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The Left, the Right and the Sequential Topology on Boolean Algebras

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Abstract. For the algebraic convergence $\lambda_{\rm s}$, which generates the well known sequential topology τ_s on a complete Boolean algebra $\mathbb{B}$, we have $\lambda_{\rm s} = \lambda_{\rm ls} \cap \lambda_{\rm li}$, where the convergences $\lambda_{\rm ls}$ and $\lambda_{\rm li}$ are defined by $\lambda_{\rm ls}(x) = \{\limsup x\}\uparrow$ and $\lambda_{\rm li}(x) = \{\liminf x\}\downarrow$ (generalizing the convergence of sequences on the Alexandrov cube and its dual). We consider the minimal topology $\mathcal{O}_{\rm lsi}$ extending the (unique) sequential topologies $\mathcal{O}_{\lambda_{\rm ls}}$ (left) and $\mathcal{O}_{\lambda_{\rm li}}$ (right) generated by the convergences $\lambda_{\rm ls}$ and $\lambda_{\rm li}$ and establish a general hierarchy between all these topologies and the corresponding a priori and a posteriori convergences. In addition, we observe some special classes of algebras and, in particular, show that in $(\omega, 2)$-distributive algebras we have $\lim_{\mathcal{O}_{\rm lsi}} = \lim_{\tau_{\rm s}} = \lambda_{\rm s}$, while the equality $\mathcal{O}_{\rm lsi} = \tau_s$ holds in all Maharam algebras. On the other hand, in some collapsing algebras we have a maximal (possible) diversity.

1. Introduction

It is known that a sequence $\langle x_n : n \in \omega\rangle$ of reals from the unit interval $I = [0,1]$ converges to a point $a \in I$ with respect to the left (resp. right, standard) topology on I if and only if $a \geq \limsup x_n$ (resp. $a \leq \liminf x_n$, $a = \liminf x_n = \limsup x_n$) and, more generally, these three properties define three convergence structures on any complete lattice or σ-complete Boolean algebra. In this paper, continuing the investigation from [8]–[12], we consider the corresponding convergences $\lambda_{\rm ls}$, $\lambda_{\rm li}$ and $\lambda_{\rm s}$ on a complete Boolean algebra $\mathbb{B}$, as well as the sequential topologies $\mathcal{O}_{\lambda_{\rm ls}}$, $\mathcal{O}_{\lambda_{\rm li}}$ and $\mathcal{O}_{\lambda_{\rm s}}$ on $\mathbb{B}$ generated by them. Having in mind that the union of the left and the right topology on I generates the standard topology on that interval, we regard the minimal topology $\mathcal{O}_{\rm lsi}$ on $\mathbb{B}$ extending $\mathcal{O}_{\lambda_{\rm ls}} \cup \mathcal{O}_{\lambda_{\rm li}}$, as well as the corresponding topological convergence $\lim_{\mathcal{O}_{\rm lsi}}$ on $\mathbb{B}$, and explore the relationship between all the topologies and convergences mentioned above. It turns out that $\lambda_{\rm s} \leq \lim_{\mathcal{O}_{\lambda_{\rm s}}} \leq \lim_{\mathcal{O}_{\rm lsi}}$ and $\mathcal{O}_{\rm lsi} \subset \mathcal{O}_{\lambda_{\rm s}}$ and that there are several possibilities consistent with these constraints. For example, if $\mathbb{B}$ is the power set algebra $P(\omega)$, then we have an analogy to the unit interval: $\lambda_{\rm s} = \lim_{\mathcal{O}_{\rm lsi}}$ and $\mathcal{O}_{\rm lsi} = \mathcal{O}_{\lambda_{\rm s}}$; if $\mathbb{B}$ is a Maharam algebra (i.e. admits a strictly positive Maharam submeasure), then $\lambda_{\rm s} < \lim_{\mathcal{O}_{\rm lsi}}$ and $\mathcal{O}_{\rm lsi} = \mathcal{O}_{\lambda_{\rm s}}$; finally, for some collapsing algebras we obtain a maximal diversity: $\lambda_{\rm s} < \lim_{\mathcal{O}_{\lambda_{\rm s}}} < \lim_{\mathcal{O}_{\rm lsi}}$ and $\mathcal{O}_{\rm lsi} \subsetneq \mathcal{O}_{\lambda_{\rm s}}$.

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We note that the topology O_{λ_s} on a complete Boolean algebra (c.B.a) $\mathbb{B}$ (traditionally called the *sequential topology* and denoted by τ_s), generated by the convergence λ_s (traditionally called the *algebraic convergence*) was widely considered in the context of the von Neumann problem [14]: Is each ccc weakly distributive c.B.a. a measure algebra? A consistent counter-example (a Suslin algebra) was given by Maharam [13]. In addition, Maharam has shown that the topology O_{λ_s} is metrizable iff $\mathbb{B}$ is a Maharam algebra and asked whether this implies that $\mathbb{B}$ admits a measure (the Control Measure Problem, negatively solved by M. Talagrand [15, 16]). Moreover, Balcar, Jech and Pazák [3] and, independently, Veličković [18], proved that it is consistent that the topology O_{λ_s} is metrizable on each complete ccc weakly distributive algebra. (See also [1, 2, 6, 17] for that topic).

Regarding the power set algebras, $P(\kappa)$, the convergence λ_s is exactly the convergence on the Cantor cube, while λ_{ls} generalizes the convergence on the Alexandrov cube in the same way (see [11]). Further, on any c.B.a., the topologies $O_{\lambda_{ls}}$ and $O_{\lambda_{li}}$ are homeomorphic (take $f(a) = a'$) and generated by some other convergences relevant for set-theoretic forcing (see [9, 10]). For obvious reasons, the topology $O_{\lambda_{ls}}$ (resp. $O_{\lambda_{li}}$) will be called the *left* (resp. the *right*) *topology on* $\mathbb{B}$ (see also Fact 2.3(i)).

2. Preliminaries

Convergence. Here we list the standard facts concerning convergence structures which will be used in the paper. (For details and proofs see, for example, [9].)

Let X be a non-empty set. Each mapping $x\colon \omega \to X$ is called a *sequence* in X. Usually, instead of $x(n)$ we write x_n and $x = \langle x_n : n \in \omega\rangle$. A *constant sequence* $\langle a, a, \dots\rangle$ is denoted shortly by $\langle a\rangle$. A sequence $y \in X^\omega$ is said to be a *subsequence* of x iff there is an increasing function $f\colon \omega \to \omega$ (notation: $f \in \omega^{\uparrow\omega}$) such that $y = x \circ f$; then we write $y \prec x$.

Each mapping $\lambda\colon X^\omega \to P(X)$ is called a *convergence*. The set $\mathrm{Conv}(X) = P(X)^{(X^\omega)}$ of all convergences on the set X ordered by the relation $\lambda_1 \leq \lambda_2$ ifand only if $\lambda_1(x) \subseteq \lambda_2(x)$, for each $x \in X^\omega$, is, clearly, a Boolean lattice and $\lambda_1 \cap \lambda_2$ will denote the infimum $\lambda_1 \wedge \lambda_2$; that is, $(\lambda_1 \cap \lambda_2)(x) = \lambda_1(x) \cap \lambda_2(x)$, for all $x \in X^\omega$. If $|\lambda(x)| \leq 1$ for each sequence x, then λ is called a *Hausdorff convergence*.

Let $\langle X, O\rangle$ be a topological space. A point $a \in X$ is a *limit point* of a sequence $x \in X^\omega$ if and only if each neighborhood of a contains all but finitely many members of x. The set of all limit points of a sequence $x \in X^\omega$ is denoted by $\lim_O(x)$ and so we obtain a convergence $\lim_O\colon X^\omega \to P(X)$, that is, $\lim_O \in \mathrm{Conv}(X)$.

Let $\mathrm{Top}(X)$ denote the lattice of all topologies on the set X. A convergence $\lambda \in \mathrm{Conv}(X)$ is called *topological*, we will write $\lambda \in \mathrm{TopConv}(X)$, if and only if there is a topology $O \in \mathrm{Top}(X)$ such that $\lambda = \lim_O$. So we establish the mapping

$$G\colon \mathrm{Top}(X) \to \mathrm{TopConv}(X), \text{ where } G(O) = \lim\nolimits_O .$$

A topology $O \in \mathrm{Top}(X)$ is called *sequential*, we will write $O \in \mathrm{SeqTop}(X)$ if and only if in the space $\langle X, O\rangle$ we have: a set $A \subset X$ is closed if and only if it is *sequentially closed* (that is, $\lim_O(x) \subset A$, for each sequence $x \in A^\omega$). If $O_1, O_2 \in \mathrm{SeqTop}(X)$ and $\lim_{O_1} = \lim_{O_2}$, then $O_1 = O_2$. (We note that this is false in general: take the discrete and the co-countable topology on the real line; in both spaces exactly the almost-constant sequences converge.) So, G is one-to-one on $\mathrm{SeqTop}(X)$.

For each convergence $\lambda \in \mathrm{Conv}(X)$ there is a (unique) maximal topology O_λ such that $\lambda \leq \lim_{O_\lambda}$. The topology O_λ is sequential; so, we obtain the mapping

$$F\colon \mathrm{Conv}(X) \to \mathrm{SeqTop}(X), \text{ defined by } F(\lambda) = O_\lambda.$$

F and G are antitone mappings, that is, $\lambda_1 \leq \lambda_2$ implies that $O_{\lambda_2} \subset O_{\lambda_1}$ and $O_1 \subset O_2$ implies $\lim_{O_2} \leq \lim_{O_1}$. Moreover, a convergence λ is topological if and only if $\lambda = \lim_{O_\lambda} (= G(F(\lambda)))$ and, by Theorem 2.6 of [9], a topology O is sequential if and only if $O = O_{\lim_O} (= F(G(O))$. We remark that, in fact, the pair F, G is an antitone Galois connection between the complete lattices $\mathrm{Conv}(X)$ and $\mathrm{Top}(X)$, because $O \subset F(\lambda) \Leftrightarrow \lambda \leq G(O)$, for each $\lambda \in \mathrm{Conv}(X)$ and $O \in \mathrm{Top}(X)$. (If $O \subset O_\lambda$, then $\lambda \leq \lim_{O_\lambda} \leq \lim_O$. Conversely, if $\lambda \leq \lim_O$, then $O \subset O_\lambda$, by the maximality of O_λ). Moreover, the restriction $F \upharpoonright \mathrm{TopConv}(X)$ is a bijection from $\mathrm{TopConv}(X)$ onto $\mathrm{SeqTop}(X)$ and $G \upharpoonright \mathrm{SeqTop}(X)$ is its inverse.

Each topological convergence λ satisfies the following conditions:
(L1) $\forall a \in X \;\; a \in \lambda(\langle a \rangle)$,
(L2) $\forall x \in X^\omega \; \forall y \prec x \; \lambda(x) \subset \lambda(y)$,
(L3) $\forall x \in X^\omega \; \forall a \in X \; ((\forall y \prec x \; \exists z \prec y \; a \in \lambda(z)) \Rightarrow a \in \lambda(x))$.
If $\lambda \in \mathrm{Conv}(X)$ satisfies (L1) and (L2), then $\mathcal{O}_\lambda = \{X \setminus F \colon F \subset X \wedge u_\lambda(F) = F\}$, where $u_\lambda \colon P(X) \to P(X)$ is the *operator of sequential closure determined by* λ, defined by $u_\lambda(A) = \bigcup_{x \in A^\omega} \lambda(x)$. In addition, the minimal closure of λ under (L1)–(L3) is given by $\lambda^*(x) = \bigcap_{f \in \omega^{\uparrow\omega}} \bigcup_{g \in \omega^{\uparrow\omega}} \lambda(x \circ f \circ g)$ and λ is called a *weakly-topological convergence* iff the convergence λ^* is topological.

Fact 2.1. ([9]) *If $\lambda \in \mathrm{Conv}(X)$ is a convergence satisfying (L1) and (L2), then*
(a) λ is weakly-topological if and only if $\lim_{\mathcal{O}_\lambda} = \lambda^*$, that is, for each $x \in X^\omega$ and $a \in X$

$$a \in \lim_{\mathcal{O}_\lambda}(x) \Leftrightarrow \forall y \prec x \;\; \exists z \prec y \;\; a \in \lambda(z)$$

(see [9, Theorem 4.1]);
(b) If λ is a Hausdorff convergence, then λ^* is Hausdorff and weakly-topological ([9, Theorem 4.2]).

Convergences on Boolean algebras. Let $\mathbb{B}$ be a complete Boolean algebra or, more generally, a complete lattice. If $\langle x_n \colon n \in \omega \rangle$ is a sequence of its elements, $\liminf x_n := \bigvee_{k \in \omega} \bigwedge_{n \geq k} x_n$ and $\limsup x_n := \bigwedge_{k \in \omega} \bigvee_{n \geq k} x_n$, then, clearly, $\liminf x_n \leq \limsup x_n$. We consider the convergences $\lambda_{\mathrm{ls}}, \lambda_{\mathrm{li}}, \lambda_{\mathrm{s}} \colon \mathbb{B}^\omega \to P(\mathbb{B})$ defined by

$$\lambda_{\mathrm{ls}}(\langle x_n \rangle) = \{\limsup x_n\}\uparrow, \tag{1}$$
$$\lambda_{\mathrm{li}}(\langle x_n \rangle) = \{\liminf x_n\}\downarrow, \tag{2}$$
$$\lambda_{\mathrm{s}}(\langle x_n \rangle) = \begin{cases} \{x\} & \text{if } \liminf x_n = \limsup x_n = x, \\ 0 & \text{if } \liminf x_n < \limsup x_n, \end{cases} \tag{3}$$

where $A\uparrow := \{b \in \mathbb{B} \colon \exists a \in A \; b \geq a\}$ and $A\downarrow := \{b \in \mathbb{B} \colon \exists a \in A \; b \leq a\}$, for $A \subset \mathbb{B}$. The following property of c.B.a.'s will play a role in this paper

$$\forall x \in \mathbb{B}^\omega \; \exists y \prec x \; \forall z \prec y \;\; \limsup z = \limsup y. \tag{$\hbar$}$$

We note that property ($\hbar$) is closely related to the cellularity of Boolean algebras. Namely, by [8], $\mathfrak{t}$-cc $\Rightarrow$ ($\hbar$) $\Rightarrow$ $\mathfrak{s}$-cc and, in particular, ccc complete Boolean algebras satisfy ($\hbar$). By [12], the set $\{\kappa \in \mathrm{Card} \colon \kappa\text{-cc} \Rightarrow (\hbar)\}$ is equal either to $[0, \mathfrak{h})$, or to $[0, \mathfrak{h}]$ and $\{\kappa \in \mathrm{Card} \colon (\hbar) \Rightarrow \kappa\text{-cc}\} = [\mathfrak{s}, \infty)$. Basic facts concerning the invariants of the continuum $\mathfrak{t}$, $\mathfrak{s}$, and $\mathfrak{h}$ can be found in [5].

Fact 2.2. ([8]) *If $\mathbb{B}$ is a complete Boolean algebra, then we have*
(a) λ_{s} is a weakly-topological Hausdorff convergence satisfying (L1) and (L2) ([8, Lemma 3]);
(b) λ_{s} is a topological convergence if and only if the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive (see [8, Theorem 2]).

Fact 2.3. ([11]) *If $\mathbb{B}$ is a complete non-trivial Boolean algebra, then*
(a) λ_{ls} and λ_{li} are non-Hausdorff convergences satisfying (L1) and (L2) ([11, Theorem 4.3]);
(b) If $\mathbb{B}$ satisfies ($\hbar$), then λ_{ls} and λ_{li} are weakly-topological convergences ([11, Theorem 6.1]);
(c) λ_{ls} is topological if and only if λ_{li} is topological if and only if the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive ([11, Theorem 3.5]);
(d) $\lambda_{\mathrm{s}} = \lambda_{\mathrm{ls}} \cap \lambda_{\mathrm{li}}$ ([11, Theorem 4.3]);
(e) $\mathcal{O}_{\lambda_{\mathrm{ls}}}, \mathcal{O}_{\lambda_{\mathrm{li}}} \subset \mathcal{O}_{\lambda_{\mathrm{s}}}$ ([11, Theorem 4.3]);
(f) $\lambda^*_{\mathrm{ls}} \leq \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}$ and $\lambda^*_{\mathrm{li}} \leq \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}$ ([11, Theorem 4.3]);
(g) $\lambda^*_{\mathrm{s}} = \lambda^*_{\mathrm{ls}} \cap \lambda^*_{\mathrm{li}}$ ([11, Theorem 4.3]);
(h) $\mathcal{O}_{\lambda_{\mathrm{ls}}}$ and $\mathcal{O}_{\lambda_{\mathrm{li}}}$ are homeomorphic, T_0, connected and compact topologies ([11, Theorem 4.4]);
(i) A set $F \subset \mathbb{B}$ is $\mathcal{O}_{\lambda_{\mathrm{ls}}}$-closed if and only if it is upward-closed and $\bigwedge_{n \in \omega} x_n \in F$, for each decreasing sequence $\langle x_n \rangle \in F^\omega$; (and dually, for $\mathcal{O}_{\lambda_{\mathrm{li}}}$-closed sets) ([11, Theorem 4.4]).

3. The Topology $\mathcal{O}_{\rm lsi}$ on Boolean Algebras

On a complete Boolean algebra $\mathbb{B}$ we consider the minimal topology containing the topologies $\mathcal{O}_{\lambda_{\rm ls}}$ and $\mathcal{O}_{\lambda_{\rm li}}$. This topology, denoted by $\mathcal{O}_{\rm lsi}$, is generated by the base $\mathcal{B}_{\rm lsi} = \{O_1 \cap O_2 : O_1 \in \mathcal{O}_{\lambda_{\rm ls}} \wedge O_2 \in \mathcal{O}_{\lambda_{\rm li}}\}$. By Fact 2.3(i), the sets from $\mathcal{O}_{\lambda_{\rm ls}}$ (resp. $\mathcal{O}_{\lambda_{\rm li}}$) are downward (resp. upward)-closed; so, the elements of $\mathcal{B}_{\rm lsi}$ are convex subsets of $\mathbb{B}$.

Theorem 3.1. *The following diagrams show the relations between the considered convergences and topologies on a non-trivial c.B.a. $\mathbb{B}$. In addition, we have*

(a) $\lambda_{\rm ls} \cap \lambda_{\rm li} = \lambda_{\rm s}$, $\lambda^*_{\rm ls} \cap \lambda^*_{\rm li} = \lambda^*_{\rm s}$ *and* $\lim_{\mathcal{O}_{\lambda_{\rm ls}}} \cap \lim_{\mathcal{O}_{\lambda_{\rm li}}} = \lim_{\mathcal{O}_{\rm lsi}}$*;*

(b) $\lambda_{\rm s} < \lambda_{\rm ls}, \lambda_{\rm li}$, $\lambda^*_{\rm s} < \lambda^*_{\rm ls}, \lambda^*_{\rm li}$, $\lim_{\mathcal{O}_{\rm lsi}} < \lim_{\mathcal{O}_{\lambda_{\rm ls}}}, \lim_{\mathcal{O}_{\lambda_{\rm li}}}$ *and* $\mathcal{O}_{\rm lsi} \supsetneq \mathcal{O}_{\lambda_{\rm ls}}, \mathcal{O}_{\lambda_{\rm li}}$*.*

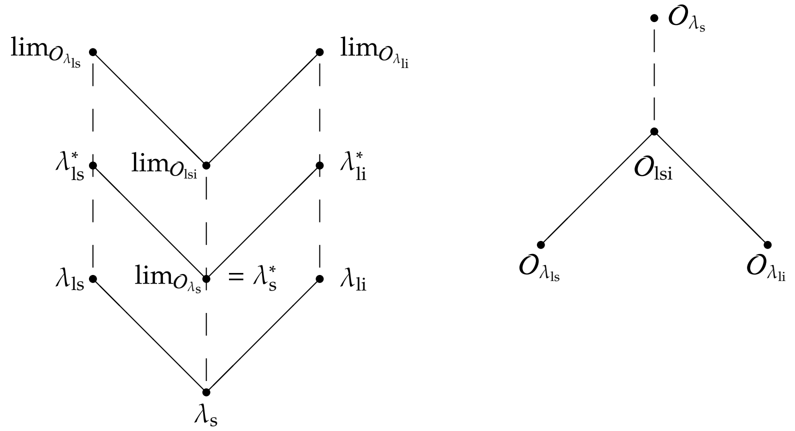

Figure 1: Convergences and topologies on $\mathbb{B}$

Proof. By Fact 2.3(e) we have $\mathcal{O}_{\lambda_{\rm ls}}, \mathcal{O}_{\lambda_{\rm li}} \subset \mathcal{O}_{\lambda_{\rm s}}$ and the inclusion $\mathcal{O}_{\rm lsi} \subset \mathcal{O}_{\lambda_{\rm s}}$ follows from the minimality of $\mathcal{O}_{\rm lsi}$. So the diagram for topologies is correct.

By Fact 2.3(d) and (g) we have $\lambda_{\rm s} = \lambda_{\rm ls} \cap \lambda_{\rm li}$ and $\lambda^*_{\rm s} = \lambda^*_{\rm ls} \cap \lambda^*_{\rm li}$, which implies $\lambda_{\rm s} \leq \lambda_{\rm ls}, \lambda_{\rm li}$ and $\lambda^*_{\rm s} \leq \lambda^*_{\rm ls}, \lambda^*_{\rm li}$. By Facts 2.2(a) and 2.3(a), $\lambda_{\rm s}$ is a Hausdorff convergence, while $\lambda_{\rm ls}$ and $\lambda_{\rm li}$ are not; thus, $\lambda_{\rm s} < \lambda_{\rm ls}, \lambda_{\rm li}$. By Fact 2.1(b) $\lambda^*_{\rm s}$ is a Hausdorff convergence and, clearly, $\lambda^*_{\rm ls}$ and $\lambda^*_{\rm li}$ are not Hausdorff; so, $\lambda^*_{\rm s} < \lambda^*_{\rm ls}, \lambda^*_{\rm li}$.

By the construction of the closure λ^* it follows that we always have $\lambda \leq \lambda^*$; thus $\lambda_{\rm ls} \leq \lambda^*_{\rm ls}$, $\lambda_{\rm li} \leq \lambda^*_{\rm li}$ and $\lambda_{\rm s} \leq \lambda^*_{\rm s}$. By Fact 2.3(f) we have $\lambda^*_{\rm ls} \leq \lim_{\mathcal{O}_{\lambda_{\rm ls}}}$ and $\lambda^*_{\rm li} \leq \lim_{\mathcal{O}_{\lambda_{\rm li}}}$. The equality $\lambda^*_{\rm s} = \lim_{\mathcal{O}_{\lambda_{\rm s}}}$ follows from Facts 2.2(a) and 2.1(a). Since $\mathcal{O}_{\rm lsi} \subset \mathcal{O}_{\lambda_{\rm s}}$ we have $\lim_{\mathcal{O}_{\lambda_{\rm s}}} \leq \lim_{\mathcal{O}_{\rm lsi}}$.

Further we prove that $\lim_{\mathcal{O}_{\rm lsi}} = \lim_{\mathcal{O}_{\lambda_{\rm ls}}} \cap \lim_{\mathcal{O}_{\lambda_{\rm li}}}$. Since $\mathcal{O}_{\lambda_{\rm ls}}, \mathcal{O}_{\lambda_{\rm li}} \subset \mathcal{O}_{\rm lsi}$, we have $\lim_{\mathcal{O}_{\rm lsi}} \leq \lim_{\mathcal{O}_{\lambda_{\rm ls}}}, \lim_{\mathcal{O}_{\lambda_{\rm li}}}$. Conversely, if $a \in \lim_{\mathcal{O}_{\lambda_{\rm ls}}}(x) \cap \lim_{\mathcal{O}_{\lambda_{\rm li}}}(x)$ and U is a $\mathcal{O}_{\rm lsi}$-neighborhood of a, then there is $O_1 \cap O_2 \in \mathcal{B}_{\rm lsi}$ such that $a \in O_1 \cap O_2 \subset U$ and, hence, there are $n_i \in \omega$, $i \in \{1, 2\}$, such that $x_n \in O_i$, for each $n \geq n_i$. Thus for each $n \geq \max\{n_1, n_2\}$ we have $x_n \in U$, so $a \in \lim_{\mathcal{O}_{\rm lsi}}(x)$.

So we have $\lim_{\mathcal{O}_{\rm lsi}} \leq \lim_{\mathcal{O}_{\lambda_{\rm ls}}}, \lim_{\mathcal{O}_{\lambda_{\rm li}}}$. Since we have $1 \in \lambda_{\rm ls}(\langle 0 \rangle)$, assuming that $\lim_{\mathcal{O}_{\rm lsi}} = \lim_{\mathcal{O}_{\lambda_{\rm ls}}}$, we would have $\lambda_{\rm ls} \leq \lim_{\mathcal{O}_{\lambda_{\rm ls}}} \leq \lim_{\mathcal{O}_{\lambda_{\rm li}}}$ and, therefore $1 \in \lim_{\mathcal{O}_{\lambda_{\rm li}}}(\langle 0 \rangle)$. Now, since the sets from $\mathcal{O}_{\lambda_{\rm li}}$ are upward-closed, for a non-empty set $O \in \mathcal{O}_{\lambda_{\rm li}}$ we would have $1 \in O$ and, since $1 \in \lim_{\mathcal{O}_{\lambda_{\rm li}}}(\langle 0 \rangle)$, $0 \in O$ as well, which would give $O = \mathbb{B}$. So $\mathcal{O}_{\lambda_{\rm li}}$ would be the antidiscrete topology which is false, because it is T_0. Thus $\lim_{\mathcal{O}_{\rm lsi}} < \lim_{\mathcal{O}_{\lambda_{\rm ls}}}$ and, similarly, $\lim_{\mathcal{O}_{\rm lsi}} < \lim_{\mathcal{O}_{\lambda_{\rm li}}}$, which implies that $\mathcal{O}_{\rm lsi} \supsetneq \mathcal{O}_{\lambda_{\rm ls}}, \mathcal{O}_{\lambda_{\rm li}}$. □

In the sequel we consider the topology $\mathcal{O}_{\rm lsi}$ and its convergence and investigate the form of the diagrams in Figure 1 for particular (classes of) Boolean algebras. In particular, it is natural to ask for which complete Boolean algebras we have

$$\mathcal{O}_{\rm lsi} = \mathcal{O}_{\lambda_{\rm s}} \ \text{ or, at least, } \ \lim_{\mathcal{O}_{\rm lsi}} = \lim_{\mathcal{O}_{\lambda_{\rm s}}} ? \tag{4}$$

First we give some sufficient conditions for these equalities.

Theorem 3.2. *Let* $\mathbb{B}$ *be a complete Boolean algebra. Then*

(a) If the algebra $\mathbb{B}$ *satisfies condition* ($\hbar$), *then* $\lim_{O_{\mathrm{lsi}}} = \lim_{O_{\lambda_{\mathrm{s}}}}$*;*

(b) If the algebra $\mathbb{B}$ *is* $(\omega, 2)$*-distributive, then* $\lim_{O_{\lambda_{\mathrm{ls}}}} = \lambda_{\mathrm{ls}}$, $\lim_{O_{\lambda_{\mathrm{li}}}} = \lambda_{\mathrm{li}}$ *and* $\lim_{O_{\mathrm{lsi}}} = \lim_{O_{\lambda_{\mathrm{s}}}} = \lambda_{\mathrm{s}}$*; so the diagram for convergences collapses to 3 nodes;*

(c) If $\lim_{O_{\mathrm{lsi}}} = \lim_{O_{\lambda_{\mathrm{s}}}}$*, then* $O_{\mathrm{lsi}} = O_{\lambda_{\mathrm{s}}}$ *if and only if* $\langle \mathbb{B}, O_{\mathrm{lsi}} \rangle$ *is a sequential space.*

Proof. (a) By Theorem 3.1 we have $O_{\mathrm{lsi}} \subset O_{\lambda_{\mathrm{s}}}$ so, $\lim_{O_{\lambda_{\mathrm{s}}}} \leq \lim_{O_{\mathrm{lsi}}}$.

Conversely, assuming that $x \in \mathbb{B}^\omega$ and $a \in \lim_{O_{\mathrm{lsi}}}(x)$, by Theorem 3.1 we have

$$a \in \lim_{O_{\lambda_{\mathrm{ls}}}}(x) \cap \lim_{O_{\lambda_{\mathrm{li}}}}(x) \tag{5}$$

and we should prove that $a \in \lim_{O_{\lambda_{\mathrm{s}}}}(x)$. Thus, by Facts 2.2(a) and 2.1(a), we have to show that for each $y \prec x$ there is $z \prec y$ such that $\limsup z = \liminf z = a$.

Let y be a subsequence of x. By Fact 2.3(b) the convergence λ_{ls} is weakly topological so, by (5) and Fact 2.1(a), there is $z' \prec y$ such that $\limsup z' \leq a$. Since $z' \prec x$ and the convergence λ_{li} is weakly topological, by (5) and Fact 2.1(a) again, there is $z \prec z'$ such that $\liminf z \geq a$. Now, we have $\limsup z \leq \limsup z' \leq a \leq \liminf z$, which implies that $\liminf z = \limsup z = a$.

(b) If the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive, then by Facts 2.2(b) and 2.3(c) we have $\lim_{O_{\lambda_{\mathrm{s}}}} = \lambda_{\mathrm{s}}$, $\lim_{O_{\lambda_{\mathrm{ls}}}} = \lambda_{\mathrm{ls}}$ and $\lim_{O_{\lambda_{\mathrm{li}}}} = \lambda_{\mathrm{li}}$. Thus, by Theorem 3.1 we have $\lim_{O_{\mathrm{lsi}}} = \lim_{O_{\lambda_{\mathrm{ls}}}} \cap \lim_{O_{\lambda_{\mathrm{li}}}} = \lambda_{\mathrm{ls}} \cap \lambda_{\mathrm{li}} = \lambda_{\mathrm{s}} = \lim_{O_{\lambda_{\mathrm{s}}}}$.

(c) The implication "$\Rightarrow$" is true because the topology $O_{\lambda_{\mathrm{s}}}$ is sequential. If $\lim_{O_{\mathrm{lsi}}} = \lim_{O_{\lambda_{\mathrm{s}}}}$, and O_{lsi} is a sequential topology, then (since a topology O is sequential if and only if $O = O_{\lim_O}$) we have $O_{\mathrm{lsi}} = O_{\lim_{O_{\mathrm{lsi}}}} = O_{\lim_{O_{\lambda_{\mathrm{s}}}}} = O_{\lambda_{\mathrm{s}}}$. □

The unit interval. Although the unit interval $I = [0, 1]$ is not a Boolean algebra, it provides obvious examples of the convergences considered in this paper. Let $O_{\leftarrow} = \{[0, a) : 0 < a \leq 1\} \cup \{\emptyset, I\}$ and $O_{\rightarrow} = \{(a, 1] : 0 \leq a < 1\} \cup \{\emptyset, I\}$ be the left and the right topology on I and let O_{st} denote the standard topology on I. It is easy to check that defining $\lambda_{\mathrm{ls}}(\langle x_n \rangle) = \{x \in I : x \geq \limsup x_n\}$ and $\lambda_{\mathrm{li}}(\langle x_n \rangle) = \{x \in I : x \leq \liminf x_n\}$ we have

$$\lim_{O_{\leftarrow}} = \lambda_{\mathrm{ls}} \quad \text{and} \quad \lim_{O_{\rightarrow}} = \lambda_{\mathrm{li}} \quad \text{and} \quad \lim_{O_{\mathrm{st}}} = \lambda_{\mathrm{s}}. \tag{6}$$

We recall that a topology O is sequential if and only if $O = O_{\lim_O}$. So since the topology $O_{\leftarrow}$ is first countable and, hence, sequential, by (6) we have $O_{\lambda_{\mathrm{ls}}} = O_{\lim_{O_{\leftarrow}}} = O_{\leftarrow}$; and similarly for the other two topologies. So

$$O_{\lambda_{\mathrm{ls}}} = O_{\leftarrow} \quad \text{and} \quad O_{\lambda_{\mathrm{li}}} = O_{\rightarrow} \quad \text{and} \quad O_{\lambda_{\mathrm{s}}} = O_{\mathrm{st}}. \tag{7}$$

Since $O_{\leftarrow} \cup O_{\rightarrow}$ is a subbase of O_{st} we have $O_{\mathrm{st}} = O(O_{\leftarrow} \cup O_{\rightarrow})$ and by (7) we have $O_{\lambda_{\mathrm{s}}} = O_{\mathrm{st}} = O(O_{\leftarrow} \cup O_{\rightarrow}) = O(O_{\lambda_{\mathrm{ls}}} \cup O_{\lambda_{\mathrm{li}}}) = O_{\mathrm{lsi}}$ and (4) is true.

Power set algebras. Let $\kappa \geq \omega$ be a cardinal. We recall that the *Alexandrov cube of weight* κ is the product of κ many copies of the two point space $2 = \{0, 1\}$ with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$. Identifying the sets $P(\kappa)$ and 2^κ via characteristic functions we obtain a homeomorphic copy $\mathbb{A}_\kappa = \langle P(\kappa), \tau_{\mathbb{A}_\kappa} \rangle$ of that space. We recall that for a sequence $\langle X_n : n \in \omega \rangle$ in $P(\kappa)$ we have

$\liminf_{n \in \omega} X_n = \bigcup_{k \in \omega} \bigcap_{n \geq k} X_n = \{x : x \in X_n \text{ for all but finitely many } n\}$,

$\limsup_{n \in \omega} X_n = \bigcap_{k \in \omega} \bigcup_{n \geq k} X_n = \{x : x \in X_n \text{ for infinitely many } n\}$. Further, the *Cantor cube of weight* κ is the product of κ many copies of the two point discrete space $2 = \{0, 1\}$ and, identifying the sets $P(\kappa)$ and 2^κ again, we obtain its homeomorphic copy $\mathbb{C}_\kappa = \langle P(\kappa), \tau_{\mathbb{C}_\kappa} \rangle$. By [11, Theorem 4.2] we have

Fact 3.3. *For the power algebra* $P(\kappa)$ *with the Aleksandrov topology we have*

(a) $\lambda_{\mathrm{ls}} = \lim_{O_{\lambda_{\mathrm{ls}}}} = \lim_{\tau_{\mathbb{A}_\kappa}}$*; thus* λ_{ls} *is a topological convergence;*

(b) $\langle P(\kappa), \tau_{\mathbb{A}_\kappa} \rangle$ *is a sequential space if and only if* $O_{\lambda_{\mathrm{ls}}} = \tau_{\mathbb{A}_\kappa}$ *if and only if* $\kappa = \omega$*;*

(c) If $\kappa > \omega$*, then* $\tau_{\mathbb{A}_\kappa} \subsetneq O_{\lambda_{\mathrm{ls}}} \not\subset \tau_{\mathbb{C}_\kappa}$.

For the power algebra $P(\kappa)$ *with the Cantor topology we have*

(d) $\lambda_{\mathrm{s}} = \lim_{O_{\lambda_{\mathrm{s}}}} = \lim_{\tau_{\mathbb{C}_\kappa}}$*; thus* λ_{s} *is a topological convergence;*

(e) $\langle P(\kappa), \tau_{\mathbb{C}_\kappa} \rangle$ *is a sequential space if and only if* $O_{\lambda_{\mathrm{s}}} = \tau_{\mathbb{C}_\kappa}$ *if and only if* $\kappa = \omega$*;*

(f) If $\kappa > \omega$*, then* $\tau_{\mathbb{C}_\kappa} \subsetneq O_{\lambda_{\mathrm{s}}}$.

Let $\tau_{\mathbb{A}_\kappa^c}$ be the topology on the power algebra $P(\kappa)$ obtained by the standard identification of $P(\kappa)$ and 2^κ with the Tychonov topology of κ many copies of the space 2 with the topology $\{\emptyset, \{1\}, \{0,1\}\}$. Then, clearly, $X \mapsto \kappa \setminus X$ is a homeomorphism from $\mathbb{A}_\kappa = \langle P(\kappa), \tau_{\mathbb{A}_\kappa}\rangle$ onto the reversed Alexandrov cube $\mathbb{A}_\kappa^c = \langle P(\kappa), \tau_{\mathbb{A}_\kappa^c}\rangle$. Replacing $\tau_{\mathbb{A}_\kappa}$ by $\tau_{\mathbb{A}_\kappa^c}$ and λ_{ls} by λ_{li} in (a), (b) and (c) of Fact 3.3 we obtain the corresponding dual statements. In addition, we have

Theorem 3.4. *For the power algebra $P(\kappa)$ we have*

(a) $\lim_{\mathcal{O}_{\mathrm{lsi}}} = \lambda_{\mathrm{s}}$*;*

(b) $\tau_{\mathbb{C}_\kappa}$ is the minimal topology containing $\tau_{\mathbb{A}_\kappa}$ and $\tau_{\mathbb{A}_\kappa^c}$;

(c) $\tau_{\mathbb{C}_\kappa} \subset \mathcal{O}_{\mathrm{lsi}}$ and so $\mathcal{O}_{\mathrm{lsi}}$ is a Hausdorff topology on $P(\kappa)$;

(d) For $\kappa = \omega$ we have $\mathcal{O}_{\mathrm{lsi}} = \tau_{\mathbb{C}_\omega} = \mathcal{O}_{\lambda_{\mathrm{s}}}$;

(e) $\mathcal{O}_{\mathrm{lsi}} = \tau_{\mathbb{C}_\kappa}$ if and only if $\kappa = \omega$.

Proof. (a) By Fact 3.3(a) and its dual we have $\lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} = \lambda_{\mathrm{ls}}$ and $\lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}} = \lambda_{\mathrm{li}}$. Now, by Theorem 3.1, $\lim_{\mathcal{O}_{\mathrm{lsi}}} = \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} \cap \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}} = \lambda_{\mathrm{ls}} \cap \lambda_{\mathrm{li}} = \lambda_{\mathrm{s}}$.

(b) Let $\mathcal{O}$ be the minimal topology containing $\tau_{\mathbb{A}_\kappa}$ and $\tau_{\mathbb{A}_\kappa^c}$. A subbase for the topology $\tau_{\mathbb{A}_\kappa}$ (resp. $\tau_{\mathbb{A}_\kappa^c}$) consists of the sets $B_i := \{X \subset \kappa : i \notin X\}$ (resp. $B_i^c := \{X \subset \kappa : i \in X\}$), where $i \in \kappa$; while the family $\mathcal{S}_{\mathbb{C}_\kappa} := \bigcup_{i\in\kappa}\{B_i, B_i^c\}$ is a subbase for the topology $\tau_{\mathbb{C}_\kappa}$. Thus $\tau_{\mathbb{A}_\kappa} \cup \tau_{\mathbb{A}_\kappa^c} \subset \tau_{\mathbb{C}_\kappa}$ and, hence, $\mathcal{O} \subset \tau_{\mathbb{C}_\kappa}$. On the other hand, $\mathcal{S}_{\mathbb{C}_\kappa} \subset \tau_{\mathbb{A}_\kappa} \cup \tau_{\mathbb{A}_\kappa^c} \subset \mathcal{O}$, which gives $\tau_{\mathbb{C}_\kappa} \subset \mathcal{O}$.

(c) By Fact 3.3 and its dual we have $\tau_{\mathbb{A}_\kappa} \subset \mathcal{O}_{\lambda_{\mathrm{ls}}}$ and $\tau_{\mathbb{A}_\kappa^c} \subset \mathcal{O}_{\lambda_{\mathrm{li}}}$. Thus $\tau_{\mathbb{A}_\kappa} \cup \tau_{\mathbb{A}_\kappa^c} \subset \mathcal{O}_{\mathrm{lsi}}$ and $\tau_{\mathbb{C}_\kappa} \subset \mathcal{O}_{\mathrm{lsi}}$, by the minimality of $\tau_{\mathbb{C}_\kappa}$ proved in (b).

(d) By (c) and Theorem 3.1, $\tau_{\mathbb{C}_\omega} \subset \mathcal{O}_{\mathrm{lsi}} \subset \mathcal{O}_{\lambda_{\mathrm{s}}}$ and we apply Fact 3.3(e).

(e) By (d) the implication "$\Leftarrow$" is true. Assuming that $\mathcal{O}_{\mathrm{lsi}} = \tau_{\mathbb{C}_\kappa}$ and $\kappa > \omega$, by Fact 3.3(c) we would have $\mathcal{O}_{\lambda_{\mathrm{ls}}} \not\subset \tau_{\mathbb{C}_\kappa}$, which gives a contradiction because $\mathcal{O}_{\lambda_{\mathrm{ls}}} \subset \mathcal{O}_{\mathrm{lsi}}$. □

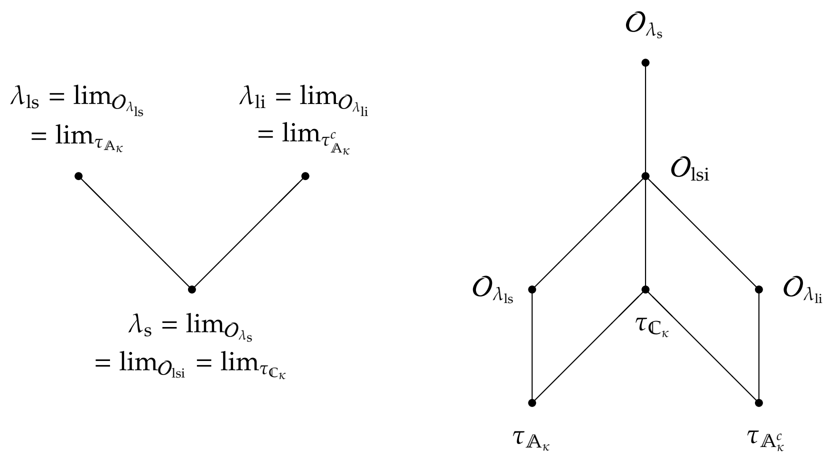

Figure 2: Convergences and topologies on the algebra $P(\kappa)$

For the power set algebras the diagrams from Figure 1 are presented in Figure 2. Namely, by Theorem 3.2(b), the diagram describing convergences collapses to three nodes. The diagram for topologies in Figure 2 contains the topologies from Figure 1 as well as the topologies of the Cantor, Alexandrov and reversed Alexandrov cube (see Fact 3.3(c) and Theorem 3.4(c)). By Fact 3.3(b) and (e), for $\kappa = \omega$ the diagram describing topologies contains exactly three different topologies. So, for the algebra $P(\omega)$ we have $\mathcal{O}_{\mathrm{lsi}} = \mathcal{O}_{\lambda_{\mathrm{s}}}$ and (4) is true.

Maharam algebras. We recall that a *submeasure* on a complete Boolean algebra $\mathbb{B}$ is a function $\mu : \mathbb{B} \to [0, \infty)$ satisfying:

(i) $\mu(0) = 0$;

(ii) $a \leq b \Rightarrow \mu(a) \leq \mu(b)$ and

(iii) $\mu(a \vee b) \leq \mu(a) + \mu(b)$.

A submeasure μ is *strictly positive* if and only if

(iv) $a > 0 \Rightarrow \mu(a) > 0$.

A submeasure μ is called a *Maharam* (or a *continuous*) *submeasure* if and only if

(v) $\lim_{n\to\infty} \mu(a_n) = 0$ holds for each decreasing sequence $\langle a_n : n \in \omega \rangle$ in $\mathbb{B}$ satisfying $\bigwedge_{n\in\omega} a_n = 0$.

It is easy to prove that if μ is a Maharam submeasure, then $\lim_{n\to\infty} \mu(a_n) = \mu(\bigwedge_{n\in\omega} a_n)$, for each decreasing sequence $\langle a_n \rangle$ in $\mathbb{B}$. A complete Boolean algebra $\mathbb{B}$ admitting a strictly positive Maharam submeasure is called a *Maharam algebra*.

Theorem 3.5. *On each Maharam algebra $\mathbb{B}$ we have $\mathcal{O}_{\mathrm{lsi}} = \mathcal{O}_{\lambda_{\mathrm{s}}}$.*

Proof. Under the assumption, $d(x, y) = \mu(x \vartriangle y)$ is a metric on $\mathbb{B}$ which generates the topology $\mathcal{O}_{\lambda_{\mathrm{s}}}$ (see [13]). For a non-empty set $O \in \mathcal{O}_{\lambda_{\mathrm{s}}}$ we show that $O \in \mathcal{O}_{\mathrm{lsi}}$. Let $a \in O$ and $r > 0$, where $B(a, r) = \{x \in \mathbb{B} : \mu(x \vartriangle a) < r\} \subset O$. Let

$$O_1 = \{x \in \mathbb{B} : \mu(x \setminus a) < r/2\} \text{ and } O_2 = \{x \in \mathbb{B} : \mu(a \setminus x) < r/2\}.$$

Then by (i) we have $a \in O_1 \cap O_2$. If $x \in O_1 \cap O_2$, then, by (iii), $\mu(x \vartriangle a) \leq \mu(x \setminus a) + \mu(a \setminus x) < r$ and, hence, $x \in B(a, r)$, thus $a \in O_1 \cap O_2 \subset O$.

Let us prove that $O_1 \in \mathcal{O}_{\lambda_{\mathrm{ls}}}$. By Fact 2.3(a) the convergence λ_{ls} satisfies (L1) and (L2), so it is sufficient to prove that $\mathbb{B} \setminus O_1$ is a closed set, which means that $u_{\lambda_{\mathrm{ls}}}(\mathbb{B} \setminus O_1) \subset \mathbb{B} \setminus O_1$. By (iii), the set $\mathbb{B} \setminus O_1$ is upward-closed and it is sufficient to show that for a sequence $\langle x_n \rangle$ in $\mathbb{B} \setminus O_1$ we have $\limsup x_n \in \mathbb{B} \setminus O_1$, that is $\mu(\limsup x_n \setminus a) \geq r/2$. By the assumption we have $\mu(x_n \setminus a) \geq r/2$, for each $n \in \omega$. Now $\limsup x_n \setminus a = \bigwedge_{k\in\omega} y_k$, where $y_k = \bigvee_{n\geq k} x_n \setminus a$, $k \in \omega$, is a decreasing sequence and $\mu(y_k) \geq r/2$ so, by the continuity of μ, $\mu(\limsup x_n \setminus a) = \lim_{k\to\infty} \mu(y_k) \geq r/2$. Similarly we prove that $O_2 \in \mathcal{O}_{\lambda_{\mathrm{li}}}$ so $O_1 \cap O_2 \in \mathcal{O}_{\mathrm{lsi}}$ and O is an $\mathcal{O}_{\mathrm{lsi}}$-neighborhood of the point a. □

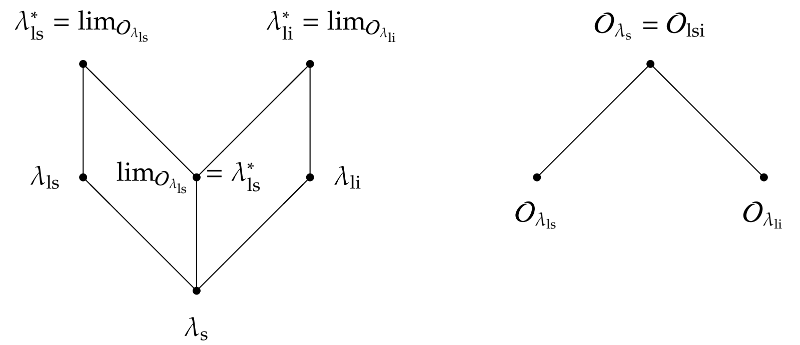

Figure 3: Non-$(\omega, 2)$-distributive Maharam algebras

Thus, if $\mathbb{B}$ is a Maharam algebra which is not $(\omega, 2)$-distributive (for example, the algebra of the Lebesgue-measurable subsets of $[0, 1]$ modulo the ideal of the sets of measure zero), then, the Figure 3 describes the corresponding diagrams. Namely, by Facts 2.2(a) and 2.1(a) we have $\lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}} = \lambda^*_{\mathrm{s}}$ and, by Fact 2.2(b), $\lambda_{\mathrm{s}} \neq \lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}}$. Since the algebras with strictly positive measure satisfy the countable chain condition the algebra $\mathbb{B}$ has $(\hbar)$. Thus, by Facts 2.3(b) and 2.1(a) we have $\lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}} = \lambda^*_{\mathrm{ls}}$ and $\lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}} = \lambda^*_{\mathrm{li}}$. By Fact 2.3(c) we have $\lambda_{\mathrm{ls}} \neq \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}$ and $\lambda_{\mathrm{li}} \neq \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}$. By Theorem 3.5 we have $\mathcal{O}_{\mathrm{lsi}} = \mathcal{O}_{\lambda_{\mathrm{s}}}$ and, hence, $\lim_{\mathcal{O}_{\mathrm{lsi}}} = \lim_{\mathcal{O}_{\lambda_{\mathrm{s}}}}$.

Collapsing algebras. We show that both equalities from (4) can fail. We recall that a family $T \subset [\omega]^\omega$ is a *tower* if and only if it is well-ordered by ${}^*\supsetneq$ and has no pseudointersection; the *tower number*, $\mathfrak{t}$, is the minimal cardinality of a tower. The *distributivity number*, $\mathfrak{h}$, is the smallest number of dense open families in the preorder $\langle [\omega]^\omega, \subset^* \rangle$ with empty intersection. A family $\mathcal{T} \subset [\omega]^\omega$ is called a *base matrix tree* if and only if $\langle \mathcal{T}, {}^*\supset \rangle$ is a tree of height $\mathfrak{h}$ and $\mathcal{T}$ is a dense set in the preorder $\langle [\omega]^\omega, \subset^* \rangle$. By a theorem of Balcar, Pelant and Simon (see [4]), such a tree always exists, its levels are maximal almost disjoint families and maximal chains in $\mathcal{T}$ are towers.

Theorem 3.6. *If* $\mathbb{B}$ *is a complete Boolean algebra satisfying* $1 \Vdash_{\mathbb{B}} (\mathfrak{h}^V)\check{} < \mathfrak{t}$ *and* $\mathrm{cc}(\mathbb{B}) > 2^{\mathfrak{h}}$, *then* $\lim_{\mathcal{O}_{\lambda_s}} < \lim_{\mathcal{O}_{\mathrm{lsi}}}$ *and* $\mathcal{O}_{\mathrm{lsi}} \subsetneq \mathcal{O}_{\lambda_s}$.

Proof. Using the construction from the proof of Theorem 6.4 from [11], we will find a sequence x in $\mathbb{B}$ such that $0 \in \lim_{\mathcal{O}_{\mathrm{lsi}}}(x) \setminus \lim_{\mathcal{O}_{\lambda_s}}(x)$.

Let $\mathcal{T} \subset [\omega]^\omega$ be a base matrix tree and $\mathrm{Br}(\mathcal{T})$ the set of its maximal branches. Since the height of $\mathcal{T}$ is $\mathfrak{h}$, the branches of $\mathcal{T}$ are of size $\leq \mathfrak{h}$; so $\kappa := |\mathrm{Br}(\mathcal{T})| \leq \mathfrak{c}^{\mathfrak{h}} = 2^{\mathfrak{h}}$ and we take a one-to-one enumeration $\mathrm{Br}(\mathcal{T}) = \{T_\alpha : \alpha < \kappa\}$.

Since $1 \Vdash (\mathfrak{h}^V)\check{} < \mathfrak{t}$, for each $\alpha < \kappa$ we have $1 \Vdash |\check{T}_\alpha| < \mathfrak{t}$ and, consequently, in each generic extension of the ground model by $\mathbb{B}$ the family T_α obtains a pseudointersection. Thus $1 \Vdash \exists X \in [\check{\omega}]^{\check{\omega}}\ \forall B \in \check{T}_\alpha\ X \subset^* B$ so, by the Maximum Principle (see [7, p. 226]), there is a name $\sigma_\alpha \in V^{\mathbb{B}}$ such that

$$1 \Vdash \sigma_\alpha \in [\check{\omega}]^{\check{\omega}} \wedge \forall B \in \check{T}_\alpha \ \ \sigma_\alpha \subset^* B. \tag{8}$$

Since $\mathrm{cc}(\mathbb{B}) > 2^{\mathfrak{h}} \geq \kappa$, there is a maximal antichain in $\mathbb{B}$ of cardinality κ, say $\{b_\alpha : \alpha < \kappa\}$. By the Mixing lemma (see [7, p. 226]) there is a name $\tau \in V^{\mathbb{B}}$ such that

$$\forall \alpha < \kappa \ \ b_\alpha \Vdash \tau = \sigma_\alpha, \tag{9}$$

and, clearly, $1 \Vdash \tau \in [\check{\omega}]^{\check{\omega}}$. Let $x = \langle x_n \rangle \in \mathbb{B}^\omega$, where $x_n := \| \check{n} \in \tau \|$, for $n \in \omega$. Then for the corresponding name $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ we have

$$1 \Vdash \tau = \tau_x. \tag{10}$$

Now, by Claims 1 and 2 from the proof of Theorem 6.4 from [11] we have

$$0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(x) \setminus \lambda^*_{\mathrm{ls}}(x).$$

By Facts 2.2(a) and 2.3(g) we have $\lim_{\mathcal{O}_{\lambda_s}}(x) = \lambda^*_s(x) = \lambda^*_{\mathrm{ls}}(x) \cap \lambda^*_{\mathrm{li}}(x)$ and, since $0 \notin \lambda^*_{\mathrm{ls}}(x)$, it follows that $0 \notin \lim_{\mathcal{O}_{\lambda_s}}(x)$.

By Theorem 3.1 we have $\lim_{\mathcal{O}_{\mathrm{lsi}}}(x) = \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(x) \cap \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}(x)$ and, since $0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}(x)$, it remains to be proved that $0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}(x)$. But, if $0 \in O \in \mathcal{O}_{\lambda_{\mathrm{li}}}$, then, since O is an upward-closed set, we have $O = \mathbb{B}$. Consequently, $x_n \in O$, for all $n \in \omega$, so $0 \in \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}(x)$. $\square$

Example 3.7. An algebra for which the diagrams describing convergences and topologies from Figure 1 contain exactly 9 and 4 different objects respectively.

If $\mathbb{B}$ is a complete Boolean algebra which collapses $2^{\mathfrak{h}}$ to ω (e.g. the collapsing algebra $\mathrm{Coll}(\omega, 2^{\mathfrak{h}}) = \mathrm{r.o.}({}^{<\omega}(2^{\mathfrak{h}}))$), then $\mathbb{B}$ satisfies the assumptions of Theorem 3.6 and, hence, $\lim_{\mathcal{O}_{\lambda_s}} < \lim_{\mathcal{O}_{\mathrm{lsi}}}$ and $\mathcal{O}_{\mathrm{lsi}} \subsetneq \mathcal{O}_{\lambda_s}$. By Theorem 6.4 from [11] the same conditions provide that the convergence λ_{ls} is not weakly topological, which, by Fact 2.1(a), gives $\lambda^*_{\mathrm{ls}} < \lim_{\mathcal{O}_{\lambda_{\mathrm{ls}}}}$. By Theorem 4.4 from [11], the mapping $h : \langle \mathbb{B}, \mathcal{O}_{\lambda_{\mathrm{ls}}} \rangle \to \langle \mathbb{B}, \mathcal{O}_{\lambda_{\mathrm{li}}} \rangle$ given by $h(b) = b'$, for each $b \in \mathbb{B}$, is a homeomorphism, so $\lambda^*_{\mathrm{li}} < \lim_{\mathcal{O}_{\lambda_{\mathrm{li}}}}$ as well. Assuming that $\lambda_{\mathrm{ls}} = \lambda^*_{\mathrm{ls}}$, by duality we would have $\lambda_{\mathrm{li}} = \lambda^*_{\mathrm{li}}$ and, by Theorem 3.1, $\lim_{\mathcal{O}_{\lambda_s}} = \lambda^*_s = \lambda^*_{\mathrm{ls}} \cap \lambda^*_{\mathrm{li}} = \lambda_{\mathrm{ls}} \cap \lambda_{\mathrm{li}} = \lambda_s$. But this is not true since the algebra $\mathbb{B}$ is not $(\omega, 2)$-distributive. Thus $\lambda_{\mathrm{ls}} < \lambda^*_{\mathrm{ls}}$ and, similarly, $\lambda_{\mathrm{li}} < \lambda^*_{\mathrm{li}}$. By Fact 2.2(b) we have $\lambda_s < \lim_{\mathcal{O}_{\lambda_s}}$. The rest follows from Theorem 3.1.

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