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Geodesic Mappings of Spaces with Affine Connnection onto Generalized Ricci Symmetric Spaces

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Abstract. The presented work is devoted to study of the geodesic mappings of spaces with affine connection onto generalized Ricci symmetric spaces. We obtained a fundamental system for this problem in a form of a system of Cauchy type equations in covariant derivatives depending on no more than $\frac{1}{2}n^2(n+1)+n$ real parameters. Analogous results are obtained for geodesic mappings of manifolds with affine connection onto equiaffine generalized Ricci symmetric spaces.

1. Introduction

The paper is devoted to study of geodesic mappings theory of special spaces with affine connection. The first idea about this theory appears in the paper [15] by T. Levi-Civita, where the problem of finding Riemannian spaces with common geodesics was defined and solved in a special coordinate system. Let us remark an interesting fact, that this theory was connected with a study of equations of dynamic of mechanical systems.

Later, the theory of geodesic mappings was developed in works by Thomas, Cartan, Eisenhart, Weyl, Shirokov, Kagan, Vranceanu, Rashevski, Solodovnikov, Sinyukov, Petrov, Prvanović, Mikeš, etc. [4–7, 10–13, 22–26, 30–45].

Geodesic mappings of Einstein, symmetric, recurrent spaces and the generalizations of these mappings were studied in [1, 2, 8, 10, 13, 16–22, 26–29, 31, 33, 36, 39, 41]. Detailed analysis of geodesic mappings of generalized symmetric recurrent manifolds is presented in work [1, 3].

In presented paper we continue to study geodesic mappings of generalized Ricci symmetric manifolds which are a natural generalization of the Ricci manifolds. We find fundamental equations of geodesic mappings of spaces with affine connection onto generalized Ricci symmetric spaces in closed Cauchy type system equations in covariant derivatives. General solutions of this system depended on finite number of real parameters. We study a special case of the above mentioned mappings.

Let us suppose, that studied object are continuous and smooth enough.

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2. Geodesic mappings of spaces with affine connection

Let $f\colon A_n \to \bar{A}_n$ be a diffeomorphism between spaces $A_n = (M, \nabla)$ and $\bar{A}_n = (\bar{M}, \bar{\nabla})$ with affine connections ∇ and $\bar{\nabla}$, where M and $\bar{M}$ are n-dimensional manifolds. We supposed that $M \equiv \bar{M}$ and therefore in coordinate neighbourhood (U, x) corresponding points $x \in M$ and $f(x) \in \bar{M}$ have "common" coordinates $(x^1, x^2, \dots, x^n)$.

We define a *deformation tensor of affine connections respective mapping* $f\colon A_n \to \bar{A}_n$ in the following form $P = \bar{\nabla} - \nabla$, i.e. their components in common coordinate system x have the following form

$$P^h_{ij}(x) = \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{1}$$

where $\Gamma^h_{ij}(x)$ and $\bar{\Gamma}^h_{ij}(x)$ are components of connections ∇ and $\bar{\nabla}$ of spaces A_n and $\bar{A}_n$, respectively.

A curve $\ell(t)$ in space A_n is a *geodesic* if there exists a vector field along ℓ which is tangent and parallel along ℓ. A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called *geodesic mapping* of A_n onto $\bar{A}_n$ if f maps any geodesic in A_n onto a geodesic in $\bar{A}_n$.

It is known [5, 6, 22, 26, 31, 41], that diffeomorphism $f\colon A_n \to \bar{A}_n$ is a geodesic mapping if and only if in common coordinate system $x = (x^1, x^2, \dots, x^n)$ the deformation tensor (1) has the following form

$$P^h_{ij}(x) = \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i, \tag{2}$$

where $\psi_i(x)$ are components of a covector ψ and δ^h_i is the Kronecker symbol. Geodesic mapping is *non trivial* if $\psi_i(x) \not\equiv 0$.

Evidently, any space A_n admit non trivial geodesic mapping onto other space $\bar{A}_n$. Analogically, the statement is not valid for geodesic mappings onto Riemannian spaces. Particularly, there are found spaces with affine connection which do not admit non trivial geodesic mappings onto (pseudo-) Riemannian spaces, see [10, 16, 17, 19–22, 26–29, 31, 39, 41].

We obtained [1, 2] that fundamental equations of geodesic mappings of spaces with affine connection onto Riemannian spaces and fundamental equations of geodesic mappings of spaces with affine connection onto Ricci symmetric spaces are formed to closed Cauchy type equations system in covariant derivative. Moreover, for geodesic mappings onto Riemannian spaces this system is linear.

3. Geodesic mappings of spaces with affine connection onto generalized Ricci symmetric spaces

In paper [3], see [26, pp. 469–473], we studied geodesic mappings onto *generalized Ricci symmetric space*. In that case, there were found the fundamental equations in the form of closed Cauchy type equations system in covariant derivative. In this section we find the fundamental equations of the geodesic mappings in a simpler form.

Let $f\colon A_n \to \bar{A}_n$ be a geodesic mapping of space with affine connection A_n onto generalized Ricci symmetric spaces $\bar{A}_n$. We supposed that $x = (x^1, x^2, \dots, x^n)$ is common coordinate system respective mapping f.

Space $\bar{A}_n$ is called *generalized Ricci symmetric space* if Ricci tensor satisfy the following condition

$$\bar{R}_{ij|k} + \bar{R}_{kj|i} = 0, \tag{3}$$

where $\bar{R}_{ij}$ are components of Ricci tensor on $\bar{A}_n$ and symbol "$|$" denotes covariant derivative on $\bar{A}_n$.

Let us note, that generalized Ricci symmetric space which is (pseudo-) Riemannian space is *Ricci symmetric* ($\bar{R}_{ij|k} = 0$). Let us remark, that the geodesic mappings of Ricci symmetric spaces were studied in [1].

Because the covariant derivative of Riemannian tensor has the following form

$$\bar{R}^h_{ijk|m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{m\alpha}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{mi}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{mj}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{mk}\bar{R}^h_{ij\alpha},$$

from condition (1) we obtain

$$\bar{R}^h_{ijk|m} = \bar{R}^h_{ijk,m} + P^h_{m\alpha}\bar{R}^\alpha_{ijk} - P^\alpha_{mi}\bar{R}^h_{\alpha jk} - P^\alpha_{mj}\bar{R}^h_{i\alpha k} - P^\alpha_{mk}\bar{R}^h_{ij\alpha}, \tag{4}$$

where " , " denotes covariant derivative on A_n, R^h_{ijk} and $\bar{R}^h_{ijk}$ are components of Riemannian tensors of A_n and $\bar{A}_n$respectively.

After contracting (4) with respect to the indices h and k we get

$$\bar{R}_{ij|m} = \bar{R}_{ij,m} - P^\alpha_{mi}\bar{R}_{\alpha j} - P^\alpha_{mj}\bar{R}_{i\alpha} \tag{5}$$

and after symmetrization (5) with respect to the indices i and m we obtain

$$\bar{R}_{ij|m} + \bar{R}_{mj|i} = \bar{R}_{ij,m} + \bar{R}_{mj,i} - 2P^\alpha_{mi}\bar{R}_{\alpha j} - P^\alpha_{mj}\bar{R}_{i\alpha} - P^\alpha_{ij}\bar{R}_{m\alpha}. \tag{6}$$

Since the space $\bar{A}_n$ is generalized Ricci symmetric it satisfies equation (3). Therefore from (6) follows

$$\bar{R}_{ij,m} + \bar{R}_{mj,i} = 2P^\alpha_{mi}\bar{R}_{\alpha j} + P^\alpha_{mj}\bar{R}_{i\alpha} + P^\alpha_{ij}\bar{R}_{m\alpha}. \tag{7}$$

Deformation tensor of affine connection P^h_{ij} has form (2), and from its formula (7) we conclude

$$\bar{R}_{ij,m} + \bar{R}_{mj,i} = 3\psi_m\bar{R}_{ij} + 3\psi_i\bar{R}_{mj} + \psi_j(\bar{R}_{im} + \bar{R}_{mi}). \tag{8}$$

It is known [41] and [26, p. 182], that between Riemannian tensors R^h_{ijk} and $\bar{R}^h_{ijk}$ of A_n and $\bar{A}_n$ the following dependence holds

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{j\alpha} - P^\alpha_{ij}P^h_{k\alpha}. \tag{9}$$

From (2) it follows $P^h_{ij,k} = \psi_{i,k}\delta^h_j + \psi_{j,k}\delta^h_i$, and from formula (9) we obtain

$$\bar{R}^h_{ijk} = R^h_{ijk} - \delta^h_j\psi_{i,k} + \delta^h_k\psi_{i,j} - \delta^h_i\psi_{j,k} + \delta^h_i\psi_{k,j} + \delta^h_j\psi_i\psi_k - \delta^h_k\psi_i\psi_j. \tag{10}$$

Finally, by contracting (10) with respect to the indices h and k we get

$$\bar{R}_{ij} = R_{ij} + n\psi_{i,j} - \psi_{j,i} + (1-n)\psi_i\psi_j, \tag{11}$$

and after alternating (11) we obtain

$$\bar{R}_{[ij]} = R_{[ij]} + (n+1)\psi_{i,j} - (n+1)\psi_{j,i}, \tag{12}$$

where $[ij]$ denotes alternation respective indices i and j.

From condition (12) we obtain

$$\psi_{i,j} - \psi_{j,i} = \frac{1}{n+1}\left(\bar{R}_{[ij]} - R_{[ij]}\right). \tag{13}$$

Analyzing (11) and (13) we get the following equation

$$\psi_{i,j} = \frac{1}{n^2-1}\left[n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})\right] + \psi_i\psi_j. \tag{14}$$

Equation (8) covariantly differentiating respective x^k in space A_n and after substituing (14) follows

$$\bar{R}_{ij,mk} + \bar{R}_{mj,ik} = 3\psi_m\bar{R}_{ij,k} + 3\psi_i\bar{R}_{mj,k} + \psi_j(\bar{R}_{im,k} + \bar{R}_{mi,k}) + T_{ijmk}, \tag{15}$$

where

$$T_{ijmk} = 3\left(\frac{1}{n^2-1}\left(n\bar{R}_{mk} + \bar{R}_{km} - (nR_{mk} + R_{km})\right) + \psi_m\psi_k\right)\bar{R}_{ij} + \\ +3\left(\frac{1}{n^2-1}\left(n\bar{R}_{ik} + \bar{R}_{ki} - (nR_{ik} + R_{ki})\right) + \psi_i\psi_k\right)\bar{R}_{mj} + \\ +\left(\frac{1}{n^2-1}\left(n\bar{R}_{jk} + \bar{R}_{kj} - (nR_{jk} + R_{kj})\right) + \psi_j\psi_k\right)(\bar{R}_{im} + \bar{R}_{mi}).$$

By alternating condition (15) with respect to the indices i and k:

$$\bar{R}_{ij,mk} - \bar{R}_{kj,mi} = \bar{R}_{\alpha j}R^{\alpha}_{mki} + \bar{R}_{m\alpha}R^{\alpha}_{jki} + 3\psi_m\bar{R}_{ij,k} - 3\psi_m\bar{R}_{kj,i} + \\ +3\psi_i\bar{R}_{mj,k} - 3\psi_k\bar{R}_{mj,i} + \psi_j(\bar{R}_{im,k} - \bar{R}_{km,i} + \bar{R}_{mi,k} - \bar{R}_{mk,i}) + T_{ijmk} - T_{kjmi}.$$

From the Ricci identity and the algebraic identy of Riemannian tensor we obtain

$$\bar{R}_{ij,km} - \bar{R}_{kj,im} = 2\bar{R}_{\alpha j}R^{\alpha}_{mki} + \bar{R}_{i\alpha}R^{\alpha}_{jkm} + \bar{R}_{k\alpha}R^{\alpha}_{jmi} + \bar{R}_{m\alpha}R^{\alpha}_{jki} + \\ +3\psi_m\bar{R}_{ij,k} - 3\psi_m\bar{R}_{kj,i} + 3\psi_i\bar{R}_{mj,k} - 3\psi_k\bar{R}_{mj,i} + \\ +\psi_j(\bar{R}_{im,k} - \bar{R}_{km,i} + \bar{R}_{mi,k} - \bar{R}_{mk,i}) + T_{ijmk} - T_{kjmi}.$$

Finally, by replacing the indices k and m in the last formula and adding to (15) we get

$$\begin{aligned} 2\bar{R}_{ijm,k} \quad = \quad & 2\bar{R}_{\alpha j}R^{\alpha}_{kim} + \bar{R}_{i\alpha}R^{\alpha}_{jmk} + \bar{R}_{m\alpha}R^{\alpha}_{jki} + \bar{R}_{k\alpha}R^{\alpha}_{jmi} + \\ & +3\psi_k(\bar{R}_{ijm} - \bar{R}_{mji}) + 3\psi_i(\bar{R}_{kjm} + \bar{R}_{mjk}) + 3\psi_m(\bar{R}_{ijk} - \bar{R}_{kji}) + \\ & +\psi_j(\bar{R}_{imk} + \bar{R}_{mik} + \bar{R}_{ikm} - \bar{R}_{mki} + \bar{R}_{kim} - \bar{R}_{kmi}) + \\ & +T_{ijkm} - T_{mjki} + T_{ijmk}, \end{aligned} \tag{16}$$

where $\bar{R}_{ijm} = \bar{R}_{ij,m}$.

Evidently, the equations (14) and (16) together with

$$\bar{R}_{ij,m} = \bar{R}_{ijm} \tag{17}$$

form closed Cauchy type system on space A_n with respect to unknown functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, and $\bar{R}_{ijk}(x)$.

We obtain the following

Theorem 3.1. *A space with affine connection A_n admits geodesic mapping onto generalized Ricci symmetric space $\bar{A}_n$ if and only if on A_n there exists a solution of closed Cauchy type system of equations in covariant derivative* (14), (16) *and* (17) *with respect to unknown functions $\psi_i(x)$, $\bar{R}_{ij}(x)$, and $\bar{R}_{ijk}(x)$.*

General solution of the system (14), (16) and (17) depends on no more than

$$n + n^2 + \frac{1}{2}n^2(n-1) \equiv \frac{1}{2}n^2(n+1) + n$$

real parameters.

From the conditions (1) and (2) it follows that the space $\bar{A}_n$ from the Theorem 3.1 has the equiaffine connection

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i\psi_j + \delta^h_j\psi_i,$$

where ψ_i is the solution of the above mentioned system (14), (16) and (17). Furthermore, it follows that $\bar{R}_{ij}$ are components of the Ricci tensor on $\bar{A}_n$.

4. Geodesic mappings of spaces with affine connection onto equiaffine generalized Ricci symmetric spaces

It is known [41], [26, pp. 85], that equiaffine spaces are defined by symmetry of Ricci tensor. We verify, that the following lemma holds.

Lemma 4.1. *Equiaffine generalized Ricci symmetric space is Ricci symmetric.*

Proof. Let $\bar{A}_n$ be a generalized Ricci symmetric space for which the condition (3) holds. Since $\bar{A}_n$ is equiaffine then

$$\bar{R}_{ij} = \bar{R}_{ji}. \tag{18}$$

After differentiating (18) we obtain

$$\bar{R}_{ij|k} = \bar{R}_{ji|k}. \tag{19}$$

Then from the properties (3) and (19) it follows

$$\bar{R}_{ij|k} \overset{(3)}{=} -\bar{R}_{ik|j} \overset{(19)}{=} -\bar{R}_{ki|j} \overset{(3)}{=} \bar{R}_{kj|i} \overset{(19)}{=} \bar{R}_{jk|i} \overset{(3)}{=} -\bar{R}_{ji|k} \overset{(19)}{=} -\bar{R}_{ij|k}.$$

Now we compare first and the last article and verify that

$$\bar{R}_{ij|k} = 0,$$

therefore $\bar{A}_n$ is Ricci symmetric. □

Geodesic mappings onto Ricci symmetric manifolds were studied in [1]. In our case the equations of such mapping would be simpler because $\bar{A}_n$ is equiaffine. This system of equations, using [1], has the form:

$$\bar{R}_{ij,m} = 2\psi_m\bar{R}_{ij} + \psi_i\bar{R}_{mj} + \psi_j\bar{R}_{im} \tag{20}$$

$$\psi_{i,j} = \frac{1}{n^2-1}\Big[(n+1)\bar{R}_{ij} - (nR_{ij} + R_{ji})\Big] + \psi_i\psi_j. \tag{21}$$

It is evident that the equations (20) and (21) in the given manifold represent a closed Cauchy type system with respect to unknown functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$.

Theorem 4.2. *A manifold A_n with affine connection admits a geodesic mapping onto an equiaffine Ricci symmetric manifold $\bar{A}_n$ if and only if in A_n there exists a solution of a closed Cauchy type equations in covariant derivative* (20) *and* (21) *with respect to unknown functions $\bar{R}_{ij}(x)$ $(= R_{ji}(x))$ and $\psi_i(x)$.*

General solution of a closed Cauchy system of equations (20) and (21) depends on no more than $\frac{1}{2}n(n+1) + n \equiv \frac{1}{2}n(n+3)$ independent real parameters.

From the conditions (1) and (2) it follows that the space $\bar{A}_n$ from the Theorem 4.2 has the equiaffine connection

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i\psi_j + \delta^h_j\psi_i,$$

where ψ_i is a solution of the above mentioned system (20) and (21). Furthermore, it follows that $\bar{R}_{ij}$ are components of the Ricci tensor on $\bar{A}_n$.

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