M^T_\lambda(x, y) + \hat{h}(fy)D(fy, Tx) + \hat{k}(gy)D(gy, Tx) + \beta(d(x, y))\Big(\frac{1}{2}\max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\} + LD(y, Tx)\Big)$$

for each $x \ \mathfrak{R} \ y$ with $x \neq y$, where $L \geq 0$. Suppose that
(i) T is monotone of type $\mathfrak{R}_S$;
(ii) there exists $x_0 \in X$ such that $\{x_0\} \ \mathfrak{R}_W \ Tx_0$.
Then $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Proof: The proof follows from Theorem 2.2 and Example 1.13-(II). □

Letting

$$\phi_3(\vec{t_1^5}) = \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\} + Lt_5,$$

where $L \geq 0$, we get a generalization of Theorem 1 of [10], Theorem 2.2 of [15] and Theorem 4 of [9].

Corollary 3.4. *Let $(X, d, \mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f, g : X \to X$ be $\mathfrak{R}$-continuous self-mappings and $\alpha, \beta : [0, \infty) \to [0, 1)$ be functions such that α is continuous and $\alpha(t) + \limsup_{s\to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0, \infty)$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} H(Tx, Ty) &\leq \alpha(d(x, y))M_\lambda^T(x, y) + \hat{h}(fy)D(fy, Tx) + \hat{k}(gy)D(gy, Tx) \\ &\quad + \beta(d(x, y))\Big(\max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(y, Tx) + D(x, Ty)}{2}\} \\ &\quad + LD(y, Tx)\Big) \end{aligned}$$

for each $x \ \mathfrak{R} \ y$ with $x \neq y$, where $L \geq 0$. Suppose that
(i) T is monotone of type $\mathfrak{R}_S$;
(ii) there exists $x_0 \in X$ such that $\{x_0\} \ \mathfrak{R}_W \ Tx_0$.
Then $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Proof: The proof follows from Theorem 2.2 and Example 1.13-(III). □

Letting

$$\lambda(\vec{t}_1^5) = \begin{cases} \frac{t_2t_3}{t_1}, & t_1 > 0, \\ 0, & t_1 = 0, \end{cases}$$

we get a generalized multivalued version of the main results of [18, 19].

Corollary 3.5. *Let $(X, d, \mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f, g : X \to X$ be $\mathfrak{R}$-continuous self-mappings and $\alpha, \beta : [0, \infty) \to [0, 1)$ be functions such that α is continuous and $\alpha(t) + \limsup_{s\to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0, \infty)$ such that

$$\begin{aligned} H(Tx, Ty) &\leq \alpha(d(x, y))\frac{D(x, Ty)D(y, Tx)}{d(x, y)} + \hat{h}(fy)D(fy, Tx) + \hat{k}(gy)D(gy, Tx) \\ &\quad + \beta(d(x, y))\Big(\max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(y, Tx) + D(x, Ty)}{2}\} \\ &\quad + LD(y, Tx)\Big) \end{aligned}$$

for each $x \ \mathfrak{R} \ y$ with $x \neq y$, where $L \geq 0$. Suppose that
(i) T is monotone of type $\mathfrak{R}_S$;
(ii) there exists $x_0 \in X$ such that $\{x_0\} \ \mathfrak{R}_W \ Tx_0$.
Then $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Proof: The proof follows from Theorem 2.2, Example 1.15-(IV) and Example 1.13-(III). □

Letting

$$\lambda(\vec{t}_1^5) = \frac{t_3(1 + t_2)}{1 + t_1},$$

we get a generalized multivalued version of main results of [11, 14].

Corollary 3.6. *Let $(X, d, \mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f, g : X \to X$ be $\mathfrak{R}$-continuous self-mappings and $\alpha, \beta : [0, \infty) \to [0, 1)$ be functions such that α is continuous and $\alpha(t) + \limsup_{s\to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*

(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0, \infty)$ such that

$$
\begin{aligned}
H(Tx, Ty) \leq{}& \alpha(d(x,y))\frac{D(y,Ty)(1 + D(x,Tx))}{1 + d(x,y)} \\
&+ \beta(d(x,y))\Big(\max\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(y,Tx) + D(x,Ty)}{2}\} \\
&+ LD(y,Tx)\Big) + \hat{h}(fy)D(fy,Tx) + \hat{k}(gy)D(gy,Tx)
\end{aligned}
$$

for each $x \ \mathfrak{R}\ y$ with $x \neq y$, where $L \geq 0$. Suppose that
(i) T is monotone of type $\mathfrak{R}_S$;
(ii) there exists $x_0 \in X$ such that $\{x_0\}\ \mathfrak{R}_W\ Tx_0$.
Then $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Proof: The proof follows from Theorem 2.2, Example 1.15-(II) and Example 1.13-(III). □

Letting

$$\lambda_4(\vec{t}_1^{5}) = t_3 + Lt_5,$$

where $L \geq 0$, we get the following result.

Corollary 3.7. *Let $(X, d, \mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f, g : X \to X$ be $\mathfrak{R}$-continuous self-mappings and $\alpha, \beta : [0, \infty) \to [0, 1)$ be functions such that α is continuous and $\alpha(t) + \limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0, \infty)$ such that

$$
\begin{aligned}
H(Tx, Ty) \leq{}& \alpha(d(x,y))\Big(D(y,Ty) + LD(y,Tx)\Big) + \beta(d(x,y))\Big(\max\{d(x,y) \\
&D(x,Tx), D(y,Ty), \frac{D(y,Tx) + D(x,Ty)}{2}\} \\
&+ LD(y,Tx)\Big) + \hat{h}(fy)D(fy,Tx) + \hat{k}(gy)D(gy,Tx),
\end{aligned}
$$

for each $x \ \mathfrak{R}\ y$ with $x \neq y$, where $L \geq 0$. Suppose that
(i) T is monotone of type $\mathfrak{R}_S$;
(ii) there exists $x_0 \in X$ such that $\{x_0\}\ \mathfrak{R}_W\ Tx_0$.
Then $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Proof: The proof follows from Theorem 2.2, Example 1.15-(III) and Example 1.13-(III). □

Letting

$$\lambda_4(\vec{t}_1^{5}) = \frac{t_3(1 + t_2)(1 + t_5)}{1 + t_1},$$

we get the following result.

Corollary 3.8. *Let $(X, d, \mathfrak{R})$ be an $\mathfrak{R}$-complete (not necessarily complete) and $\mathfrak{R}$-regular metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f, g : X \to X$ be $\mathfrak{R}$-continuous self-mappings and $\alpha, \beta : [0, \infty) \to [0, 1)$ be functions such that α is continuous and $\alpha(t) + \limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy = gy : y \in Tx\} \subseteq Tx$;
(a_2) there exist two functions $\hat{h}, \hat{k} : X \to [0, \infty)$ such that

$$
\begin{aligned}
H(Tx, Ty) \leq{}& \alpha(d(x,y))\frac{D(y,Ty)(1 + D(x,Tx))(1 + D(y,Tx))}{1 + d(x,y)} \\
&+ \beta(d(x,y))\Big(\max\{d(x,y)D(x,Tx), D(y,Ty), \frac{D(y,Tx) + D(x,Ty)}{2}\} + LD(y,Tx)\Big) \\
&+ \hat{h}(fy)D(fy,Tx) + \hat{k}(gy)D(gy,Tx)
\end{aligned}
$$

for each $x \ \mathfrak{R} \ y$ *with* $x \neq y$, *where* $L \geq 0$. *Suppose that*
(i) *T is monotone of type* $\mathfrak{R}_S$;
(ii) *there exists* $x_0 \in X$ *such that* $\{x_0\} \ \mathfrak{R}_W \ Tx_0$.
Then $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Proof: The proof follows from Theorem 2.2, Example 1.15-(I) and Example 1.13-(III). □

4. Some examples

The following simple examples show the generality of our main theorem over Theorem 1 of [10], Theorem 3.2 of [23], Theorem 2.2 of [15], Theorem 4 of [9], Theorem 5 of [21] and Theorem 2.2 of [1].

Example 4.1. *Let* $X = (-1, \infty)$ *be endowed with the Euclidean metric* $d(x, y) = |x - y|$ *for* $x, y \in X$ *and suppose that* $x \ \mathfrak{R} \ y$ *if and only if* $x = 0$. *Let* $T : X \to CB(X)$ *be given by* $Tx = \{2x\}$ *whenever* $x \in [0, 1)$ *and* $Tx = \{x, 5|x|\}$ *whenever* $x \notin [0, 1)$. *Hence* $(X, d, \mathfrak{R})$ *is an* $\mathfrak{R}$*-complete and* $\mathfrak{R}$*-regular metric space. It is easy to see that T is monotone of type* $\mathfrak{R}_S$ *and* $\{0\} \ (\mathfrak{R}_W) \ T0$. *On the other hand, for* $x, y \in X$ *with* $x \ \mathfrak{R} \ y$, *we have*

$$H(Tx, Ty) \leq \beta(d(x, y))\Big(\hat{\alpha} d(x, y) + LD(y, Tx)\Big),$$

where $\hat{\alpha} = 1$, $L = 12$ *and* $\beta : [0, \infty) \to [0, 1)$ *is defined by* $\beta(t) = \frac{1}{2}$, $t \in [0, \infty)$. *Hence, by Corollary 3.1, for arbitrary functions* $\hat{h}, \hat{k} : X \to [0, \infty)$ *and* $f, g : X \to X$ *satisfying conditions of Corollary 3.1, we conclude that* $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Notice that the mapping T does not satisfy the assumptions of Theorem 3.2 of [23], Theorem 1 of [10], Theorem 2.2 of [15], Theorem 4 of [9], Theorem 5 of [21] and Theorem 2.2 of [1]. For this reason take $x = 2$ *and* $y = 10$.

Example 4.2. *Let* $X = (0, 1]$ *be endowed with the Euclidean metric* $d(x, y) = |x - y|$ *for* $x, y \in X$ *and suppose that* $x \ \mathfrak{R} \ y$ *if and only if* $y = 1$. *Let* $T : X \to CB(X)$ *be given by* $Tx = [\frac{x}{2}, x]$ *whenever* $x \in (0, \frac{1}{2})$ *and* $Tx = \{1\}$ *whenever* $x \in [\frac{1}{2}, 1]$. *Now we can easily show that*
(1) *X is an* $\mathfrak{R}$*-complete and* $\mathfrak{R}$*-regular metric space;*
(2) *there exists* $x_0 \in X$ *such that* $\{x_0\} \ (\mathfrak{R}_W) \ Tx_0$;
(3) *T is monotone of type* $\mathfrak{R}_S$;
(4) *the inequality*

$$\begin{aligned} H(Tx, Ty) \leq{} & \alpha(d(x, y))\Big(LD(y, Tx)\Big) \\ & + \beta(d(x, y)) \max\Big\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}\Big(D(x, Ty) + D(y, Tx)\Big)\Big\} \end{aligned}$$

holds for $x, y \in X$ *with* $x \ \mathfrak{R} \ y$, *where* $\alpha, \beta : [0, \infty) \to [0, 1)$ *is defined by* $\alpha(t) = \frac{1}{4}, \beta(t) = \frac{1}{2}$, $t \in [0, \infty)$ *and* $L = 5$. *Hence, by Corollary 3.7, for arbitrary functions* $\hat{h}, \hat{k} : X \to [0, \infty)$ *and* $f, g : X \to X$ *satisfying conditions of Corollary 3.7, we conclude that* $\mathrm{COP}(f, g, T) \bigcap \mathrm{Fix}(T) \neq \emptyset$.

Notice that the mapping T does not satisfy the assumptions of Theorem 3.2 of [23], Theorem 1 of [10], Theorem 2.2 of [15], Theorem 4 of [9], Theorem 5 of [21] and Theorem 2.2 of [1]. For this reason take $x = \frac{1}{4}$ *and* $y = \frac{1}{8}$.

5. Application: A generalization of the Kelisky-Rivlin theorem

In this section, as application of Corollary 3.2, we present a generalization of Theorem 4.1 of [20] and Theorem 6 of [26] in the following.

Theorem 5.1. *Let* $(X, \|.\|)$ *be a norm space (not necessarily a Banach space) and* X_0 *be a complete subspace of X. Let* $f : X \to X$ *be a map (not necessarily linear) and* $\beta : [0, \infty) \to [0, 1)$ *be function such that* $\limsup_{s \to t^+} \beta(s) < 1$ *for each* $t \geq 0$. *Assume that*

$$d(fx, fy) \leq \beta(d(x, y)\Big(\hat{\alpha} d(x, y) + \hat{\beta} d(x, fx) + \hat{\gamma} d(y, fy) + \hat{\delta} d(x, fy)\Big) \tag{12}$$

whenever $x - y \in X_0$ with $x \neq y$, where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \geq 0$, $\hat{\alpha} + \hat{\beta} + \hat{\gamma} + 2\hat{\delta} = 1$, $\hat{\gamma} \neq 1$ and the metric d induced by $\|.\|$, i.e., $d(x, y) = \|x - y\|$ for each $x, y \in X$. If $(I - f)(X) \subseteq X_0$, then for each $x \in X$, $\{f^n(x)\}$ converges to a fixed point of f. Moreover, $(x + X_0) \cap Fixf = \{\lim_{n\to\infty} f^n x\}$ for each $x \in X$.

Proof: Consider the relation $\mathfrak{R}$ defined on X as follows:

$$x\mathfrak{R}y \Longleftrightarrow \Big(x - y \in X_0\Big).$$

We shall separate the proof into several steps.

Step 1. $(X, d, \mathfrak{R})$ is an $\mathfrak{R}$-complete metric space.

Let $\{x_n\}$ ba a Cauchy $\mathfrak{R}$-sequence in $(X, d, \mathfrak{R})$. By definition of the relation $\mathfrak{R}$, we obtain $x_{n+k} - x_n \in X_0$ for each $n, k \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ be a fixed element. Since $\{x_n\}$ is a Cauchy sequence, also $\{y_k := x_{n_0+k} - x_{n_0}\}$ is a Cauchy sequence in X_0. Let $\lim_{k\to\infty} y_k = y^* \in X_0$. This shows that $\{x_{n_0+k}\}$ is a convergent sequence. Let $\lim_{k\to\infty} x_{n_0+k} = x^*$, for some $x^* \in X$. On the other hand, $\{x_n\}$ is a Cauchy sequence and has a convergent subsequence, then it converges to x^*. Hence, $x^* = y^* + x_{n_0} \in X$.

Step 2. $(X, d, \mathfrak{R})$ is an $\mathfrak{R}$-regular metric space.

Take $\mathfrak{R}$-sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$. Since $\{x_n\}$ is an $\mathfrak{R}$-sequence, for each $n, k \in \mathbb{N}$, $x_{n+k} - x_n \in X_0$ which gives rise to $x - x_n \in X_0$ for all $n \in \mathbb{N}$, i.e., $x_n \ \mathfrak{R}\ x$ for all $n \in \mathbb{N}$. Hence $(X, d, \mathfrak{R})$ is an $\mathfrak{R}$-regular metric space.

Step 3. f is monotone and $x \ \mathfrak{R}\ fx$ for each $x \in X$.

Let $x \ \mathfrak{R}\ y$. Now, $fx - fy = (y - fy) - (x - fx) + (x - y) \in X_0$, since $(I - f)(X) \subseteq X_0$. Hence $fx \ \mathfrak{R}\ fy$. Also, since $(I - f)(X) \subseteq X_0$, $x \ \mathfrak{R}\ fx$ for each $x \in X$.

Let $x \in X$ be arbitrary. Then by Corollary 3.2, $\{f^n(x)\}_{n=0}^{\infty}$ converges to a fixed point f. Besides, $\{f^n(x)\}_{n=0}^{\infty}$ is an $\mathfrak{R}$-sequence, then, for each $m \in \mathbb{N} \cup \{0\}$, $f^m x \ \mathfrak{R}\ \lim_{n\to\infty} f^n x$. Hence $x \ \mathfrak{R}\ \lim_{n\to\infty} f^n x$. This shows that $\lim_{n\to\infty} f^n x \in (x + X_0) \cap \mathrm{Fix} f$.

Let $z_1, z_2 \in (x + X_0) \cap \mathrm{Fix} f$, then $z_1 \ \mathfrak{R}\ z_2$. Here, by applying Corollary 3.2, $z_1 = z_2$. Therefore, we concede that $(x + X_0) \cap \mathrm{Fix} f = \{\lim_{n\to\infty} f^n x\}$ for each $x \in X$. □

Consider the space of all continuous real functions on the closed unit interval $C[0, 1]$ and $X := \{x \in C[0, 1] : \big|x(t)\big| < 1, \forall t \in [0, 1]\}$. We endow X with the sup norm $\|.\|_\infty$. Notice that $(X, \|.\|_\infty)$ is not a Banach space.

We now consider the operator $\hat{B}_n$ where $n \in \mathbb{N}$ on X defined by

$$\hat{B}_n(x)(t) = \sum_{k=0}^{n} \Big|x(\frac{k}{n})\Big|\binom{n}{k}t^k(1 - t)^{n-k} \text{ for } x \in X \text{ and } t \in [0, 1]. \tag{13}$$

Let $x \in X$ and $L := \max_{0\leq k\leq n}\{x(\frac{k}{n})\}$. Then $L < 1$ and also

$$\Big|\hat{B}_n(x)(t)\Big| = \Big|\sum_{k=0}^{n} \Big|x(\frac{k}{n})\Big|\binom{n}{k}t^k(1 - t)^{n-k}\Big| \leq L\sum_{k=0}^{n}\binom{n}{k}t^k(1 - t)^{n-k} = L < 1, \forall t \in [0, 1].$$

Hence $\hat{B}_n : X \to X$ is a nonlinear self mapping.

Corollary 5.2. *Let $n \in \mathbb{N}$ and $\hat{B}_n$ be an operator on X defined by* (13). *Let $\bar{X} := \{x \in X : x(0) \geq 0 \ \text{ and } \ x(1) \geq 0\}$. Then, for each $x \in X$, $\{\hat{B}_n^j\}_{j\in\mathbb{N}}$ converges to a fixed point of $\hat{B}_n$. Moreover, for each $x \in \bar{X}$, we have*

$$\lim_{j\to\infty} \hat{B}_n^j x = x(0)(1 - t) + x(1)t \ \text{ for } \ t \in [0, 1].$$

Proof: Let $X_0 := \{x \in X : x(0) = x(1)\}$. Let $x, y \in X$ be elements such that $x - y \in X_0$. Then, for each $t \in [0,1]$,

$$\begin{aligned} \left|\hat{B}_n(x)(t) - \hat{B}_n(y)(t)\right| &= \Big|\sum_{k=0}^{n}\left|x(\frac{k}{n})\right|\binom{n}{k}t^k(1-t)^{n-k} - \sum_{k=0}^{n}\left|y(\frac{k}{n})\right|\binom{n}{k}t^k(1-t)^{n-k}\Big| \\ &\le \sum_{k=0}^{n}\left|(x-y)(\frac{k}{n})\right|\binom{n}{k}t^k(1-t)^{n-k} \\ &\le \sum_{k=1}^{n-1}\|x-y\|_\infty\binom{n}{k}t^k(1-t)^{n-k} \\ &\le \Big(1 - t^n - (1-t)^n\Big)\|x-y\|_\infty \le \Big(1 - \frac{1}{2^{n-1}}\Big)\|x-y\|_\infty. \end{aligned} \tag{14}$$

Hence, $\|\hat{B}_n(x) - \hat{B}_n(y)\|_\infty \le \beta(\|x-y\|_\infty)\|x-y\|_\infty$, where $\beta(t) = \Big(1 - \frac{1}{2^{n-1}}\Big)$ for each $t \ge 0$. It is easy to see that $(I - \hat{B}_n)(X) \subseteq X_0$. Hence by Theorem 5.1, for each $x \in X$, $\{\hat{B}_n^j x\}_{j\in\mathbb{N}}$ converges to a fixed point of $\hat{B}_n$ and $(x + X_0) \cap \mathrm{Fix}\hat{B}_n = \{\lim_{j\to\infty}\hat{B}_n^j x\}$. Since $e_0, e_1 \in \mathrm{Fix}\hat{B}_n$, where $e_i(t) = t^i$ for $i = 0, 1$ and $t \in [0,1]$, then $x(0)(1-t) + x(1)t \in \mathrm{Fix}\hat{B}_n$ for each $x \in \bar{X}$. Also, for each $x \in \bar{X}$, $x(0)(1-t) + x(1)t \in x + X_0$. Hence $x(0)(1-t) + x(1)t \in (x + X_0) \cap \mathrm{Fix}\hat{B}_n$. □

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