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Sequentially Right-Like Properties on Banach Spaces

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Abstract. In this paper, we study first the concept of p-sequentially Right property, which is p-version of the sequentially Right property. Also, we introduce a new class of subsets of Banach spaces which is called p-Right* set and obtain the relationship between p-Right subsets and p-Right* subsets of dual spaces. Furthermore, for $1 \leq p < q \leq \infty$, we introduce the concepts of properties $(SR)_{p,q}$ and $(SR^*)_{p,q}$ in order to find a condition such that every Dunford-Pettis q-convergent operator is Dunford-Pettis p-convergent. Finally, we apply these concepts and obtain some characterizations of the p-Dunford-Pettis relatively compact property of Banach spaces and their dual spaces.

1. Introduction

The concept of sequentially Right property on Banach spaces was introduced by Peralta et al. [26]. Later on Kacena [21], by introducing the notion of Right set in the dual of X, gave a characterization of those Banach spaces which have the sequentially Right property. Recently, Cilia and Emmanuele [8] and Ghenciu [18], obtained some sufficient conditions implying that the projective tensor product of two Banach spaces has the sequentially Right property. Moreover, they introduced the concept sequentially Right* property on Banach spaces and gave its characterization. For more information and examples of those spaces with the sequentially Right property and sequentially Right* property, we refer to [8, 17, 21, 26]. Recently, Ghenciu [19], by introducing the concepts of Dunford-Pettis p-convergent operators, p-Right sets and p-sequentially Right property obtained some characterizations of these notions. Numerous authors, by studying localized properties, e.g., (V)-sets, (V^*)-sets, Dunford-Pettis sets, (L)-sets, point evaluations sets and Right sets, showed how these notions can be used to study more global structure properties. For instance, we know that, every Dunford-Pettis subset of a dual space is an (L) subset, while the converse of this implication is false. The relationship between (L) subsets and Dunford-Pettis subsets of dual spaces was obtained by Bator et al. [3]. Recently, the authors [8, 18], by using Right topology, proved that a sequence $(x_n)_n$ in a Banach space X is Right null if and only if it is Dunford-Pettis and weakly null. Also, they showed that a sequence $(x_n)_n$ in a Banach space X is Right Cauchy if and only if it is Dunford-Pettis and weakly Cauchy.
Motivated by these facts, in Section 3, we introduce the concepts of p-Right null and p-Right Cauchy sequences, p-Right* sets and p-sequentially Right* property on Banach space. Also, we obtain some characterizations of these properties. Moreover, we obtain the relationship between p-Right subsets and p-Right* subsets of dual spaces. Finally, the stability of the p-sequentially Right property for some subspaces of the space of bounded linear operators and the projective tensor product between two Banach spaces are

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investigated.
In the Section 4, for $1 \leq p < q \leq \infty$, motivated by the class $\mathcal{P}_{p,q}$ [27], for those Banach spaces in which relatively p-compact sets are relatively q-compact, we introduce the concepts of properties $(SR)_{p,q}$ and $(SR^*)_{p,q}$ for those Banach spaces in which Dunford-Pettis q-convergent operators are Dunford-Pettis p-convergent operators. Finally, by using these concepts some characterizations for the p-version of the Dunford-Pettis relatively compact property of Banach spaces and their dual spaces are investigated.

2. Definitions and Notions

Let $1 \leq p < \infty$. A sequence $(x_n)_n$ in X is said to be weakly p-summable, if $(x^*(x_n))_n \in \ell_p$ for each $x^* \in X^*$. We denote the set of all weakly p-summable sequences in X by $\ell_p^w(X)$ [12]. Note that, a sequence $(x_n)_n$ in X is said to be weakly p-convergent to $x \in X$ if $(x_n - x)_n \in \ell_p^w(X)$. A bounded subset K of X is a Dunford-Pettis set, if every weakly null sequence $(x_n^*)_n$ in X^*, converges uniformly to zero on the set K [2]. It is easy to verify that the class of Dunford-Pettis sets strictly contains the class of relatively compact sets, but in general the converse is not true. Let us recall from [13], that a Banach space X has the Dunford-Pettis relatively compact property (in short X has the $(DPrcP)$), if every Dunford-Pettis subset of X is relatively compact. A bounded linear T from a Banach space X to a Banach space Y is called Dunford-Pettis completely continuous, if it transforms Dunford-Pettis and weakly null sequences to norm null ones. The class of Dunford-Pettis completely continuous operators from X to Y is denoted by $DPcc(X,Y)$[30]. A bounded linear operator T between two Banach spaces is called p-convergent, if it transforms weakly p-summable sequences into norm null sequences [5]. We denote the class of p-convergent operators from X into Y by $C_p(X,Y)$. A Banach space X has the p-Schur property (in short $X \in C_p$), if every weakly p-summable sequence in X is norm null. It is clear that, X has the ∞-Schur property if and only if every weakly null sequence in X is norm null. So the ∞-Schur property coincides with the classical Schur property. A Banach space X has the Dunford-Pettis property of order p (X has the (DPP_p)), if every weakly compact operator on X is p-convergent [6]. A subset K of a Banach space X is called relatively weakly p-compact, if each sequence in K admits a weakly p-convergent subsequence with limit in X. If the "limit point" of each weakly p-convergent subsequence lies in K, then we say that K is a weakly p-compact set. A bounded linear operator $T : X \to Y$ is called weakly p-compact, if $T(B_X)$ is a relatively weakly p-compact set in Y [6]. The set of weakly p-compact operators $T : X \to Y$ will be denoted by $W_p(X,Y)$. Let us recall from [29], that $\ell_p(X)$ denote the set of all sequences $(x_n)_n$ in X such that $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. A set $K \subset X$ is said to be relatively p-compact if there is a sequence $(x_n)_n$ in $\ell_p(X)$ such that $K \subset \{\sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n) \in B_{\ell_{p^*}}\}$. An operator $T \in L(X,Y)$ is said to be p-compact if $T(B_X)$ is a relatively p-compact set in Y.

Throughout this paper $1 \leq p < \infty$, $1 \leq p < q \leq \infty$, except for those cases where we consider other assumptions. Also, we suppose X, Y and Z are arbitrary Banach spaces, p^* is the Hölder conjugate of p; if $p = 1$, ℓ_{p^*} plays the role of c_0. The unit coordinate vector in ℓ_p (resp. c_0 or ℓ_∞) is denoted by e_n^p (resp. e_n). The space X embeds in Y, if X is isomorphic to a closed subspace of Y (in short we denote $X \hookrightarrow Y$). We denote two isometrically isomorphic spaces X and Y by $X \cong Y$. The word "operator" will always means a bounded linear operator. For any Banach space X, the dual space of bounded linear functionals on X will be denoted by X^*. Also we use $\langle \cdot, \cdot \rangle$ for the duality between X and X^*. We denote the closed unit ball of X by B_X and the identity operator on X is denoted by id_X. For a bounded linear operator $T : X \to Y$, the adjoint of the operator T is denoted by T^*. The space of all bounded linear operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X,Y)$, $W(X,Y)$, and $K(X,Y)$, respectively. The space of all w^*-w continuous (resp. compact) operators from X^* to Y will be denoted by $L_{w^*}(X^*,Y)$ (resp. $K_{w^*}(X^*,Y)$). The projective tensor product of two Banach spaces X and Y will be denoted by $X\widehat{\bigotimes}_\pi Y$. We recall the following well-known isometries ([28, page 60]):
(1) $L_{w^*}(X^*,Y) \simeq L_{w^*}(Y^*,X)$ and $K_{w^*}(X^*,Y) \simeq K_{w^*}(Y^*,X)$ $(T \mapsto T^*)$.
(2) $W(X,Y) \simeq L_{w^*}(X^{**},Y)$ and $K(X,Y) \simeq K_{w^*}(X^{**},Y)$ $(T \mapsto T^{**})$.
We refer the reader for undefined terminologies to the classical references [1, 11].

3. p-sequentially Right and p-sequentially Right* properties on Banach spaces

In this section, we introduce the notion of p-Right* sets and obtain the relationship between p-Right subsets and p-Right* subsets of dual spaces. In addition, the stability of the p-sequentially Right property for some subspaces of bounded linear operators and the projective tensor product between two Banach spaces are investigated.
The main goal of this section is to obtain some characterizations of p-Right sets that are relatively weakly q-compact, whenever $1 \leq p < q \leq \infty$.

Let us recall some concepts from [19]:

- An operator $T : X \to Y$ is said to be Dunford-Pettis p-convergent, if it takes Dunford-Pettis weakly p-summable sequences to norm null sequences. The class of Dunford-Pettis p-convergent operators from X into Y is denoted by $DPC_p(X, Y)$.
- A Banach space X has the p-Dunford-Pettis relatively compact property (X has the p-$(DPrcP)$), if every Dunford-Pettis weakly p-summable sequence $(x_n)_n$ in X is norm null. Note that, the ∞-$(DPrcP)$ is precisely the $(DPrcP)$.
- A bounded subset K of X^* is called a p-Right set, if every Dunford-Pettis weakly p-summable sequence $(x_n)_n$ in X converges uniformly to zero on K, that is,

$$\lim_n \sup_{x^* \in K} |x^*(x_n)| = 0.$$

- A Banach space X has the p-sequentially Right property (X has the p-(SR)), if every p-Right set in X^* is relatively weakly compact. Note that, the ∞-sequentially Right property is precisely the sequentially Right property.

Inspired by Right null and Right Cauchy sequences [8, 18], we introduce the following definition:

Definition 3.1. (i) *We say that a sequence $(x_n)_n$ in X is p-Right null, if $(x_n)_n$ is Dunford-Pettis weakly p-summable.*
(ii) *A sequence $(x_n)_n$ in X is p-Right Cauchy, if $(x_n)_n$ is Dunford-Pettis weakly p-Cauchy.*

It is clear that, the ∞-Right null sequences are precisely the Right null sequences and ∞-Right Cauchy sequences are precisely the Right Cauchy sequences.

Recently, the notions of p-(V) sets and p-(V) property as an extension of the notions (V) sets and Pelczyński's property (V) introduced by Li et al. [22] as:

- A bounded subset K of X^* is a p-(V) set, if $\lim_{n\to\infty} \sup_{x^* \in K} |x^*(x_n)| = 0$, for every weakly p-summable sequence $(x_n)_n$ in X.
- A Banach space X has Pelczyński's property (V) of order p (p-(V) property), if every p-(V) subset of X^* is relatively weakly compact.

Remark 3.2. (i) Every relatively weakly compact subset of a dual Banach space is a p-Right set, but the converse, in general, is false. For example, ℓ_1 has the Schur property and so, ℓ_1 has the p-$(DPrcP)$. Hence, the closed unit ball B_{ℓ_∞} of ℓ_∞ is a p-Right set, while it is not relatively weakly compact.
(ii) Every p-(V) set in X^* is a p-Right set, but the converse, in general, is false. For example, since ℓ_2 has the 2-$(DPrcP)$, the closed unit ball B_{ℓ_2} of ℓ_2 is a 2-Right set, while it is not 2-(V) set.
(iii) Every weakly p-compact operator is Dunford-Pettis p-convergent, but in general the converse is not true. For example, the identity operator on ℓ_1 is Dunford-Pettis p-convergent, while it is not weakly p-compact.
(iv) It is easy to verify, if K is a infinite compact Hausdorff metric space, then the Banach space $C(K)$ of all continuous functions on K has Pelczyński's property (V) of order p. On the other hand $C(K)$ has the (DPP_p). Hence, by ([19, Corollary 3.19 (ii)]) $C(K)$ has the p-(SR) property. In particular, c_0 and ℓ_∞ have the p-(SR)

property. However, ([26, Example 8]) shows that ℓ_1 as a subspace of ℓ_∞ does not have the sequentially Right property and so, does not have the p-(SR) property.
(v) It is clear that, every reflexive Banach space has the p-(SR) property. But, there exists a non reflexive Banach space with the p-(SR) property. For example, c_0 has the p-(SR) property, while c_0 is not reflexive space.

Definition 3.3. (i) *A bounded subset K of a Banach space X is said to be p-Right* set, if for every p-Right null sequence $(x_n^*)_n$ in X^* one has:*

$$\lim_n \sup_{x\in K} |x_n^*(x)| = 0.$$

(ii) *We say that X has the p-sequentially Right* property (in short X has the p-(SR^*) property), if every p-Right* set is relatively weakly compact.*

Note that, ∞-Right* sets are precisely Right* sets. Also, the ∞-$(SR)^*$ property is precisely the sequentially Right* property.
If K is a bounded subset of X, the Banach space of all bounded real-valued functions defined on K is denoted by $B(K)$. Inspired by Theorem 3.1 of [3], we obtain some characterizations of notions p-Right sets and p-Right* sets by using evaluation maps.

Lemma 3.4. (i) *If $T \in L(X, Y)$, then T is Dunford-Pettis p-convergent iff $T^*(B_{Y^*})$ is a p-Right subset of X^*.*
(ii) *A bounded subset K of X^* is a p-Right set iff $E_X : X \to B(K)$ defined by $E_X(x) = x^*(x)$ is Dunford-Pettis p-convergent.*
(iii) *If $T \in L(X, Y)$, then T^* is Dunford-Pettis p-convergent iff $T(B_X)$ is a p-Right* subset of Y.*
(iv) *X^* has the p-$(DPrcP)$ iff every bounded subset of X is a p-Right* set.*
(v) *A bounded subset K of X is a p-Right* set iff $E : X^* \to B(K)$ defined by $E(x^*)(x) = x^*(x)$ is Dunford-Pettis p-convergent.*
(vi) *A subset K of X is a p-Right* set iff there is a Banach space Y and a bounded linear operator $T : Y \to X$ so that T and T^* are Dunford-Pettis p-convergent and $K \subseteq T(B_Y)$.*

Proof. (i) Suppose that $T : X \to Y$ is a bounded linear operator. Clearly, $T^*(B_{Y^*})$ is a p-Right set iff

$$\lim_n \|T(x_n)\| = \lim_n(\sup\{|\langle y^*, T(x_n)\rangle| : y^* \in B_{Y^*}\})$$

$$= \lim_n(\sup\{|\langle T^*(y^*), x_n\rangle| : y^* \in B_{Y^*}\}) = 0$$

for each p-Right null sequence $(x_n)_n$ in X, i.e., iff T is a Dunford-Pettis p-convergent operator.
(ii) Let K be a bounded subset of X^*. The evaluation map $E_X : X \to B(K)$ is Dunford-Pettis p-converging iff $\|E_X(x_n)\| \to 0$ for each p-Right null sequence $(x_n)_n$ in X iff

$$\lim_n(\sup\{|x^*(x_n)| : x^* \in K\}) = 0,$$

for each p-Right null sequence $(x_n)_n$ in X iff K is a p-Right set.
(iii) Suppose that $T : X \to Y$ is a bounded linear operator. Clearly, $T(B_X)$ is a p-Right* set iff

$$\lim_n \|T^*(y_n^*)\| = \lim_n(\sup\{|\langle x, T^*(y_n^*)\rangle| : x \in B_X\})$$

$$= \lim_n(\sup\{|\langle T(x), y_n^*\rangle| : x \in B_X\}) = 0,$$

for each p-Right null sequence $(y_n^*)_n$ in X^*, i.e., iff T^* is a Dunford-Pettis p-convergent operator.
(iv) is clear.
(v) Suppose that K is a bounded subset of X and $E : X^* \to B(K)$ is a Dunford-Pettis p-convergent operator. Thus E^* maps the unit ball of $B(K)^*$, to a p-Right set in X^{**}. However, if $k \in K$ and δ_k denotes the point mass at k, then $E^*(\{\delta_k : k \in K\}) = K$, and so K is a p-Right set in X^{**}. Hence K is a p-Right* set in X. Conversely,

suppose that K is a p-Right* set in X, and let $E : X^* \to B(K)$ be the evaluation map. If $(x_n^*)_n$ is a p-Right null sequence in X^*, then

$$\lim_n \|E(x_n^*)\| = \lim_n (\sup\{|x_n^*(x)| : x \in K\}) = 0,$$

and E is a Dunford-Pettis p-convergent operator.
(vi) Suppose that K is a p-Right* set, and let $\overline{aco}(K)$ denote the closed absolutely convex hull of K. Note that $\overline{aco}(K)$ is also a p-Right* set. Let $Y = \ell_1(K)$, and define $T : Y \to X$ by $T(f) = \sum_{k\in K} f(k)k$ for $f \in \ell_1(K)$. It is clear that T is a bounded linear operator, and $K \subseteq T(B_{\ell_1(K)}) \subseteq \overline{aco}(K)$. Since $\ell_1(K)$ has the p-Schur property, the operator T is p-convergent and so, T is Dunford-Pettis p-convergent. Moreover, T^* is the evaluation map $E : X^* \to B(K)$, and T^* is Dunford-Pettis p-convergent by (v). □

Let us recall from [22], that:

- A bounded subset K of X is a p-(V^*) set, if $\lim\limits_{n\to\infty} \sup\limits_{x\in K} |x_n^*(x)| = 0$, for every weakly p-summable sequence $(x_n^*)_n$ in X^*.
- A Banach space X has Pelczyński's property (V^*) of order p (p-(V^*) property), if every p-(V^*) subset of X is relatively weakly compact.

Since the proof of the following proposition is similar to the proof of Theorem ([19, Corollary 3.19 (ii)]), its proof is omitted

Proposition 3.5. *X^* has the (DPP_p) iff each p-Right* set in X is a p-(V^*) set.*

The relationship between (L) subsets and Dunford-Pettis subsets of dual spaces was obtained by Bator et al. [3]. In fact, they showed that every (L) subset of X^* is a Dunford-Pettis set in X^* iff T^{**} is completely continuous whenever Y is an arbitrary Banach space and $T : X \to Y$ is a completely continuous operator.
Obviously, for each $1 \le p \le \infty$, every p-Right* subset of a dual space is a p-Right set, while the converse of implication is false. The following theorem continues our study of the relationship between p-Right subsets and p-Right* subsets of dual spaces.

Theorem 3.6. *Let $1 \le p \le \infty$. Every p-Right subset of X^* is a p-Right* set in X^* iff T^{**} is a Dunford-Pettis p-convergent operator whenever $T : X \to Y$ is a Dunford-Pettis p-convergent operator.*

Proof. Suppose that $T : X \to Y$ is a Dunford-Pettis p-convergent operator. The part (i) of Lemma of 3.4, yields that $T^*(B_{Y^*})$ is a p-Right set. By our hypothesis $T^*(B_{Y^*})$ is a p-Right* set. By applying the Lemma 3.4 (iii), we see that T^{**} is a Dunford-Pettis p-convergent operator. Conversely, suppose that K is a p-Right subset of X^* and consider the natural evaluation map $E_X : X \to B(K)$ defined by $E_X(x)(x^*) = x^*(x)$. The part (ii) of Lemma 3.4, implies that E_X is Dunford-Pettis p-convergent. Therefore, E_X^{**} is Dunford-Pettis p-convergent. Hence, if the unit ball of $B(K)^*$ is denoted by S, then $E_X^*(S)$ is a p-Right* set. Since $K \subset E_X^*(S)$, K is a p-Right* set in X^*. □

Corollary 3.7. *Every Right subset of X^* is a Right* set in X^* iff T^{**} is Dunford-Pettis completely continuous whenever $T : X \to Y$ is a Dunford-Pettis completely continuous operator.*

The finite regular Borel signed measures on the compact space K is denoted by $M(K)$.

Corollary 3.8. *If K is a compact Hausdorff space, then every p-Right subset of $M(K)$ is a p-Right* set in $M(K)$.*

Proof. Suppose that K is a compact Hausdorff space and $T : C(K) \to Y$ is a Dunford-Pettis p-convergent operator. Since $C(K)$ has the p-sequentially Right property, T is weakly compact and so, T^{**} is weakly compact. Therefore, T^{**} is Dunford-Pettis p-convergent. Hence, Theorem 3.6 implies that, every p-Right subset of $M(K)$ is a p-Right* set in $M(K)$. □

For each two Banach spaces X and Y, by meaning of [9], let $\mathcal{U}(X,Y)$ be the component of operator ideal $\mathcal{U}$ of all operators from X to Y that belongs to $\mathcal{U}$. If $\mathcal{M}$ is a closed subspace of $\mathcal{U}(X,Y)$, then for each arbitrary elements $x \in X$ and $y^* \in Y^*$ the point evaluation maps $\phi_x : \mathcal{M} \to Y$ and $\psi_{y^*} : \mathcal{M} \to X^*$ on $\mathcal{M}$ are defined by

$$\phi_x(T) = Tx, \quad \psi_{y^*}(T) = T^*y^*, \quad T \in \mathcal{M}.$$

Also, the point evaluation sets related to $x \in X$ and $y^* \in Y^*$ are the images of the closed unit ball $B_{\mathcal{M}}$ of $\mathcal{M}$, under the evaluation operators ϕ_x and ψ_{y^*} and are denoted by $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$, respectively. Note that, if we speak about $\mathcal{U}(X,Y)$ or its linear subspace $\mathcal{M}$, then the related norm is the ideal norm $\mathcal{A}(.)$ while, the operator norm $\|.\|$ is applied when the space is considered as a linear subspace of $L(X,Y)$.

Theorem 3.9. *Suppose that $1 \le p \le \infty$ and the dual $\mathcal{M}^*$ of a closed subspace $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ has the p-(DPrcP). Then, $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are p-Right sets.*

Proof. We first show that ϕ_x^* is a Dunford-Pettis p-convergent operator. For this purpose, let $(y_n^*)_n$ be a p-Right null sequence. It is clear that $(\phi_x^*(y_n^*))_n$ is a p-Right null sequence in $\mathcal{M}^*$. If we consider the evaluation map $E : \mathcal{M}^* \to B(B_{\mathcal{M}})$ as $E(f)(T) = f(T)$ for all $T \in B_{\mathcal{M}}$ and $f \in \mathcal{M}^*$, then by Lemma 3.4, E is a Dunford-Pettis p-convergent operator. Therefore,

$$\lim_{n\to\infty} \|\phi_x^*(y_n^*)\| = \lim_{n\to\infty} \|E(\phi_x^*(y_n^*))\| = 0.$$

Hence, ϕ_x^* is a Dunford-Pettis p-convergent operator.
On the other hand,

$$\|\phi_x^*(y_n^*))\| = \sup\{|\phi_x^* y_n^*(T)| : T \in B_{\mathcal{M}}\} = \sup\{|y_n^*(T(x))| : T \in B_{\mathcal{M}}\}.$$

Hence, $\mathcal{M}_1(x)$ is a p-Right set in Y, for all $x \in X$. A similar proof shows that $\widetilde{\mathcal{M}_1}(y^*)$ is a p-Right set in X^*. □

In the following, we obtain some sufficient conditions for which $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are relatively weakly compact for all $x \in X$ and all $y^* \in Y^*$.
Motivated by a result in [25], here we obtain a similar result for the case of Dunford-Pettis p-convergent operator.

Theorem 3.10. *Suppose that $1 \le p \le \infty$ and X^{**} and Y^* have the p-(SR) property. If $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is a closed subspace so that the natural restriction operator $R : \mathcal{U}(X,Y)^* \to \mathcal{M}^*$ is Dunford-Pettis p-convergent, then $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are relatively weakly compact.*

Proof. It is enough to show that ϕ_x and ψ_{y^*} are weakly compact operators. For this purpose, suppose that $T \in \mathcal{U}(X,Y)$. Since $\|T\| \le \mathcal{A}(T)$, it is not difficult to show that the operator $\psi : X^{**}\widehat{\bigotimes}_\pi Y^* \to \mathcal{U}(X,Y)^*$ which is defined by

$$\vartheta \mapsto tr(T^{**}\vartheta) = \sum_{n=1}^{\infty} \langle T^{**}x_n^{**}, y_n^* \rangle$$

is linear and continuous, where $\vartheta = \sum_{n=1}^{\infty} x_n^{**} \otimes y_n^*$. Fix now an arbitrary element $x \in X$, we define $U_x : Y^* \to X^{**}\widehat{\bigotimes}_\pi Y^*$ by $U_x(y^*) = x \otimes y^*$. It is clear that the operator $\phi_x^* = R \circ \psi \circ U_x$ is Dunford-Pettis p-convergent. Since Y^* has the p-(SR) property, we conclude that ϕ_x^* is a weakly compact operator. Hence, ϕ_x is weakly compact. Similarly, we can see that ψ_{y^*} is weakly compact. □

A sequence $(x_n)_n$ in a Banach space X is weakly p-Cauchy if for each pair of strictly increasing sequences $(k_n)_n$ and $(j_n)_n$ of positive integers, the sequence $(x_{k_n} - x_{j_n})_n$ is weakly p-summable in X[7]. Notice that, every weakly p-convergent sequence is weakly p-Cauchy, and the weakly ∞-Cauchy sequences are precisely the weakly Cauchy sequences. A subset K of a Banach space X is called weakly p-precompact, if every sequence from K has a weakly p-Cauchy subsequence. The weakly ∞-precompact sets are precisely the weakly precompact sets. An operator $T : X \to Y$ is called weakly-p-precompact if $T(B_X)$ is weakly-p-precompact. The set of all weakly-p-precompact operators $T : X \to Y$ is denoted by $WPC_p(X, Y)$.

Theorem 3.11. *Let X be a Banach space and $1 \le p < q \le \infty$. The following statements are equivalent:*
(i) *For every Banach space Y, if $T : X \to Y$ is a Dunford-Pettis p-convergent operator, then T has a weakly q-precompact (weakly q-compact, q-compact) adjoint;*
(ii) *Same as* (i) *with $Y = \ell_\infty$;*
(iii) *Every p-Right subset of X^* is weakly q-precompact (relatively weakly q-compact, q-compact).*

Proof. We only prove the relatively weakly q-compact case. The proof of the other cases are similar.
(i) $\Rightarrow$ (ii) It is obvious.
(ii) $\Rightarrow$ (iii) Let K be a p-Right subset of X^* and let $(x_n^*)_n$ be a sequence in K. Define $T : X \to \ell_\infty$ by $T(x) = (x_n^*(x))_n$. Let $(x_n)_n$ be a p-Right null sequence in X. Since K is a p-Right set,

$$\lim_{n\to\infty} \|T(x_n)\| = \lim_{n\to\infty} \sup_m |x_m^*(x_n)| = 0.$$

Therefore, T is Dunford-Pettis p-convergent. Hence, T^* is weakly q-compact, and $(T^*(e_n^1))_n = (x_n^*)_n$ has a weakly q-convergent subsequence.
(iii) $\Rightarrow$ (i) Let $T : X \to Y$ be a Dunford-Pettis p-convergent operator. Then $T^*(B_{Y^*})$ is a p-Right subset of X. Therefore $T^*(B_{Y^*})$ is relatively weakly q-compact, and thus T^* is weakly q-compact. □

Let A and B be nonempty subsets of a Banach space X, we define the ordinary and non-symmetrized Hausdorff distances respectively, by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}, \qquad \hat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

Let X be a Banach space and K be a bounded subset of X^*. For $1 \le p \le \infty$, we set

$$\zeta_p(K) = \inf\{\hat{d}(A, K) : A \subset X^* \text{ is a } p\text{-Right set}\}.$$

We can conclude that $\zeta_p(K) = 0$ iff $K \subset X^*$ is a p-Right set. Now, let K be a bounded subset of a Banach space X. The de Blasi measure of weak non-compactness of K is defined by

$$\omega(K) = \inf\{\hat{d}(K, A) : \emptyset \neq A \subset X \text{ is weakly-compact}\}.$$

It is clear that $\omega(K) = 0$ iff K is relatively weakly compact. For a bounded linear operator $T : X \to Y$, we denote $\zeta_p(T^*(B_{Y^*})), \omega(T^*(B_{Y^*})$ by $\zeta_p(T^*), \omega(T^*)$ respectively.

Corollary 3.12. *Let X be a Banach space and $1 \le p \le \infty$. The following statements are equivalent:*
(i) *X has the p-(SR) property;*
(ii) *For each Banach space Y, the adjoint of every Dunford-Pettis p-convergent $T : X \to Y$ is weakly compact;*
(iii) *Same as* (ii) *with $Y = \ell_\infty$;*
(iv) *$\omega(T^*) \le \zeta_p(T^*)$ for every bounded linear operator T from X into any Banach space Y;*
(v) *$\omega(K) \le \zeta_p(K)$ for every bounded subset K of X^*.*

Corollary 3.13. *If X has the (DPP_p), then X has Pelczyński's property (V) of order p iff X has the p-(SR) property.*

Proof. Suppose that X has Pelczyński's property (V) of order p. We show that for each Banach space Y, the adjoint of every Dunford-Pettis p-convergent $T : X \to Y$ is weakly compact. Let $T \in DPC_p(X, Y)$. The part (i) of Lemma 3.4, implies that $T^*(B_{Y^*})$ is a p-Right set in X^*. By our hypothesis $X \in (DPP_p)$. So, $T^*(B_{Y^*})$ is a p-(V) set in X^*. Since X has Pelczyński's property (V) of order p, T^* is weakly compact. Hence, Corollary 3.12 implies that X has the p-(SR) property. Conversely, If X has the p-(SR) property, then X has Pelczyński's property (V) of order p, since every p-(V) set in X^* is a p-Right set. □

Suppose that X is a Banach space and Y is a subspace of X^*. We define

$$^{\perp}Y := \{x \in X : y^*(x) = 0 \quad \text{for all} \quad y^* \in Y^*\}.$$

Corollary 3.14. (i) *If X is an infinite dimensional non reflexive Banach space with the p-Schur property, then X does not have the p-(SR) property.*
(ii) *If every separable subspace of X has the p-(SR) property, then X has the same property.*
(iii) *Let Y be a reflexive subspace of X^*. If $^{\perp}Y$ has the p-(SR) property, then X has the same property.*

Proof. (i) Since $X \in C_p$, the identity operator $id_X : X \to X$ is p-convergent and so, it is Dunford-Pettis p-convergent. It is clear that id_X is not weakly compact. Hence, Corollary 3.12 implies that X does not have the p-(SR) property.
(ii) Let $(x_n)_n$ be a sequence in B_X and let $Z = [x_n : n \in \mathbb{N}]$ be the closed linear span of $(x_n)_n$. Since Z is a separable subspace of X, Z has the p-(SR) property. Now, let $T : X \to Y$ be a Dunford-Pettis p-convergent operator. It is clear that $T_{|Z}$ is a Dunford-Pettis p-convergent operator. Therefore, Corollary 3.12, implies that $T_{|Z}$ is weakly compact. Hence, there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ so that $T(x_{n_k})$ is weakly convergent. Thus T is weakly compact. Now an appeal to Corollary 3.12 completes the proof.
(iii) Suppose that $^{\perp}Y$ has the p-(SR) property. Let $Q : X^* \to \frac{X^*}{Y}$ be the quotient map and $i : \frac{X^*}{Y} \to (^{\perp}Y)^*$ be the natural surjective isomorphism ([24, Theorem 1.10.6]). It is known that $i \circ Q : X^* \to (^{\perp}Y)^*$ is w^*-w^* continuous, since $iQ(x^*)$ is the restriction of x^* to $^{\perp}Y$(see ([24, Theorem 1.10.6])). Therefore there is an operator $S : {}^{\perp}Y \to X$ so that $S^* = i \circ Q$.
Let $T : X \to Z$ be a Dunford-Pettis p-convergent operator. Since $^{\perp}Y$ has the p-(SR) property and the operator $T \circ S : {}^{\perp}Y \to Z$ is Dunford-Pettis p-convergent, it has a weakly compact adjoint (by Corollary 3.12). Since $S^* \circ T^* = i \circ Q \circ T^*$ is weakly compact and i is a surjective isomorphism, $Q \circ T^*$ is weakly compact. Now, we show that T^* is weakly compact. Indeed, suppose that $(x_n^*)_n$ is a sequence in B_{X^*}. By passing to a subsequence, we can assume that $(QT^*(x_n^*))_n$ is weakly convergent. Hence, by ([20, Theorem 2.7]) $(T^*(x_n^*))_n$ has a weakly convergent subsequence. Therefore, T^* is weakly compact. Then by Corollary 3.12 the proof is complete. □

The following result appeared in [19]. Here, we include a new proof for the convenience of the reader.

Theorem 3.15. Let $1 \leq p \leq \infty$. Eevery p-Right set in X^* is relatively compact iff $DPC_p(X, Y) = K(X, Y)$ for every Banach space Y.

Proof. Suppose that every p-Right set in X^* is relatively compact. If $T \in DPC_p(X, Y)$, then the part (i) of Lemma 3.4, implies that $K = T^*(B_{Y^*})$ is a p-Right set. Hence, K is relatively compact and so T^* is compact. Therefore, T is compact.
Conversely, let K be a p-Right set in X^*. Suppose that $T : X \to B(K)$ is defined by $T(x)x^* = x^*(x)$, $x \in X$, $x^* \in K$ for all $x \in X$ and $x^* \in X^*$. It is clear that T is a Dunford-Pettis p-convergent operator. Therefore, T is compact and so $T^*(B_{B(K)^*})$ is a relatively compact subset of X^*. On the other hand $T^*(\delta_{x^*}) = x^*$, where δ_{x^*} is the point evaluation mass at $x^* \in K$. Hence, $K \subseteq T^*(B_{B(K)^*})$ and so K is relatively compact. □

Corollary 3.16. Suppose that X is infinite dimensional. If every p^*-Right subset of X^* is relatively compact, then X does not have the p^*-$(DPrcP)$.

Theorem 3.17. *Assume that $1 \leq p \leq \infty$, then the following statements are equivalent:*

(i) *$DPC_p(X^*, Y^*) = K(X^*, Y^*)$, for every Banach space Y;*
(ii) *Same as* (i) *with $Y = \ell_1$;*
(iii) *Every p-Right* set in X is relatively compact.*

Proof. We will show that (i) ⇒ (ii) ⇒ (iii) ⇒ (i).
(i) ⇒ (ii) is obvious.
(ii) ⇒ (iii) Let K be a p-Right* subset of X and let $(x_n)_n$ be a sequence in K. Define $T : \ell_1 \to X$ by $T(b) = \sum b_i x_i$. It is clear that that $T^*(x^*) = (x^*(x_i))_i$. Let $(x_n^*)_n$ be a p-Right null sequence in X^*. Since K is a p-Right* set, then

$$\lim_n \|T^*(x_n^*)\| = \lim_n \sup_i |x_n^*(x_i)| = 0.$$

Therefore, T^* is Dunford-Pettis p-convergent and so T^* is compact. Hence, $(T(e_n^1))_n = (x_n)_n$ has a norm convergent subsequence.
(iii) ⇒ (i) Let $T : Y \to X$ be an operator such that T^* is Dunford-Pettis p-convergent. Therefore if $(x_n^*)_n$ is a p-Right null sequence in X^*, then

$$\lim_{n\to\infty} \sup_{y\in B_Y} |\langle T(y), x_n^*\rangle| \leq \lim_{n\to\infty} \|T^*(x_n^*)\| = 0.$$

Hence, $T(B_Y)$ is a p-Right* subset of X and so $T(B_Y)$ is relatively compact. Thus T is compact. □

Since every p-Right* set in X is a p-Right set in X^{**} from Theorem 3.15 one can obtain implication (i) ⇒ (iii).

Theorem 3.18. *Suppose that $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$. If X and Y have the p-(SR) property, then $K_{w^*}(X^*, Y)$ has the same property.*

Proof. Suppose X and Y have the p-(SR) property. Let H be a p-Right subset of $K_{w^*}(X^*, Y)$. For fixed $x^* \in X^*$, the map $T \mapsto T(x^*)$ is a bounded operator from $K_{w^*}(X^*, Y)$ into Y. It can easily be seen that continuous linear images of p-Right sets are p-Right sets. Therefore, $H(x^*) := \{T(x^*) : T \in H\}$ is a p-Right subset of Y, hence relatively weakly compact. For fixed $y^* \in Y^*$ the map $T \mapsto T^*(y^*)$ is a bounded linear operator from $K_{w^*}(X^*, Y)$ into X. Therefore, $H^*(y^*) := \{T^*(y^*) : T \in H\}$ is a p-Right subset of X, hence relatively weakly compact. Then, by ([14, Theorem 4. 8]), H is relatively weakly compact. □

Corollary 3.19. *Suppose that $L(X, Y) = K(X, Y)$. If X^* and Y have the p-(SR) property, then $K(X, Y)$ has the same property.*

Next, the stability of the p-sequentially Right property for projective tensor products between Banach spaces is investigated.

Theorem 3.20. *Suppose that X has the p-(SR) property and Y is a reflexive space. If $L(X, Y^*) = K(X, Y^*)$, then $X\widehat{\bigotimes}_\pi Y$ has the p-(SR) property.*

Proof. Let H be a p-Right subset of $(X\widehat{\bigotimes}_\pi Y)^* \simeq L(X, Y^*)$. Let $(T_n)_n$ be an arbitrary sequence in H and let $x \in X$. We show that $\{T_n(x) : n \in \mathbb{N}\}$ is a p-Right subset of Y^*. Let $(y_n)_n$ be a p-Right null sequence in Y. For each $n \in \mathbb{N}$,

$$\langle T_n(x), y_n\rangle = \langle T_n, x \otimes y_n\rangle.$$

We claim that $(x \otimes y_n)_n$ is a p-Right null sequence in $X\widehat{\bigotimes}_\pi Y$. If $T \in (X\widehat{\bigotimes}_\pi Y)^* \simeq L(X, Y^*)$, then

$$(|\langle T, x \otimes y_n\rangle|)_n = (|\langle T(x), y_n\rangle|)_n \in \ell_p,$$

since $(y_n)_n$ is weakly p-summable. Thus $(x \otimes y_n)_n$ is weakly p-summable in $X\widehat{\bigotimes}_\pi Y$. Let $(A_n)_n$ be a weakly null sequence in $(X\widehat{\bigotimes}_\pi Y)^* \simeq L(X, Y^*)$. Consider the operator $\Theta_x : L(X, Y^*) \to Y^*$ defined by $\Theta_x(T) = T(x)$ for each $T \in L(X, Y^*)$. It is clear that the operator Θ_x, is linear and bounded. Hence, $(A_n(x))_n$ is weakly null in Y^*. Therefore

$$|\langle A_n, x \otimes y_n\rangle| = |\langle A_n(x), y_n\rangle| \to 0,$$

since $(y_n)_n$ is a Dunford-Pettis sequence in Y. Therefore, $(x \otimes y_n)_n$ is a Dunford-Pettis sequence in $X\widehat{\bigotimes}_\pi Y$. Hence, $(x \otimes y_n)_n$ is p-Right null in $X\widehat{\bigotimes}_\pi Y$ and so, the equivalence between (i) and (v) in ([19, Theorem 3.26]) implies that $\{T_n(x) : n \in \mathbb{N}\}$ is a p-Right set in Y^*. Therefore, $\{T_n(x) : n \in \mathbb{N}\}$ is relatively weakly compact. Now, let $y \in Y^{**} = Y$ and $(x_n)_n$ be a p-Right null sequence in X. By a similar argument as above, we can show that $(x_n \otimes y)_n$ is a p-Right null sequence in $X\widehat{\bigotimes}_\pi Y$. Therefore, by applying ([19, Theorem 3.26]) we have

$$\sup_k |\langle T_k^*(y), x_n\rangle| = \sup_k |\langle T_k(x_n), y\rangle| = \sup_k |\langle T_k, x_n \otimes y\rangle| \to 0,$$

since $(T_n)_n$ is a p-Right set. Therefore, $\{T_n^*(y) : n \in \mathbb{N}\}$ is a p-Right subset of X^*. Hence, $\{T_n^*(y) : n \in \mathbb{N}\}$ is relatively weakly compact. Then, by ([16, Theorem 3.8]), H is relatively weakly compact. □

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. If $1 \le r < \infty$ the space of all vector-valued sequences $(\sum_{n=1}^{\infty} \oplus X_n)_{\ell_r}$ is called the infinite direct sum of X_n in the sense of ℓ_r, consisting of all sequences $x = (x_n)_n$ with values in X_n such that $\|x\|_r = (\sum_{n=1}^{\infty} \|x_n\|^r)^{\frac{1}{r}} < \infty$. For every $n \in \mathbb{N}$, we denote the canonical projection from $(\sum_{n=1}^{\infty} \oplus X_n)_{\ell_r}$ into X_n by π_n. Also, we denote the canonical projection from $(\sum_{n=1}^{\infty} \oplus X_n^*)_{\ell_{r^*}}$ onto X_n^* by P_n.

Using the ([19, Corollary 3.19]), and ([22, Theorem 3.1]), we obtain the following result:

Theorem 3.21. *Let $1 < p < \infty$ and $(X_n)_n$ be a sequence of Banach spaces with (DPP_p) and let $X = (\sum_{n=1}^{\infty} \oplus X_n)_{\ell_p}$. The following are equivalent for a bounded subset K of X^*:*
(i) *K is a p^*-Right set;*
(ii) *$P_n(K)$ is a p^*-Right set for each $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \sup\{\sum_{k=n}^{\infty} \|P_k x^*\|^{p^*} : x^* \in K\} = 0.$$

Theorem 3.22. *Let $1 < p < \infty$ and $(X_n)_n$ be a sequence of Banach spaces. If $X = (\sum_{n=1}^{\infty} \oplus X_n)_{\ell_p}$ and $1 \le q < p^*$, then a bounded subset K of X^* is a q-Right set iff each $P_n(K)$ is a q-Right set.*

Proof. It can easily seen that continuous linear images of a q-Right set is a q-Right set. Therefore, we only prove the sufficient part. Assume that K is not a q-Right set. Therefore, there exist $\varepsilon_0 > 0$, a q-Right null sequence $(x_n)_n$ in X and a sequence $(x_n^*)_n$ in K such that

$$|\langle x_n^*, x_n\rangle| = |\sum_{k=1}^{\infty} \langle P_k x_n^*, \pi_k x_n\rangle| > \varepsilon_0, \qquad n = 1, 2, 3, ... \qquad (*)$$

By our assumption, we obtain

$$\lim_{n\to\infty} |\langle P_k x_n^*, \pi_k x_n\rangle| = 0, \qquad k = 1, 2, 3, ... \qquad (**)$$

By induction on n in $(*)$ and k in $(**)$, there exist two strictly increasing sequences $(n_j)_j$ and $(k_j)_j$ of positive integers such that

$$|\sum_{k=k_{j-1}+1}^{k_j} \langle P_k x_{n_j}^*, \pi_k x_{n_j}\rangle| > \frac{\varepsilon_0}{2}, \qquad j = 1, 2, 3, ...$$

For each $j = 1, 2, ...$, we consider $y_j = x_{n_j}$ and $y_j^* \in X^*$ by

$$P_k y_j^* = \begin{cases} P_{k_j} x_{n_j}^* & \text{if } k_{j-1} + 1 \le k \le k_j, \\ 0 & \text{otherwise .} \end{cases}$$

It is clear that $(y_j)_j$ is a q-Right null sequence in X such that

$$|\langle y_j^*, y_j\rangle| = |\sum_{k=k_{j-1}+1}^{k_j} \langle P_k x_{n_j}^*, \pi_k x_{n_j}\rangle| > \frac{\varepsilon_0}{2}, \qquad j = 1, 2, 3, ...$$

Since the sequence $(y_j^*)_j$ has pairwise disjoint supports, Proposition 6.4.1 of [1] implies that $(y_j^*)_j$ is equivalent to the unit vector basis $(e_j^{p^*})_j$ of ℓ_{p^*}. Suppose that R is an isomorphic embedding from ℓ_{p^*} into X^* such that $R(e_j^{p^*}) = y_j^*(j = 1, 2, ...)$. Now, let T be a bounded linear operator from ℓ_{q^*} into X. By Pitts Theorem [1], the operator $T^* \circ R$ is compact and so the sequence $(T^*(y_j^*))_j = (T^*R(e_j^*))_j$ is relatively norm compact. Hence, Theorem 2.3 of [22] implies that the sequence $(y_j^*)_j$ is a q-(V) set and so is a q-Right set. Since $(y_j)_j$ is q-Right null, we have

$$|\langle y_n^*, y_n\rangle| \leq \sup_j |\langle y_j^*, y_n\rangle| \to 0 \text{ as } \quad n \to \infty,$$

which is a contradiction. □

Theorem 3.23. *Let $(X_n)_n$ be a sequence of Banach spaces. If $1 < r < \infty$ and $1 \leq p < \infty$, then $X = (\sum_{n=1}^{\infty} \oplus X_n)_{\ell_r}$ has the p-(SR) property iff each X_n has the same property.*

Proof. It is clear that if X has the p-(SR) property, then each X_n has the p-(SR) property. Conversely, let K be a p-Right subset of X^*. Since continuous linear images of p-Right sets are p-Right sets, each $P_n(K)$ is also a p-Right set. Since X_n has the p-(SR) property for each $n \in \mathbb{N}$, each $P_n(K)$ is relatively weakly compact. It follows from Lemma 3.4 [22] that K is relatively weakly compact. □

Proposition 3.24. Let $1 \leq p \leq \infty$. The following statements hold:
(i) If X has the p-(SR) property, then X^* has the p-(SR^*) property.
(ii) If X has the p-(SR) property, then X has the p-(V) property.
(iii) If X^* has the p-(SR) property, then X has the p-(SR^*) property,
(iv) If X has the p-(SR^*) property, then X has the p-(V^*) property.

Theorem 3.25. *Let Y be a reflexive subspace of X. If $\frac{X}{Y}$ has the p-(SR^*) property, then X has the same property.*

Proof. Let $Q : X \to \frac{X}{Y}$ be the quotient map. Let K be a p-Right* set in X and $(x_n)_n$ be a sequence in K. Then $(Q(x_n))_n$ is a p-Right* set in $\frac{X}{Y}$, and thus relatively weakly compact. By passing to a subsequence, suppose $(Q(x_n))_n$ is weakly convergent. By ([20, Theorem 2.7]), $(x_n)_n$ has a weakly convergent subsequence. Thus X has the p-(SR^*) property. □

Let K be a bounded subset of X. For $1 \leq p \leq \infty$, we set

$$\vartheta_p(K) = \inf\{\hat{d}(A, K) : A \subset X \text{ is a } p\text{-Right}^* \text{ set}\}.$$

We can conclude that $\vartheta_p(K) = 0$ iff $K \subset X$ is a p-Right* set. For a bounded linear operator $T : Y \to X$, we denote $\vartheta_p(T(B_Y))$ by $\vartheta_p(T)$.

The proof of the following theorem is similar to the proof of Theorem 3.11, so its proof is omitted.

Theorem 3.26. *Let X be a Banach space and $1 \leq p < q \leq \infty$. The following statements are equivalent:*
(i) *For every Banach space Y, if $T : Y \to X$ is an operator such that T^* is a Dunford-Pettis p-convergent operator, then T is a weakly q-precompact (weakly q-compact, q-compact);*
(ii) *Same as* (i) *with $Y = \ell_\infty$;*
(iii) *Every p-Right* subset of X is weakly q-precompact (relatively weakly q-compact, q-compact).*

Corollary 3.27. *Let X be a Banach space and $1 \leq p < \infty$. The following statements are equivalent:*
(i) *For every Banach space Y, if $T : Y \to X$ is an operator such that T^* is a Dunford-Pettis p-convergent operator, then T is weakly compact;*
(ii) *Same as* (i) *with $Y = \ell_1$;*
(iii) *X has the p-(SR^*) property;*
(iv) *$\omega(T) \leq \vartheta_p(T)$ for every operator T from any Banach space Y into X;*
(v) *$\omega(K) \leq \vartheta_p(K)$ for every bounded subset K of X.*

Corollary 3.28. (i) *If X^* has the p-(DPrcP) and Y has the p-(SR^*) property, then $L(X,Y) = W(X,Y)$.*
(ii) *If X^* has the (DPP_p), then X has Pelczyński's property (V^*) of order p iff X has the p-(SR^*) property.*

Proof. (i) It can easily be seen that the continuous linear image of each p-Right null sequence is a p-Right null sequence. Therefore, if $T \in L(X,Y)$ and $(y_n^*)_n$ is a p-Right null sequence in Y^*, then $(T^*(y_n^*))_n$ is a p-Right null sequence in X^*. Since X^* has the p-(*DPrcP*), $\|T^*(y_n^*)\| \to 0$ and so, $T^* \in DPC_p(Y^*, X^*)$. Hence, Corollary 3.27 implies that $T \in W(X,Y)$.
(ii) Suppose that X has Pelczyński's property (V^*) of order p. We show that for each Banach space Y, if $T : Y \to X$ is an operator such that T^* is a Dunford-Pettis p-convergent operator, then T is weakly compact. Let $T^* \in DPC_p(X^*, Y^*)$. The part (iii) of Lemma 3.4, implies that $T(B_Y)$ is a p-Right* set in X. By hypothesis $X^* \in (DPP_p)$. So, $T(B_Y)$ is a p-(V^*) set in X. Since X^* has Pelczyński's property (V^*) of order p, T is weakly compact. Hence, Corollary 3.27 implies that X has the p-(SR^*) property. Conversely, if X has the p-(SR^*) property, then X has Pelczyński's property (V^*) of order p, since every p-(V^*) set in X is a p-Right* set. □

Proposition 3.29. *Let $(X_n)_n$ be a sequence of Banach spaces. If $1 < r < \infty$ and $1 \le p < \infty$, then each X_n has the p-(SR^*) property iff $X = (\sum_{n=1}^{\infty} \oplus X_n)_{\ell_r}$ has the same property.*

Proof. It is clear that if $X = (\sum_{n=1}^{\infty} \oplus X_n)_{\ell_r}$ has the p-(SR^*) property, then each X_n has this property. Conversely, let K be a p-Right* subset of X. It is clear that each $\pi_n(K)$ is also a p-Right set. Since X_n has the p-(SR^*) property for each $n \in \mathbb{N}$, each $\pi_n(K)$ is relatively weakly compact. It follows from Lemma 3.4 [22] that K is relatively weakly compact. □

4. (p,q)-sequentially Right property on Banach spaces

In this Section, motivated by the class $\mathcal{P}_{p,q}$ in [27] for those Banach spaces in which relatively p-compact sets are relatively q-compact, we introduce the concepts of properties $(SR)_{p,q}$ and $(SR^*)_{p,q}$ in order to find a condition such that every Dunford-Pettis q-convergent operator is Dunford-Pettis p-convergent.

Definition 4.1. *We say that X has the (p,q)-sequentially Right property (in short X has the $(SR)_{p,q}$ property), if each p-Right set in X^* is a q-Right set in X^*.*

Definition 4.2. *We say that X has the (p,q)-sequentially Right property (in short X has the $(SR^*)_{p,q}$ property), if each p-Right* set in X is a q-Right* set in X.*

From Definitions 4.1 and 4.2, we have the following result. Since its proof is obvious, the proof is omitted.

Proposition 4.3. *If X^* has the $(SR)_{p,q}$ property, then X has the $(SR^*)_{p,q}$ property.*

Theorem 4.4. *Let $1 \le p < q \le \infty$. The following statements are equivalent:*

(i) X has the $(SR)_{p,q}$ property;
(ii) $DPC_p(X,Y) \subseteq DPC_q(X,Y)$, for every Banach space Y;
(iii) Same as (ii) *for $Y = \ell_\infty$.*

Proof. (i) ⇒ (ii) Let $T \in DPC_p(X,Y)$. Then by Lemma 3.4(i), $T^*(B_{Y^*})$ is a p-Right set. Since X has the $(SR)_{p,q}$ property, $T^*(B_{Y^*})$ is a q-Right set. Therefore, by Lemma 3.4(i), $T \in DPC_q(X,Y)$.
(ii) ⇒ (iii) is obvious.
(iii) ⇒ (i) Suppose that K is a p-Right set in X^* and $(x_n^*)_n$ is a sequence in K. Assume that $T : \ell_1 \to X^*$ is defined by $T(b_n) = \sum_{n=1}^{\infty} b_n x_n^*$. It is clear that $T^*(x) = (x_i^*(x))_i$ for all $x \in X$. Suppose that $(x_n)_n$ is a p-Right null sequence in X. Since, K is a p-Right set, we have

$$\lim_n \sup_i |x_i^*(x_n)| = 0.$$

So, $\lim_n \|T^*(x_n)\| = 0$ which implies that $T^*_{|X}$ is a Dunford-Pettis p-convergent operator. Hence, by the assumption $T^*_{|X}$ is a Dunford-Pettis q-convergent operator. Now, assume that $(x_n)_n$ is a q-Right null sequence in X and $y \in B_{\ell_1}$. So,

$$|T(y)(x_n)| = |T^*(x_n)(y)| \leq \|T^*(x_n)\| \to 0.$$

Therefore, $T(B_{\ell_1})$ is a q-Right set in X^*, from which it follows that $(x^*_n)_n$ is also a q-Right set in X^*. Since $(x^*_n)_n$ is an arbitrary sequence in K, then K is a q-Right set. Thus, X has the $(SR)_{p,q}$ property. □

Corollary 4.5. *If every p-Right set in X^* is relatively compact, then X has the $(SR)_{p,q}$ property.*

Now, by applying Proposition 4.3 and Theorem 4.4, we give those Banach spaces which have the $(SR)_{p,q}$ and $(SR^*)_{p,q}$ properties.

Example 4.6. (i) If K is a compact Hausdorff space, then $C(K)$ has the $(SR)_{p,q}$ property.
(ii) If (Ω, Σ, μ) is any σ-finite measure space, then $L_1(\mu)$ has the $(SR^*)_{p,q}$ property.

Corollary 4.7. *Let $1 \leq p < q \leq \infty$. The following statements hold:*

(i) If X has both properties $(SR)_{p,q}$ and p-$(DPrcP)$, then X has the q-$(DPrcP)$;
(ii) If X^{**} has both properties $(SR)_{p,q}$ and p-$(DPrcP)$, then X has the q-$(DPrcP)$;
(iii) If X has the p-(SR) property, then X has the $(SR)_{p,q}$ property.

Proof. (i) Suppose that $T : X \to Y$ is a bounded linear operator. Since X has the p-$(DPrcP)$, then $T \in DPC_p(X, Y)$. On the other hand, X has property $(SR)_{p,q}$, thus by Theorem 4.4, $T \in DPC_q(X, Y)$. Hence, X has the q-$(DPrcP)$.
(ii) By part (i), X^{**} has the q-$(DPrcP)$. Hence, X has the q-$(DPrcP)$.
(iii) Let $T \in DPC_p(X, Y)$. From part (i) of Lemma 3.4, $T^*(B_{Y^*})$ is a p-Right set. Since X has the p-(SR) property, $T^*(B_{Y^*})$ is relatively weakly compact and so T^* is a weakly compact operator. So, T^* is a Dunford-Pettis completely continuous operator. Thus, T^* is Dunford-Pettis q-convergent. Hence, by Theorem 4.4, X has the $(SR)_{p,q}$ property. □

Theorem 4.8. *Let $1 < p \leq \infty$. The following statements are equivalent:*

(i) X has the p-$(DPrcP)$;
(ii) X has the $(SR)_{1,p}$ property and X contains no isomorphic copy of c_0.

Proof. (i) ⇒ (ii) is obvious.
(ii) ⇒ (i) Since X contains no isomorphic copy of c_0, X has the 1-Schur property; (see Theorem 2.4 in [10]) and so has the 1-$(DPrcP)$. Hence, B_{X^*} is a 1-Right subset of X^*. Since X has the $(SR)_{1,p}$ property, B_{X^*} is a p-Right set. It is easy to verify that X has the p-$(DPrcP)$. □

In the sequel, we characterize property $(SR^*)_{p,q}$. Since the proof of the following theorem is similar to the proof of Theorem 4.4, its proof is omitted.

Theorem 4.9. *Let $1 \leq p < q \leq \infty$. The following statements are equivalent:*

(i) X has the $(SR^*)_{p,q}$ property;
(ii) $Dpc_p(X^*, Y^*) \subseteq DPC_q(X^*, Y^*)$, For every Banach space Y;
(iii) Same as (ii) *for $Y = \ell_1$.*

Corollary 4.10. *If X^* has both properties q-$(DPrcP)$ and p-(SR), then X has the $(SR^*)_{p,q}$ property.*

Proof. Let Y be an arbitrary Banach space and $T \in L(Y, X)$ such that T^* be a Dunford-Pettis p-convergent operator. Therefore, by Lemma 3.4(iii), $T(B_Y)$ is a p-Right* set in X. Since X^* has the p-(SR) property, part (iii) of Proposition 3.24 implies that X has the p-(SR^*) property. Hence, $T(B_Y)$ is a relatively weakly compact set in X and so T is weakly compact. Thus, T^* is a Dunford-Pettis q-convergent operator. Hence, as an immediate consequence of Theorem 4.9, we can conclude that X has the $(SR^*)_{p,q}$ property. □

Finally, we conclude this section by the dual version of Theorem 4.8.

Theorem 4.11. *If $1 < p \leq \infty$, then the following statements are equivalent:*

(i) X^* *has the* p-$(DPrcP)$;

(ii) X *has the* $(SR^*)_{1,p}$ *property and* X^* *contains no isomorphic copy of* c_0.

Proof. (i) $\Rightarrow$ (ii) Suppose that X^* has the p-$(DPrcP)$. Therefore, by the part (ii) of Theorem 4.8, X^* has the $(SR)_{1,p}$ property and X^* contains no isomorphic copy of c_0. Thus, Proposition 4.3 implies that X has the $(SR^*)_{1,p}$ property.
(ii) $\Rightarrow$ (i) By our hypothesis, X^* contains no isomorphic copy of c_0. Therefore Theorem 2.4 in [10] implies that X^* has the 1-Schur property and so X^* has the 1-$(DPrcP)$. Therefore, by Lemma 3.4(iii), B_X is a 1-Right* set in X. Since X has the $(SR^*)_{1,p}$ property, B_X is a p-Right* set. Hence, by Lemma 3.4(iii), X^* has the p-$(DPrcP)$. $\square$

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